# Characterizing Law of Demand Violations for Separable Utility

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**Abstract**

The question of when the Law of Demand for multiple goods holds or fails to hold has been studied for more than thirty years starting with Milleron, Mitjuschin and Polterovich followed by Kannai, Mas-Colell and Quah. Conditions for the Law of Demand for individuals to hold were formulated in terms of utility functions representing underlying preferences generating the demand. However, relatively few explicit classes of utility functions have been proposed for which the Law of Demand fails to hold. In this paper, we put forward an easily computable test for determining whether or not the necessary and sufficient conditions and not just the sufficient conditions for the Law of Demand are satisfied. This enables us to construct explicit families of preference relations for which one can determine easily which members of the family satisfy monotonicity and actually identify specific regions of violation. The three families of preferences analyzed in this paper will be recognized to be simple extensions of classic CES preferences. Technically, the use of least concave representations and the set of minimum concavity points are key to our simplification for separable utilities.

# 1 Introduction

*We would like to thank Herakles Polemarchakis for his insightful suggestions. Ear-

The question of when the Law of Demand for multiple goods holds or fails to hold has been studied for more than thirty years starting with Milleron [12] and Mitjuschin and Polterovich [13] followed by Kannai [7], Mas-Colell [11] and Quah [17]-[18]. Conditions for the Law of Demand for individuals to hold (i.e., demand is monotone) were formulated in terms of utility functions representing underlying preferences generating the demand. However, relatively few explicit classes of utility functions have been proposed for which the Law of Demand fails to hold. This is related to the fact that the necessary and sufficient conditions were not readily verifiable (unlike the more manageable sufficient conditions). In this paper, we put forward an easily computable test for determining whether or not the Law of Demand is satisfied. This enables us to construct explicit families of preference relations for which one can determine easily which members of the family fail to satisfy monotonicity and in which specific regions in the consumption space.

Additional simplification is achieved by considering the least concave representations of the underlying preferences and sets of minimum concavity points. It is well-known that for each direct concavifiable utility function [6] defined on a specific domain, there exists a unique (up to an affinely equivalent transformation) least concave form. For the least concave form, the Hessian determinant vanishes at a certain set of the points, which we refer to as minimum concavity points. In general, the set of the minimum concavity points will vary based on the type of the utility function and the domain over which the utility is defined. We characterize the case when the set of minimum concavity points is the entire domain and give specific examples where it is not. We point out that at a minimum concavity point, the classic Mitjuschin-Polterovich-Milleron ([13]-[12]) sufficient condition for the monotonicity of demand becomes necessary and sufficient. We follow Quah [18] in exploiting the simplification that results from assuming that the underlying preferences are separable. The three families of preferences analyzed in this paper will be recognized to be simple extensions of classic CES preferences. Furthermore we consider extensions to Lancanster preferences defined over attributes [10], and to habit formation models (e.g., Pollak [14]).

In Section 2, we introduce the least concave form, define the set of minimum concavity points and state a simple computable test for monotonicity (Theorem 1). We discuss additively separable utility in Section 3 and show

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in Proposition 1 the relation between the set of minimum concavity points and homotheticity and quasihomotheticity. In Section 4, one of the families referred to above, is the following variation of CES utility

$$U(\mathbf{c}) = -\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i}, \tag{1}$$

where $\mathbf{c} =_{def} (c_1, ..., c_n)$ and $\delta_i > -1$ $(i = 1, ..., n)$. This family of utility functions is parameterized by $\{\delta_i\}$. Since for the case of $n = 2$, the elasticity of substitution is shown to be a natural weighted average of the $\delta_i$, it will be referred to as WAES (weighted average elasticity of substitution) utility. This class of utility exhibits a number of very special and interesting demand properties which are discussed in Section 5 and in Kannai and Selden [8]. This example provides a totally transparent view of the nature of violations of monotonicity. Section 5 contains as well a verification of Theorem 1 and discussion of when the conditions for the monotonicity of demand hold for two additional specific forms of utility, translated CES and translated WAES. For each case we derive highly transparent necessary and sufficient conditions for monotonicity to hold and identify regions of violation (in parameter space). Selected non separable cases (such as the Lancaster attribute and habit formation models) are discussed briefly in Sections 3 and 5. Section 6 provides extension of the key results to the case of $n > 2$ commodities. Concluding comments are provided in Section 7. Finally in Appendix A, we provide a Table of classes of utility discussed in the paper and in Appendix B we study the geometry of the set of minimum concavity points and provide a proof of Proposition 1.

## 2 Least concave representations

Let a convex preference ordering $\preceq$ (defined on a convex subset $\Omega$ of the space of $n$ commodities $\mathbf{c} =_{def} (c_1, ..., c_n)$) be represented by a utility function $U(\mathbf{c})$. We assume that there exists at least one such concave $U$, i.e., that $\preceq$ is concavifiable. Recall that a utility function $u = F(U)$ is a *least concave* utility function for the given $\succeq$ if for every concave utility function $v$ representing $\succeq$, the function $v \circ u^{-1}$ is concave (Debreu [2]). A well-known computation shows that a necessary condition for $u = F(U)$ to be concave is that for every non-zero vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$, the inequality

$$\frac{F''(U(\mathbf{c}))}{F'(U(\mathbf{c}))} \leq -\frac{(\partial^2 U \boldsymbol{\xi}, \boldsymbol{\xi})}{(\partial U, \boldsymbol{\xi})^2} \tag{2}$$

holds. Let

$$a(U, \mathbf{c}) =_{def} \sup_{\boldsymbol{\xi} \in \mathbb{R}^n} \frac{(\partial^2 U \boldsymbol{\xi}, \boldsymbol{\xi})}{(\partial U, \boldsymbol{\xi})^2}. \tag{3}$$

The inequality (2) has to hold for all non-zero vectors $\boldsymbol{\xi} \in \mathbb{R}^n$. Hence we have to know the value of the infimum of the right hand side of (2), i.e., the value of $-a(U, \mathbf{c})$. While this value is computed in [3], here we derive a simple expression for completeness. In particular, we derive a simple expression for $a(U, \mathbf{c})$ in terms of the Hessian determinant $H$ and bordered Hessian determinant $B_H$ of $U$.

The right hand side of (2) is stationary with respect to $\boldsymbol{\xi}$ if and only if $(\partial U, \boldsymbol{\xi})\partial^2 U \boldsymbol{\xi} = \partial U(\partial^2 U \boldsymbol{\xi}, \boldsymbol{\xi})$. Inverting the matrix $\partial^2 U$ we see that at the stationary points,

$$\frac{(\partial^2 U \boldsymbol{\xi}, \boldsymbol{\xi})}{(\partial U, \boldsymbol{\xi})^2} = (\partial^2 U^{-1} \partial U, \partial U)^{-1}. \tag{4}$$

We now have the well know fact.

**Claim 1** *For every non-singular $n \times n$ matrix $A = (a_{ij})$ and for every non-zero vector $\mathbf{x} \in \mathbb{R}^n$,*

$$(A^{-1}x, x) = -\frac{\det(B)}{\det(A)} \tag{5}$$

*where $B$ is the bordered matrix*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & x_1 \\ a_{21} & \cdots & a_{2n} & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & x_n \\ x_1 & \cdots & x_n & 0 \end{pmatrix}. \tag{6}$$

To prove the claim, choose coordinates such that $\mathbf{x} = |x|\mathbf{e}_n$. Let $A'$ denote the $(n-1) \times (n-1)$ matrix $(a_{ij}),\ i, j = 1, \cdots, n-1$. Then the $n \times n$ element of $A^{-1}$ is given by $\det(A')/\det(A)$. Hence $(A^{-1}x, x) = |x|^2 \det(A')/\det(A)$. On the other hand, in this coordinate system

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ a_{21} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & |x| \\ 0 & \cdots & |x| & 0 \end{pmatrix} \tag{7}$$

so that $\det(B) = -|x|^2 \det(A')$, proving (5). (Observe that in our case $A$ is symmetric and (5) can be proved by diagonalization.)

It follows that

$$a(U,\mathbf{c}) = -\frac{H}{B_H}. \tag{8}$$

Hence a necessary condition for $u = F(U)$ to be concave is that

$$\frac{F''(U(\mathbf{c}))}{F'(U(\mathbf{c}))} \leq -a(U,\mathbf{c}) = \frac{H}{B_H}. \tag{9}$$

The right hand side is well-defined if the curvature of the indifference surface is non-zero. (This condition is satisfied in all of the examples considered in this paper.) However, the left hand side of (9) depends on $\mathbf{c}$ only via $U(\mathbf{c})$ and so is constant on each indifference surface $U(\mathbf{c}) = t$. It follows that $u(\mathbf{c}) = F(U(\mathbf{c}))$ is concave if and only if $F''(t)/F'(t)$ is majorized by the infimum of $-a(U,\mathbf{c})$ over the indifference surface $U(\mathbf{c}) = t$. We denote this infimum by $G(t)$. It is well-known (see e.g., [6]) that if $F''(t)/F'(t) = G(t)$ for all $t \in U(\Omega)$ then $u(\mathbf{c}) = F(U(\mathbf{c}))$ is a least concave representation of the preference ordering $\succeq$ on $\Omega$. Thus we need to evaluate $G(t)$.

Observe that if the least concave utility $u = F(U)$ then

$$a(u,\mathbf{c}) = \frac{[a(U,\mathbf{c}) + \frac{F''(U(\mathbf{c}))}{F'(U(\mathbf{c}))}]}{F'(U(\mathbf{c}))}. \tag{10}$$

In particular, if there exists a point $\mathbf{c}^\star$ such that $F''(t)/F'(t) = -a(U,\mathbf{c}^\star)$ with $t = U(\mathbf{c}^\star)$, then $a(u,\mathbf{c}^\star) = 0$. In this case, we may say that $\mathbf{c}^\star$ is a "minimum concavity point" and $u$ is pointwise least concave at $\mathbf{c}^\star$. (This is equivalent to $-a(U,c^*)$ being the minimum of $-a(U,c)$ on the indifference surface $U(c) = U(c^*)$.) To characterize the set of minimum concavity points we need the following somewhat broader definition.

**Definition 1** *If the least concave form $u(\mathbf{c})$ is defined on $\Omega$, then the set of the minimum concavity points is*

$$\left\{\mathbf{c}^\star \in \overline{\Omega} \ \mid a(u,\mathbf{c}^\star) = 0\right\}, \tag{11}$$

*where $\overline{\Omega}$ is a compact expansion of $\Omega$. In the special case $\Omega = \mathbb{R}^n_+$, $\overline{\Omega}$ is defined as the Cartesian product of the non-negative line with $+\infty$ added.*

It should be noted that if the set of the minimum concavity points is all of $\Omega$, then the least concave form is the same for all subdomains. On the other hand if the set of minimum concavity points is not all of $\Omega$, then the least concave form may very well be different for subdomains of $\Omega$.

For the case where $n = 2$, Table 1 in [9] (reproduced in Appendix A below) illustrates the set of minimum concavity points for 8 forms of utility including the classic CES and HARA (hyperbolic absolute risk aversion) cases.[1] Rows 6 and 7 correspond to the WAES class introduced in Section 4 and row 8 corresponds to the Wold-Jureen form. The set of minimum concavity points corresponds to the column labeled $(c_1^*, c_2^*)$. In Section 4, we will discuss the WAES Row 7 case, where the set of minimum concavity points is not the whole choice space.

Our analysis of the Law of Demand hinges crucially on properties of $a(U, \mathbf{c})$ (and on the specialization $a(u, \mathbf{c})$). Let the preference ordering $\succeq$ generate a differentiable demand function $\mathbf{h}(\mathbf{p})$, where we normalize the income by setting $(\mathbf{h}(\mathbf{p}), \mathbf{p}) = 1$. The demand is *monotone* at $\mathbf{p}$ if for every nonzero $x \in R^n$ the inequality

$$\sum_{i,j}^{n} \frac{\partial h_i}{\partial p_j} x_i x_j < 0 \tag{12}$$

holds for every $x \in \mathbb{R}^n$, where $\mathbf{h} = (h_1, \cdots, h_n)$. A key tool in the explicit testing for monotonicity is the following variant of the Mitjuschin-Polterovich [13] result which we demonstrate and apply in Section 5.

**Theorem 1** *The demand* $\mathbf{h}(\mathbf{p})$ *is monotone at* $\mathbf{p}$ *if and only if*

$$(\mathbf{c}, \partial U(\mathbf{c}))a(U, \mathbf{c}) - \frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} < 4. \tag{13}$$

Our condition (13) yields a verifiable test for monotonicity. Mitjuschin-Polterovich [13] put forward a sufficient condition for monotonicity (formula (34) below). Now it follows in particular that for the minimum concavity point $\mathbf{c} = \mathbf{c}^*$, the term $(\mathbf{c}, \partial u(\mathbf{c}))a(u, \mathbf{c})$ vanishes and the formula (13) simplifies greatly. Then the Mitjuschin-Polterovich sufficient condition becomes necessary at this point as well.

It should be noted that in general the Mitjuschin-Polterovich sufficient condition evaluated for a least concave direct utility function becomes necessary for monotonicity to hold everywhere only when the set of the minimum concavity points is the whole space. This is not the case in the first example analyzed in Section 5 corresponding to

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{c_2^{-\delta_2}}{\delta_2} \qquad \delta_1, \delta_2 > -1, \tag{14}$$

where we will see that it is possible to identify the regions for monotonicity and non-monotonicity using Theorem 1 while it is not possible to do so using any of the known sufficient conditions.

[1] In Table 1 where the set of minimum concavity points is indicated as the entire choice space, it should be understood to be $\overline{\Omega}$ including the limit points.

## 3 Separable utility

The condition in Theorem 1 is easier to compute for additively separable utility functions. Let the preference ordering $\preceq$ on $\Omega$ be represented by the utility function

$$U(\mathbf{c}) = \sum_{i=1}^{n} U_i(c_i). \tag{15}$$

In this case the Hessian determinant of $U$, i.e., the determinant of the matrix $\partial^2 U$ of second order derivatives of $U$, is given by

$$\mathrm{H}(U) = \det(\partial^2 U) = \prod_{i=1}^{n} U_i''(c_i). \tag{16}$$

(We denote by primes derivatives of functions of a *single* variable.) The bordered Hessian $B_H$ of $U$ is the determinant of the matrix

$$\begin{pmatrix} U_1'' & & & & U_1' \\ & U_2'' & & & U_2' \\ & & \ddots & & \\ & & & U_n'' & U_n' \\ U_1' & & & U_n' & 0 \end{pmatrix}. \tag{17}$$

Expanding the determinant of (17) by the last row (or column), we get the well-known formula

$$\mathrm{B_H} = -\sum_{i=1}^{n} (U_i')^2 \prod_{j \neq i} U_j'' \tag{18}$$

so that the ratio of the bordered Hessian determinant to the Hessian determinant is

$$M = -\frac{B_H}{H} = \sum_{i=1}^{n} \frac{(U_i')^2}{U_i''} \tag{19}$$

(where we assume of course that $H \neq 0$). Now we can express $a(U, \mathbf{c})$ in terms of $M$

$$a(U, c) = \frac{1}{M} \tag{20}$$

implying

$$\frac{F''(U(\mathbf{c}))}{F'(U(\mathbf{c}))} \leq -\frac{1}{M}. \tag{21}$$

The stationary vectors $\xi$ on the right hand side of eqn. (2) take a very simple form in the separable case. In fact, the condition for stationarity becomes $(\sum_{j=1}^{n} u_j\xi_j)u_i''\xi_i = u_i'(\sum_{j=1}^{n} u_j''\xi_j^2)$ , so that the stationary vectors are proportional to $(u_1'/u_1'', \cdots, u_n'/u_n'')$.

We now illustrate the general discussion by considering several examples of separable utility functions. Observe that in Table 1 (in Appendix A) there is a sharp dichotomy between the CES class of utility (e.g., Row 2), where the set of minimum concavity points is the full consumption space, and the WAES case (e.g., Rows 6 and 7), where the set shrinks to a line. Concerning the more general separable case with regard to similar properties of these sets, it is proved in Appendix B that if the preferences are additively separable, then the rows 1 through 5 in Table 1 cover **all** cases where every point in the choice set is a point of minimum concavity.

We reformulate these observations with the following, where the terms homothetic and quasihomothetic are defined as is standard (see [1], pp. 143-5).

**Proposition 1** *Given $\preceq$ defined on a convex subset $\Omega$ of the space of $n$ commodities, suppose that the utility function $U$ is ordinally additively separable. Then the set of the minimum concavity points is the whole space $\mathbb{R}_+^n$ if and only if $U$ is homothetic or quasihomothetic.*

This assertion (and others) is proved in Appendix B.

**Remark 1** *Our approach to characterizing violations of monotonicity is applicable to a range of cases in which preferences do not appear at first sight to be separable. In each instance, the utility function depends on linear functions of the original commodities. The relation between the original and transformed commodities is give by $\mathbf{c}' = A\mathbf{c}$, where $\det A \neq 0$. Possible applications include the following: (1) packages of goods (Hurwicz, Jordan and Kannai [4]), (2) attributes (Lancaster [10] and Rustichini and Siconolfi [15]) and (iii) habit formation (Pollak [14]). The budget equation*

$$(\mathbf{p}, \mathbf{c}) = (\mathbf{p}', \mathbf{c}') \tag{22}$$

*implies that $\mathbf{p} = A^t\mathbf{p}'$, where $A^t$ is the transpose of $A$. Let $h'(\mathbf{p}')$ denote the demand function. By the chain rule,*

$$\frac{\partial h_i'}{\partial p_j'} = \sum \frac{\partial h_k}{\partial p_l}\frac{\partial h_i'}{\partial h_k}\frac{\partial p_l}{\partial p_j'} = \sum \frac{\partial h_k}{\partial p_l} a_{ik}a_{jl}. \tag{23}$$

*Hence,*

$$\sum_{i,j}^{n} \frac{\partial h_i'}{\partial p_j'} x_i x_j = \sum_{i,j}^{n} \frac{\partial h_k}{\partial p_l} a_{ik}a_{jl}x_i x_j = \sum_{i,j}^{n} \frac{\partial h_k}{\partial p_l} y_k y_l, \tag{24}$$

*where $y_k = \sum a_{ik}x_i$. Thus $\mathbf{h}'$ is monotone at $\mathbf{c}'$ if and only if $\mathbf{h}$ is monotone at $\mathbf{c}$.*

# 4 Weighted Average Elasticity of Substitution utility

Suppose that $U$ defined on $\Omega$ takes the form (1) in Section 1,

$$U(\mathbf{c}) = \sum_{i=1}^{n} U_i = -\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i}, \tag{25}$$

where $\delta_i > -1$.[2] Consider first the case where $n = 2$. It is well-known that the elasticity of substitution $\eta$ is given by the formula

$$\eta = \frac{U_1' U_2' (c_1 U_1' + c_2 U_2')}{c_1 c_2 B_H}. \tag{26}$$

If $U$ is given by (25), then

$$\mathrm{B_H} = \frac{\sum_{i=1}^{2} c_i^{-\delta_i}/(1+\delta_i)}{\prod_{i=1}^{2} c_i^{\delta_i+2}/(1+\delta_i)}. \tag{27}$$

Hence

$$\eta = \frac{c_1^{\delta_1} + c_2^{\delta_2}}{c_1^{\delta_1}(\delta_1 + 1) + c_2^{\delta_2}(\delta_2 + 1)}. \tag{28}$$

Observe that in the homothetic case, $\delta_1 = \delta_2 = \delta$ and $\eta = 1/(1+\delta)$. If $\delta_1 \neq \delta_2$, note that $1/\eta$ is a weighted average of $\delta_1 + 1$ and $\delta_2 + 1$, so that $\eta \in (1/(1+\delta_1), 1/(1+\delta_2))$.[3] Because of this property, the utility function (1) will be said to represent WAES (weighted average elasticity of substitution) preferences.

Explicitly, the elasticity of substitution for general $\delta_1$ and $\delta_2$ is just the harmonic mean of the elasticities for the CES cases $\delta = \delta_1$ and $\delta = \delta_2$ with the weights $c_1^{\delta_1}$ and $c_2^{\delta_2}$.

The manageability of computations involving WAES utility enables us to obtain interesting new economic insights quite unlike those in the CES case where $\delta_1 = \delta_2 = \delta$ (see [8]).

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[2] We will assume that $U_i = \log(c_i)$ $(i = 1, 2)$ if $\delta_i = 0$. The Cobb-Douglas case will be considered as a limiting case as $\delta_i \to 0$ via a logarithmic transformation of $U_i = c_i^{\alpha_i}$.

[3] Note that one may generalize eqn. (25) slightly by setting

$$U_i = -\frac{\lambda_i c_i^{-\delta_i}}{\delta_i}$$

where $\lambda_i$ $(i = 1, ..., n)$ are positive constants. We will indicate in the sequel how certain results are affected by this generalization. (Observe that as long as $\delta_i$ is not equal to zero, we may apply the transformation $c_i \to \lambda_i^{-1/\delta_i} c_i$; in the logarithmic case $\delta_i = 0$, $c_i \to c_i^{\lambda_i}$.) In particular, the result $\eta \in (1/(1+\delta_1), 1/(1+\delta_2))$ remains valid.

Consider next the case where $n > 2$ (strictly speaking this case should not be referred to as WAES). Eqn. (27) generalizes to the following

$$B_{\mathrm{H}} = \frac{\sum_{i=1}^{n} c_i^{-\delta_i}/(1+\delta_i)}{\prod_{i=1}^{n} c_i^{\delta_i+2}/(1+\delta_i)}. \tag{29}$$

Moreover

$$M = -\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i + 1}. \tag{30}$$

We will assume in the sequel (unless otherwise stated) that $\delta_i \neq 0, \delta_i > -1$, for all $i = 1, \cdots, n$. The Cobb-Douglas case will be considered as a limiting case as $\delta_i \to 0$ via a logarithmic transformation of $U_i = \ln c_i$.

To find the least concave utility for the WAES utility, we apply (30) to find that

$$\frac{F''(U(\mathbf{c}))}{F'(U(\mathbf{c}))} \leq -a(U, \mathbf{c}) = \left( \sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i + 1} \right)^{-1}. \tag{31}$$

If the constants $\delta_i$ are not all of the same sign, assume without loss of generality that $\delta_1 > 0, \delta_2 < 0$. Assuming that $\Omega$ is the positive orthant and fixing the values of $c_i, i = 3, \cdots, n$, we see that on every indifference surface we can let $c_1$ tend to zero (by letting $c_2$ tend to $\infty$). Hence the right hand side of (31) assume arbitrarily small (positive) values, so that $G(t) = 0$. Choosing $F(t) = t$ we conclude that in this case $U(\mathbf{c})$ itself is a least concave utility function for $\succeq$.

If all constants $\delta_i$ are of the same sign, set $x_i = \frac{c_i^{-\delta_i}}{\delta_i+1}, i = 1, \cdots, n$. Thus we wish to find the infimum of $(\sum_{i=1}^{n} x_i)^{-1}$ over the set $\sum_{i=1}^{n} -((\delta_i + 1)/\delta_i)x_i = -t$ and $x_i > 0, i = 1, \cdots, n$. There is no interior minimum. Hence the infimum is attained at a vertex. But this infimum is the reciprocal of the supremum, i.e, the reciprocal of the maximum of $\sum_{i=1}^{n} x_i$ over the closed simplex, so that we seek the maximum of $-(\delta_i t)/(\delta_i + 1),\ i = 1, \cdots, n$. Assume without loss of generality that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Then the maximum is attained at $(x_1, 0, \cdots, 0)$ if every $\delta_i$ is positive and at $(0, \cdots, 0, x_n)$ if each $\delta_i$ is negative. This implies that $G(t) = -(\delta_1 + 1)/(\delta_1 t)$ in the positive case and $G(t) = -(\delta_n + 1)/(\delta_n t)$ in the negative case. Integrating, we find that the function $F(t) = -|t|^{-1/\delta_1}$ ( $F(t) = -|t|^{-1/\delta_n}$) satisfies $F''(t)/F'(t) = G(t)$ in the positive (negative) case, or that

$$u(\mathbf{c}) = |\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i}|^{-1/\delta_1} (= |\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i}|^{-1/\delta_n}), \tag{32}$$

respectively, are least concave representations of $\succeq$ in the positive orthant. Similar considerations apply if $\Omega$ is a simple set, such as the product of an interval with half-lines. It is also possible to analyze in the same manner several other separable orderings, such as the HARA and Wold-Jureen utility functions, or if one of the summands is logarithmic.

Finally, we want to return to our observation in Section 2 that the specific form of the least concave utility $u$ can vary depending on subdomains of $U$ if the set of minimum concavity points is not all of $\Omega$. Consider the particular form of WAES utility where $\delta_1 > 0 > \delta_2$, Row 7 in Table 1 in Appendix A. From the Table, we can see that the least concave form will depend on the value of $a$ or $b$ if the domain of $U$ is $[a, b] \times (0, \infty)$. This is important, since certain expressions, such as those which characterize monotonicity, can be simplified at minimum concavity points $\mathbf{c}^\star$. We will discuss this issue in more detail in the next Section.

# 5 Characterizing violations of monotonicity

## 5.1 Monotonicity and a modified Mitjuschin-Polterovich coefficient

In this subsection we prove Theorem 1, and exhibit computationally convenient forms for checking monotonicity in the separably-additive case. This enables us to exhibit one specific class of preference orderings in this Subsection and two additional classes in Subsection 5.2 for which demand can be both monotone and non-monotone. For each class we characterize monotone and non-monotone demand behavior in terms of parameters of the family.

It is proved in [13] that $\mathbf{h}$ is monotone if and only if the inequality[4]

$$\frac{(\mathbf{c}, \partial U(\mathbf{c}))}{(\partial^2 U^{-1}(\mathbf{c}) \partial U(\mathbf{c}), \partial U(\mathbf{c}))} - \frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} < 4 \tag{33}$$

holds for all $\mathbf{c} \in \Omega$. It was observed in [13] that if $U$ is concave, the first term in the left hand side of (33) is non-positive, so that a *sufficient* condition for monotonicity of $\mathbf{h}$ is

$$-\frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} < 4 \tag{34}$$

and the left hand side of (34) was referred to as the Mitjuschin-Polterovich coefficient. (See also Milleron [12]).

[4] It should be noted that Milleron [12], Mas-Colell ([11], p. 282) and Quah [18] define the monotonicity condition eqn. (12) as a strict inequality. Mas-Colell for instance refers to this as strict monotonicity. Alternatively, Mitjuschin and Polterovich [13] and Kannai [7] define monotonicity as a weak inequality. For the latter $h$ is monotone if and only if the left hand side of eqn. (33) is $\leq 4$.

The paper [13] is not readily accessible. A proof of (34) is supplied in [11], p. 282. Inspection of the argument shows that actually (33) is proved, if account is taken of the identity [13] $1/(\partial^2 U^{-1}(\mathbf{c})\partial U(\mathbf{c}), \partial U(\mathbf{c})) = \sup(\partial^2 U(\mathbf{c})\mathbf{y}, \mathbf{y})$ where the supremum is over the set of $\mathbf{y}$ for which $(\mathbf{y}, \partial U(\mathbf{c})) = 1$. Alternatively, using the same identity one can easily show that (33) is equivalent to equation (3) in [7].

It follows from (5) and the definition (19) of $M$ as the ratio of the bordered Hessian to the Hessian, that

$$(\partial^2 U^{-1}(\mathbf{c})\partial U(\mathbf{c}), \partial U(\mathbf{c})) = M. \tag{35}$$

By (8), (33), and (35), demand is monotone at $\mathbf{c}$ if and only if

$$\frac{(\mathbf{c}, \partial U(\mathbf{c}))}{M} - \frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} = (\mathbf{c}, \partial U(\mathbf{c}))a(U, \mathbf{c}) - \frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} < 4 \tag{36}$$

thus verifying Theorem 1.

We denote the middle term of (36) by $\tilde{M}_U$, and refer to it as the modified Mitjuschin-Polterovich coefficient. (This coefficient was introduced in Quah [18]; we exhibit below computationally convenient forms.) As observed earlier, eqn. (36) is necessary and sufficient and the second term is sufficient for demand to be monotone so long as $U$ is concave. Because (36) is invariant under increasing monotonic transformations of $U$, we can choose in particular $U = u$ and rewrite (36) in terms of the least concave utility $u$ as follows

$$\tilde{M}_u = (\mathbf{c}, \partial u(\mathbf{c}))a(u, \mathbf{c}) - \frac{(\partial^2 u(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial u(\mathbf{c}))} < 4. \tag{37}$$

The form (37) may be more convenient than (36) because $a(u, \mathbf{c})$ might vanish somewhere; in the latter case the Mitjuschin-Polterovich condition (34) becomes necessary as well. Recall that a point $\mathbf{c}^*$ for which $a(u, \mathbf{c}^*) = 0$ is a point of minimum concavity (see also Quah [18]). Therefore, we can use the original Mitjuschin-Polterovich sufficient condition (34) with the least concave form as a necessary and sufficient condition for montonicity everywhere in the case where each point in $\Omega$ is a point of minimum concavity. This can be seen also directly from inspection of eqn. (33): the numerator of the first term on the left hand side cannot vanish since $u$ is increasing in $\mathbf{c}$, hence this term can vanish if and only if the value of the denominator is infinite, i.e., the Hessian matrix has a zero eigenvalue and so the Hessian determinant vanishes.

We now provide an example in which one can determine for each point in the consumption space whether monotonicity holds or not. The characterization is actually quite simple. Domains of monotonicity and non-monotonicity are illustrated in Figures (1) and (2) for the full commodity space and the parameter space, respectively. Also we show that except in a very special case, the set of the minimum concavity points is not the whole space.

**Example 1** *Consider the $n = 2$ WAES utility function*

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{c_2^{-\delta_2}}{\delta_2}, \tag{38}$$

*where $\delta_1, \delta_2 > -1$. The WAES utility is neither homothetic nor quasihomothetic, hence by Proposition 1 the set of minimum concavity point is not the whole space. Straightforward computation yields*

$$\tilde{M}_U = \frac{(\delta_1 - \delta_2)^2 c_1^{\delta_1} c_2^{\delta_2}}{(c_1^{\delta_1} + c_2^{\delta_2})((1+\delta_1)c_1^{\delta_1} + (1+\delta_2)c_2^{\delta_2})}. \tag{39}$$

*To find out when $\tilde{M}_U < 4$, define $t = c_1^{\delta_1}/c_2^{\delta_2}$. Then*

$$\tilde{M}_U = \frac{(\delta_1 - \delta_2)^2 t}{(1+t)((1+\delta_2) + (1+\delta_1)t)}. \tag{40}$$

*Solving the inequality $\tilde{M}_U < 4$ and investigating the roots of $\tilde{M}_U = 4$, we find that if $(\delta_1 - \delta_2)^2 < 8(\delta_1 + \delta_2)$ then (36) holds for all positive $t$, i.e., demand is monotone over all of the positive orthant. If, however, $(\delta_1 - \delta_2)^2 \geq 8(\delta_1 + \delta_2)$ and $\delta_1, \delta_2 > -1$, then elementary calculation shows that the roots $t_1, t_2$ of the quadratic equation $\tilde{M}_U = 4$ are real. If, in addition, $\delta_1 + \delta_2 > 2$, then the roots are strictly positive and monotonicity is violated for positive $(c_1, c_2)$-pairs*[5] *– namely, for those consumption pairs for which*

$$t_1 < \frac{c_1^{\delta_1}}{c_2^{\delta_2}} < t_2 \tag{41}$$

*holds. The resulting region defined by eqn. (41) corresponds to a subset of the consumption space bounded by two curves (generalized parabolas or parabolas). See Figure 1. (See also [8] for a discussion of the nature and properties of this region.) Hence, the necessary and sufficient condition for monotonicity to hold everywhere in $\Omega$ is*

$$(\delta_1 - \delta_2)^2 < 8(\delta_1 + \delta_2) \qquad \text{or} \qquad \delta_1 + \delta_2 \leq 2. \tag{42}$$

*The regions of monotonicity holding and failing are illustrated in Figure 2. Note that if $\delta_1 = \delta_2$ then $\tilde{M}_U = 0$ implying that demand is monotone.*

[5] Note that one may generalize the utility function slightly by setting $U_i = -\frac{\lambda_i c_i^{-\delta_i}}{\delta_i}$ $(i = 1, 2)$ where $\lambda_i$ are positive constants. The inequality $(\delta_1 - \delta_2)^2 < 8(\delta_1 + \delta_2)$ characterizing monotonicity will still hold.

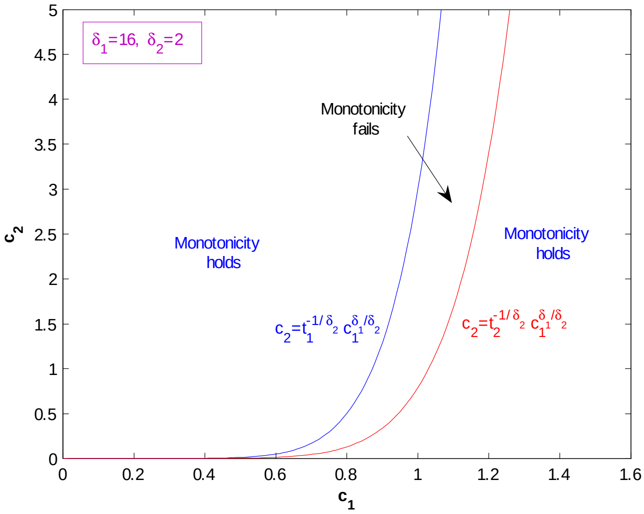



Figure 1:

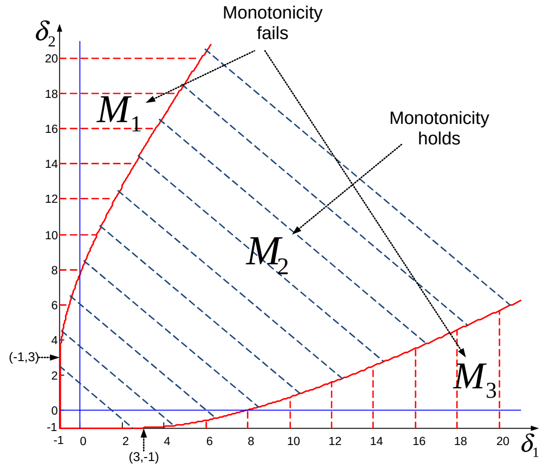



Figure 2:

*This can be seen in the Figure where the $\delta_1 = \delta_2$ ray lies completely in the monotonicity region. Also, it is clear that if the values of $\delta_1$ and $\delta_2$ lie in the southwest triangle of Figure 2 formed by the points $(-1, 3)$, $(-1, -1)$ and $(3, -1)$, then $\delta_1 + \delta_2 < 2$ and demand will be monotone. It should be noted that for WAES utility, if the necessary and sufficient condition for monotonicity is violated then there exists a curve in the commodity space defined by $c_1^{\delta_1} = tc_2^{\delta_2}$, where $t \neq 0$ and each point along the curve has the same $\tilde{M}_U$ value $> 4$. Note that this curve, which lies between the boundary curves in Figure 1, intersects every vertical or horizontal line in $\Omega$.*

## 5.2 Contrasting characterizations and sufficient conditions

As we saw in Example 1, not every point in $\Omega$ is a point of minimum concavity even in simple cases. It will be convenient to consider the set of maximal values for $\tilde{M}_U$

$$\left\{ \mathbf{c} \in \overline{\Omega} \,\middle|\, \tilde{M}_U(\mathbf{c}) = \tilde{M}_U^{\max} \right\}. \tag{43}$$

Note that the sets of points that satisfy $\tilde{M}_U(\mathbf{c}) = \tilde{M}_U^{\max}$ and $\tilde{M}_u(\mathbf{c}) = \tilde{M}_u^{\max}$ are the same. (Because it will generally be easier to compute $\tilde{M}_U^{\max}$, we will base the calculation on $\tilde{M}_U$.) Unless stated explicitly otherwise, all orderings are defined on the full orthant.

We will make frequent use of the following Corollary:

**Corollary 1** *A necessary and sufficient condition for monotonicity to hold everywhere is that for $\forall \mathbf{c} \in \{\mathbf{c}^\star \mid a(u, \mathbf{c}^\star) = 0\}$*

$$-\frac{(\partial^2 u(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial u(\mathbf{c}))} < 4, \tag{44}$$

*where $u$ is a least concave representation of the ordering defined by $U$, if and only if*

$$\left\{ \mathbf{c} \,\middle|\, \tilde{M}_U(\mathbf{c}) = \tilde{M}_U^{\max} \right\} \cap \{\mathbf{c}^\star \mid a(u, \mathbf{c}^\star) = 0\} \neq \emptyset. \tag{45}$$

Next we investigate whether Corollary 1 can be applied to the $n = 2$ WAES utility function.

**Remark 2** *In Example 1 in the prior Subsection, we obtained the necessary and sufficient condition for monotonicity to hold for the WAES class of utility. We next examine whether the set of minimum concavity points and $\tilde{M}_U^{\max}$ points intersect. Maximizing eqn. (40) with respect to $t$ yields*

$$\tilde{M}_U^{\max} = \left( \sqrt{1 + \delta_1} - \sqrt{1 + \delta_2} \right)^2. \tag{46}$$

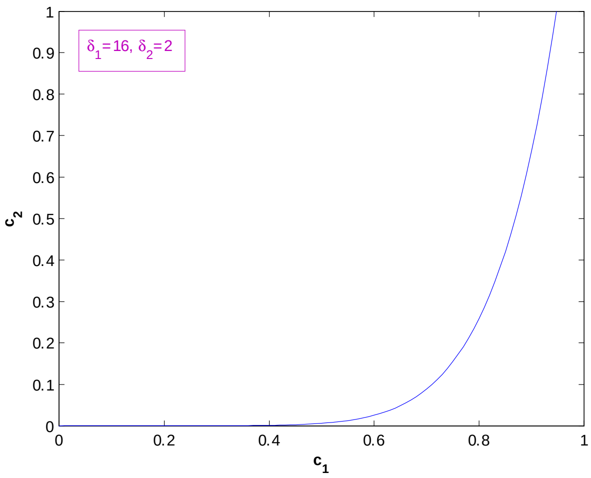



Figure 3: Set of $\tilde{M}_U = \tilde{M}_U^{\max}$

*with the maximum at*

$$t = \sqrt{\frac{1+\delta_2}{1+\delta_1}}. \tag{47}$$

*Therefore, we have*

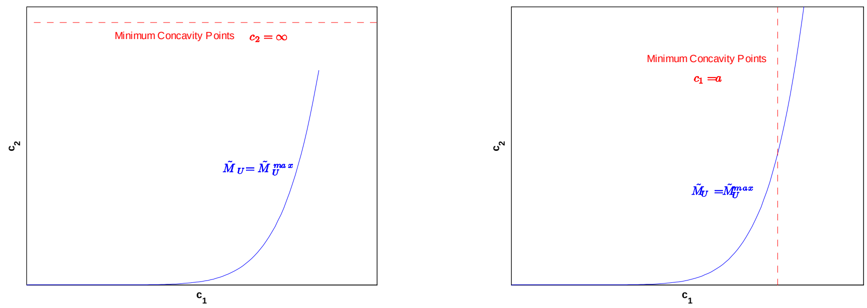
$$\left\{ \mathbf{c} \;\middle|\; \tilde{M}_U(\mathbf{c}) = \tilde{M}_U^{\max} \right\} = \left\{ (c_1, c_2) \;\middle|\; c_2 = \left( \sqrt{\frac{1+\delta_1}{1+\delta_2}} c_1^{\delta_1} \right)^{1/\delta_2} \right\}. \tag{48}$$

*This set is plotted in Figure 3. If the utility function is defined on* $(0,\infty) \times (0,\infty)$ *and* $\delta_1 > \delta_2 > 0$*, the set of minimum concavity points is* $\{(c_1, \infty)\}$*, which has no intersection with the set of* $\tilde{M}_U^{\max}$ *(see Fig. 4(a)). Therefore, we cannot apply Corollary 1 to conclude that eqn. (44) becomes necessary as well as sufficient because* $a(u, \mathbf{c})$ *does not vanish on the set of* $\tilde{M}_U^{\max}$*. However we can apply Corollary 1 if we change the domain to* $[a,b] \times (0,\infty)$*. It can be verified that the least concave form is*

$$u(c_1, c_2) = \left( \frac{\delta_2}{\delta_1} \left( c_1^{-\delta_1} - a^{-\delta_1} \right) + c_2^{-\delta_2} + \frac{\delta_2 + 1}{\delta_1 + 1} a^{-\delta_1} \right)^{-1/\delta_2} \tag{49}$$

*for* $\delta_1 > \delta_2 > 0$*. The set of the minimum concavity points is* $\{(a, c_2)\}$ *and eqn. (45) is satisfied (see Fig. 4(b)) (see Table 1, Row 7). Setting* $t_a = a^{\delta_1}/c_2^{\delta_2}$*, we see from Corollary 1 that*

$$\frac{(\delta_1 - \delta_2)^2 t_a}{(1+t_a)((1+\delta_2) + (1+\delta_1)t_a)} < 4 \tag{50}$$




(a) $\Omega = (0, \infty) \times (0, \infty)$ (b) $\Omega = [a, b] \times (0, \infty)$

Figure 4: $\tilde{M}_U^{max}$ Set and Set of Minimum Concavity Points

*will give the same necessary and sufficient condition as obtained in eqn. (42).*

As is well known, in the CES form of utility $U(c_1, c_2) = -\frac{c_1^{-\delta}}{\delta} - \frac{c_2^{-\delta}}{\delta}$, where $\delta \geq -1$, every point is a minimum concavity point and monotonicity holds. We next consider two additional variations of the CES class. One case where monotonicity does not prevail but every point in the space is a minimum concavity point is given by Example 2 below. Cases where monotonicity can fail but not every point in the space is a minimum concavity point are Examples 1 and 3. They differ by whether or not the sets of minimum concavity and $\tilde{M}_U$ maximal points intersect (depending on the specification of the domain of $U$).

Suppose we consider the case of CES preferences with translated origins.[6]

**Example 2** *Consider the $n = 2$ utility function*

$$U(c_1, c_2) = -\frac{(c_1 + q_1)^{-\delta}}{\delta} - \frac{(c_2 + q_2)^{-\delta}}{\delta}, \tag{51}$$

*where $q_1, q_2 \geq 0$ $(q_1^2 + q_2^2 \neq 0)$ and $\delta > -1$. The set of the minimum concavity points is again the whole space and so we can use original Mitjuschin-Polterovich sufficient condition (34) with the least concave form as a necessary and sufficient condition for monotonicity. The least concave form is given by*

$$u(c_1, c_2) = \left((c_1 + q_1)^{-\delta} + (c_2 + q_2)^{-\delta}\right)^{-1/\delta} \tag{52}$$

[6]It should be noted that the translated CES case can be easily be generalized to the HARA form indicated in Table 1, Row 1 where the formulas become a bit more involved.

*(see Table 1, Row 1 where $b = e = 1$ and $a = q_1$ and $d = q_2$). It can be verified that*

$$\tilde{M}_u = -\frac{(\partial^2 u(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial u(\mathbf{c}))} = \frac{\left(\frac{(1+\delta)c_1}{c_1+q_1} - \frac{(1+\delta)c_2}{c_2+q_2}\right)^2}{(1+\delta)\, AB}. \tag{53}$$

*where*

$$\begin{aligned} A &= (c_1+q_1)^\delta + (c_2+q_2)^\delta, & (54)\\ B &= (c_1+q_1)^{-1-\delta} c_1 + (c_2+q_2)^{-1-\delta} c_2. & (55) \end{aligned}$$

*If the preference ordering is defined on $(0,\infty)\times(0,\infty)$, then*

$$\begin{aligned} \tilde{M}_u &\le (1+\delta)\frac{\left(\frac{c_1}{c_1+q_1} - \frac{c_2}{c_2+q_2}\right)^2}{\frac{c_1}{c_1+q_1} + \frac{c_2}{c_2+q_2}} \\ &\le (1+\delta)\left|\frac{c_1}{c_1+q_1} - \frac{c_2}{c_2+q_2}\right| \le 1+\delta, \end{aligned} \tag{56}$$

*where the equal sign can be obtained if and only if $\delta > 0$ and $(c_1, c_2) = (0,\infty)$ or $(\infty, 0)$, the necessary and sufficient condition for monotonicity everywhere in the $(c_1, c_2)$ plane is*[7]

$$\tilde{M}_u < 4 \Leftrightarrow \delta \le 3. \tag{57}$$

*If the preference ordering is defined on $[a,b]\times(0,\infty)$, the necessary and sufficient for monotonicity can be simplified to*

$$\tilde{M}_u \le \frac{\left(\frac{(1+\delta)b}{b+q_1}\right)^2}{\frac{(1+\delta)b}{b+q_1} + \frac{(1+\delta)q_2^\delta b}{(b+q_1)^{1+\delta}}} \le 4. \tag{58}$$

*This expression illustrates that although the least concave form is the same for any domain, the necessary and sufficient condition for monotonicity to hold can vary depending on the domain of $U$. It should be noted that while the center term increases with $b$ when $\delta > 0$, $\tilde{M}_u$ tends to the limit $1+\delta$ as $b \to \infty$ . Thus we have the same restriction as in the case (57) where the domain for $c_1$ is $(0,\infty)$. However for $b < \infty$, the condition for violation becomes more restrictive.*

[7]Since in this case and others considered below where the necessary and sufficient condition for monotonicty to hold, $\tilde{M}_U < 4$, can be reached only at a limit point, we will express the condition as $\tilde{M}_U \le 4$.

We saw in Example 1 that readily characterized violations of monotonicity occur without the presence of translated origins. It should be stressed that for this example none of the known sufficient conditions can identify the necessary and sufficient condition obtained using Theorem 1.

**Remark 3** *It should be noted that for Example 1, the Mitjuschin condition (44) is only sufficient and not necessary in general. Assume $\delta_1 > \delta_2 > 0$. The least concave function can be written as more explicitly*

$$u = (-\delta_2 U)^{-\frac{1}{\delta_1}}. \tag{59}$$

*Computing* $-\dfrac{(\partial^2 u(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial u(\mathbf{c}))}$ *for (59) and setting* $t = c_1^{\delta_1}/c_2^{\delta_2}$*, we obtain a quadratic inequality in $t$ which is valid for all $t$ if and only if*

$$\delta > \frac{3}{2}, e < \frac{\sqrt{32\delta + 1} - 1}{4};\ 0 < \delta < \frac{3}{2}, e < \delta, \tag{60}$$

*where $\delta = \frac{\delta_1 + \delta_2}{2}$ and $e = \frac{\delta_1 - \delta_2}{2}$. This condition is clearly different from the necessary and sufficient condition we obtained above.*

**Remark 4** *In [18], Quah provides a simple sufficient condition for monotonicity to hold in the case of additively separable utility functions. Let $U$ assume the form in eqn. (15). Then Quah shows that a sufficient condition for monotonicity to hold is*

$$B_U = \max_{1 \le i \le n} \left( -\frac{x_i U_i''(x_i)}{U_i'(x_i)} \right) - \min_{1 \le i \le n} \left( -\frac{x_i U_i''(x_i)}{U_i'(x_i)} \right) < 4. \tag{61}$$

*Applying this condition to the $n = 2$ WAES case, one obtains*

$$|\delta_1 - \delta_2| < 4. \tag{62}$$

*In Figure 2, this inequality defines the region between two $45^\circ$ rays, which start from the points $(-1, 3)$ and $(3, -1)$. We highlight this area, labeled $M_2$, in Figure 5. Clearly this region is smaller than the region for monotonicity given by the necessary and sufficient condition (42).*

Next we consider WAES utility with translated origin and find that the characterization of monotonicity takes a completely different form from the WAES case. Moreover the set of minimum concavity points and $\tilde{M}_U^{\max}$ points do not intersect either for $(0, \infty) \times (0, \infty)$ or $[a, b] \times (0, \infty)$, quite unlike the WAES case. Consequently, Corollary 1 can not be applied.

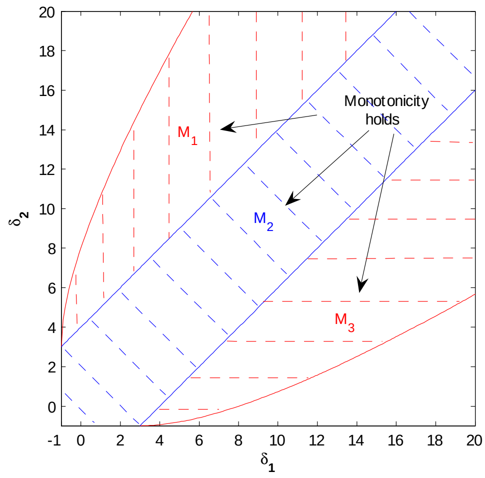



Figure 5:

**Example 3** *Consider the $n = 2$ WAES utility function with a shifted origin*

$$U(c_1, c_2) = -\frac{(c_1 + q_1)^{-\delta_1}}{\delta_1} - \frac{(c_2 + q_2)^{-\delta_2}}{\delta_2}, \tag{63}$$

*where $q_1, q_2 > 0$, $\delta_1, \delta_2 > -1$.*[8] *We have*

$$\tilde{M}_U = \frac{\left(\frac{(1+\delta_1)c_1}{c_1+q_1} - \frac{(1+\delta_2)c_2}{c_2+q_2}\right)^2}{AB}, \tag{64}$$

*where*

$$A = (1 + \delta_1)(c_1 + q_1)^{\delta_1} + (1 + \delta_2)(c_2 + q_2)^{\delta_2}, \tag{65}$$
$$B = (c_1 + q_1)^{-1-\delta_1} c_1 + (c_2 + q_2)^{-1-\delta_2} c_2. \tag{66}$$

*If the utility function is defined on $(0, \infty) \times (0, \infty)$, then we can obtain*

$$\tilde{M}_U \leq \frac{\left(\frac{(1+\delta_1)c_1}{c_1+q_1} - \frac{(1+\delta_2)c_2}{c_2+q_2}\right)^2}{\frac{(1+\delta_1)c_1}{c_1+q_1} + \frac{(1+\delta_2)c_2}{c_2+q_2}} \tag{67}$$
$$\leq \left| \frac{(1 + \delta_1) c_1}{c_1 + q_1} - \frac{(1 + \delta_2) c_2}{c_2 + q_2} \right| \leq 1 + \max(\delta_1, \delta_2), \tag{68}$$

*where the equal sign can be obtained if and only if $\delta_1, \delta_2 > 0$ and $(c_1, c_2) = (0, \infty)$ or $(\infty, 0)$ depending on $\delta_1 < \delta_2$ or $\delta_1 > \delta_2$. Therefore, the necessary*

[8] This form of utility can easily be generalized to the HARA form in Table 1, Row 1.

*and sufficient condition for monotonicity to hold is*

$$\max\left(\delta_1, \delta_2\right) \le 3. \tag{69}$$

*Next, we will show that the set of minimum concavity points and the set of* $\tilde{M}_U^{\max}$ *points do not intersect. Without loss of generality, assume that* $\delta_1 > \delta_2 > 0$. *Since the utility function is defined on* $(0, \infty) \times (0, \infty)$, *it can be verified that the set of minimum concavity points is* $\{(c_1, \infty)\}$. *However, from the above calculation we can see that the set of* $\tilde{M}_U^{\max}$ *points is* $\{(\infty, 0)\}$. *Hence eqn. (45) is not satisfied and we cannot apply Corollary 1. Furthermore, even if the preference ordering is defined on* $[a, b] \times (0, \infty)$, *the set of minimum concavity points and the set of* $\tilde{M}_U^{\max}$ *points do not intersect. If one continues to assume that* $\delta_1 > \delta_2 > 0$, *it can be verified that the least concave form is*

$$u\left(c_1, c_2\right) = \left(\frac{\delta_2}{\delta_1}\left(\left(c_1 + q_1\right)^{-\delta_1} - \left(a + q_1\right)^{-\delta_1}\right) + \left(c_2 + q_2\right)^{-\delta_2} + \frac{\delta_2 + 1}{\delta_1 + 1}\left(a + q_1\right)^{-\delta_1}\right)^{-1/\delta_2}. \tag{70}$$

*The set of the minimum concavity points is* $\{(a, c_2)\}$. *However, the set of* $\tilde{M}_U^{\max}$ *can be* $\{(b, 0)\}$. *Therefore, in general, there will be no intersection between these sets.*

**Remark 5** *There exists an interesting discontinuity and asymmetry when* $q_1$ *and/or* $q_2$ *approach the origin. If both are positive the necessary and sufficient condition for monotonicity is (69). This continues to hold if* $q_1 = 0$. *If both vanish, then the condition (42) applies. If* $q_1 > 0$ *and* $q_2 = 0$, *then the condition is*

$$\begin{aligned} \delta_1 + \delta_2 &< 2 \text{ or} \\ \delta_1 + \delta_2 &\ge 2 : (\delta_1 - \delta_2)^2 \le 8(\delta_1 + \delta_2) \text{ and } \delta_2 \le 3. \end{aligned} \tag{71}$$

*Here we assume that* $\delta_1 > \delta_2$.

**Remark 6** *It should be noted that for the CES/WAES cases with a shifted origin (Examples 2 and 3), the sufficient condition for monotonicity given by Quah [18] for additively separable utility functions, eqn. (61), becomes necessary and sufficient. To be more explicit, from [18], for the WAES case with a shifted origin (for the CES case, we can simply take* $\delta_1 = \delta_2 = \delta$), *the sufficient condition for monotonicity to hold can be written as*

$$B_U = \left|-\frac{c_1 U_1''}{U_1'} + \frac{c_2 U_2''}{U_2'}\right| = \left|\frac{(1+\delta_1)\, c_1}{c_1 + q_1} - \frac{(1+\delta_2)\, c_2}{c_2 + q_2}\right| < 4. \tag{72}$$

*Since*

$$0 < \frac{c_1}{c_1 + q_1} < 1,\ 0 < \frac{c_2}{c_2 + q_2} < 1 \tag{73}$$

*we have*

$$B_U^{\max} = 1 + \max(\delta_1, \delta_2) \tag{74}$$

*The maximum value can be reached only in the limit. Therefore, a sufficient condition for the monotonicity to hold is*

$$B_U^{\max} \leq 4 \Leftrightarrow 1 + \max(\delta_1, \delta_2) \leq 4 \Leftrightarrow \max(\delta_1, \delta_2) \leq 3, \tag{75}$$

*which is the same as the necessary and sufficient condition we have obtained in the above examples. Again it should be emphasized that this is very different from the WAES case since for that case $B_U < 4$ only gives a sufficient, but not necessary condition for monotonicity.*

# 6 Monotonicity condition for n commodities

We next extend the case of $n = 2$ commodities to the more general $n > 2$ case and obtain essentially the same characterization of monotonicity. Consider first the $n > 2$ WAES utility function.

**Example 4** *For the WAES utility function*[9]

$$U = -\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i}, \tag{76}$$

*assume without loss of generality that $\delta_1 \geq \delta_2 \geq ... \geq \delta_n > -1$. The sufficient and necessary condition for monotonicity to hold is*

$$\delta_1 + \delta_n \leq 2 \qquad or \qquad \delta_1 + \delta_n > 2,\ (\delta_1 - \delta_n)^2 < 8(\delta_1 + \delta_n). \tag{77}$$

**Proof.** It can be verified that

$$\tilde{M}_U = \frac{\sum_{i=1}^{n} c_i^{-\delta_i}(1 + \delta_i)}{\sum_{i=1}^{n} c_i^{-\delta_i}} - \frac{\sum_{i=1}^{n} c_i^{-\delta_i}}{\sum_{i=1}^{n} c_i^{-\delta_i}/(1 + \delta_i)}. \tag{78}$$

If we denote

$$\pi_i = \frac{c_i^{-\delta_i}}{\sum_{i=1}^{n} c_i^{-\delta_i}}, \tag{79}$$

[9] If we assume the utility function has a more general form $U = -\sum_{i=1}^{n} \frac{\lambda_i c_i^{-\delta_i}}{\delta_i}$, the conclusions below continue to hold.

then we can rewrite $\tilde{M}_U$ as

$$\tilde{M}_U = \sum_{i=1}^{n} \pi_i(1+\delta_i) - \frac{1}{\sum_{i=1}^{n} \dfrac{\pi_i}{1+\delta_i}}. \tag{80}$$

Using the inequality provided in Shisha and Mond [16], the necessary and sufficient condition for monotonicity to hold is

$$\tilde{M}_U \leq \tilde{M}_U^{\max} = \left(\sqrt{1+\delta_1} - \sqrt{1+\delta_n}\right)^2 < 4, \tag{81}$$

which is equivalent to

$$\delta_1 + \delta_n < 2 \text{ } or \text{ } \delta_1 + \delta_n \geq 2, \ (\delta_1 - \delta_n)^2 < 8(\delta_1 + \delta_n). \tag{82}$$

■

**Remark 7** *It should be noted that eqn. (80) can be interpreted as implying that the Mitjuschin-Polterovich coefficients* $\tilde{M}_U$ *is the difference between the arithmetic mean and the harmonic mean of the positive numbers* $(1+\delta_1), ..., (1+\delta_n)$ *with weights* $\pi_i, ..., \pi_n$.

Next we consider the WAES utility function with a shifted origin. The natural generalization of the result in Example 3 is the following.

**Example 5** *For the WAES utility function with a shifted origin*

$$U = -\sum_{i=1}^{n} \frac{(c_i + q_i)^{-\delta_i}}{\delta_i} \tag{83}$$

*where* $q_i > 0, \delta_i > -1$ $(i = 1, 2, ...n)$*, the necessary and sufficient condition for monotonicity to hold is*

$$\max(\delta_1, \delta_2, ...\delta_n) \leq 3. \tag{84}$$

# 7 Concluding comments

The explicit characterization of violations of the Law of Demand particularly in the case of WAES preferences raises a number of natural questions. Detailed analysis of (i) the regions where demand violates monotonicity, (ii) Engel curves and (iii) the interaction of the expansion path and the violation boundaries are examined in a sequel working paper (Kannai and Selden [8]).

Appendix

# A Table of Utility Classes

The following Table from Kannai, Selden and Kang [9] lists for 8 classes of utility, the corresponding least concave utility functions $u$, the direction of minimum concavity $(\xi_1^*, \xi_2^*)$ and the minimum concavity points $(c_1^*, c_2^*)$. The column corresponding to $\overline{u}$ indicates the least concave utility in the specific direction of $(\xi_1, \xi_2) = (0, 1)$ which arises in problems characterized by uncertainty only in the second good. This is explained in considerable detail in [9].

# B Sets of Minimum Concavity Points

In this Appendix we first prove Proposition 1. Then we comment on a case not covered.

Observe that in Table 1 (in Appendix A) there is a sharp dichotomy between the CES class of utility (e.g., row 2), where the set is the full consumption space, and the WAES case (e.g., Rows 6 and 7), where the set shrinks to a line. Here we analyze the more general separable case with regard to similar properties of these sets.

It follows from (20) and (21) that if $U$ is separable, then

$$a(U, \mathbf{c}) = \left[ \sum_{i=1}^{n} \frac{(U_i')^2}{U_i''} \right]^{-1}. \tag{85}$$

We are looking for points where the function $-a(U, \mathbf{c})$ is minimal relative to the indifference surface $U =$ const. (These points may be found even if we don't know explicitly $a(u, \mathbf{c})$.) Application of the Lagrange multiplier method leads to the system

$$\frac{\partial a}{\partial c_i} = \lambda \frac{\partial U}{\partial c_i}, \; i = 1, \cdots, n$$

or

$$-\frac{1}{M^2} \frac{\partial}{\partial c_i} \frac{(U_i')^2}{U_i''} = \lambda U_i', i = 1, \cdots, n. \tag{86}$$

Evaluating (86), we get the system

$$-\frac{1}{M^2} [2U_i' - \frac{U_i'''(U_i')^2}{(U_i'')^2}] = \lambda U_i', i = 1, \cdots, n$$

Table 1: LCRs for classic forms of U

| | $U$ | Restriction | $(\xi_1^*, \xi_2^*)$ | $u$ | $(c_1^*, c_2^*)$ | $\overline{u}$ | $x^*$ |
|---|---|---|---|---|---|---|---|
| 1 | $\frac{(a+bc_1)^{-\delta}}{-\delta}+\frac{(d+ec_2)^{-\delta}}{-\delta}$ <br> $C_1=(0,\infty), C_2=(0,\infty)$ | $\delta > 0$ <br> $\delta < 0$ | $\left(\frac{a}{b}+c_1, \frac{d}{e}+c_2\right)$ | $\left\{(a+bc_1)^{-\delta}+(d+ec_2)^{-\delta}\right\}^{-\frac{1}{\delta}}$ | $C_1 \times C_2$ | $\left\{(a+bc_1)^{-\delta}+(d+ec_2)^{-\delta}\right\}^{-\frac{1}{\delta}}$ <br> $\left\{(a+bc_1)^{-\delta}+(d+ec_2)^{-\delta}-a^{-\delta}\right\}^{-\frac{1}{\delta}}$ | $\infty$ <br> $0$ |
| 2 | $-\frac{c_1^{-\delta}}{\delta}-\frac{c_2^{-\delta}}{\delta}$ <br> $C_1=(0,\infty), C_2=(0,\infty)$ | $\delta > 0$ <br> $\delta < 0$ | $(c_1, c_2)$ | $\left(c_1^{-\delta}+c_2^{-\delta}\right)^{-1/\delta}$ | $C_1 \times C_2$ | $\left(c_1^{-\delta}+c_2^{-\delta}\right)^{-1/\delta}$ | $\infty$ <br> $0$ |
| 3 | $-\frac{c_1^{-\delta}}{\delta}-\frac{c_2^{-\delta}}{\delta}$ <br> $C_1=[a,b], C_2=(0,\infty)$ | $\delta > 0$ <br> $\delta < 0$ | $(c_1, c_2)$ | $\left(c_1^{-\delta}+c_2^{-\delta}\right)^{-1/\delta}$ | $C_1 \times C_2$ | $\left(c_1^{-\delta}+c_2^{-\delta}-b^{-\delta}\right)^{-1/\delta}$ <br> $\left(c_1^{-\delta}+c_2^{-\delta}-a^{-\delta}\right)^{-1/\delta}$ | $b$ <br> $a$ |
| 4 | $c_1^a c_2^b$ <br> $C_1=(0,\infty), C_2=(0,\infty)$ | $a, b > 0$ | $(c_1, c_2)$ | $c_1^{\frac{a}{a+b}} c_2^{\frac{b}{a+b}}$ | $C_1 \times C_2$ | $c_1^{\frac{a}{b}} c_2$ | $(0, \infty)$ |
| 5 | $-\frac{e^{-\alpha_1 c_1}}{\alpha_1}-\frac{e^{-\alpha_2 c_2}}{\alpha_2}$ <br> $C_1=(0,\infty), C_2=(0,\infty)$ | $\alpha_1, \alpha_2 > 0$ | $(\alpha_2, \alpha_1)$ | $-\ln\left(\frac{e^{-\alpha_1 c_1}}{\alpha_1}+\frac{e^{-\alpha_2 c_2}}{\alpha_2}\right)$ | $C_1 \times C_2$ | $-\ln\left(\frac{e^{-\alpha_1 c_1}}{\alpha_1}+\frac{e^{-\alpha_2 c_2}}{\alpha_2}\right)$ | $\infty$ |
| 6 | $-\frac{c_1^{-\delta_1}}{\delta_1}-\frac{c_2^{-\delta_2}}{\delta_2}$ <br> $C_1=(0,\infty), C_2=(0,\infty)$ | $\delta_1 > \delta_2 > 0$ <br> $\delta_2 > \delta_1 > 0$ <br> $0 > \delta_1 > \delta_2$ <br> $0 > \delta_2 > \delta_1$ | $\left(\frac{c_1}{\delta_1+1}, \frac{c_2}{\delta_2+1}\right)$ | $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_1}$ <br> $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ <br> $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ <br> $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_1}$ | $c_2^* = \infty$ <br> $c_1^* = \infty$ <br> $c_1^* = 0$ <br> $c_2^* = 0$ | $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ <br> $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ <br> $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ <br> $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ | $\infty$ <br> $\infty$ <br> $0$ <br> $0$ |
| 7 | $-\frac{c_1^{-\delta_1}}{\delta_1}-\frac{c_2^{-\delta_2}}{\delta_2}$ <br> $C_1=[a,b], C_2=(0,\infty)$ | $\delta_1 > 0 > \delta_2$ <br> $\delta_2 > 0 > \delta_1$ | $\left(\frac{c_1}{\delta_1+1}, \frac{c_2}{\delta_2+1}\right)$ | $\left(\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-a^{-\delta_1}\right)+c_2^{-\delta_2}+\frac{\delta_2+1}{\delta_1+1}a^{-\delta_1}\right)^{\frac{-1}{\delta_2}}$ <br> $\left(\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-b^{-\delta_1}\right)+c_2^{-\delta_2}+\frac{\delta_2+1}{\delta_1+1}b^{-\delta_1}\right)^{\frac{-1}{\delta_2}}$ | $c_1^* = a$ <br> $c_1^* = b$ | $\left(\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-a^{-\delta_1}\right)+c_2^{-\delta_2}\right)^{\frac{-1}{\delta_2}}$ <br> $\left(\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-b^{-\delta_1}\right)+c_2^{-\delta_2}\right)^{\frac{-1}{\delta_2}}$ | $a$ <br> $b$ |
| 8 | $U=\frac{(c_1-1)}{(c_2-2)^2}$ <br> $C_1=(1,\infty), C_2=(0,2)$ | | $(c_1-1, c_2-2)$ | $-\frac{(c_2-2)^2}{(c_1-1)}$ | $C_1 \times C_2$ | $\frac{c_2-2}{\sqrt{c_1-1}}$ | $(1, \infty)$ |

* $\delta, \delta_1, \delta_2 > -1$ for all the examples on the table.

or, equivalently,

$$2-\frac{U_i'''U_i'}{(U_i'')^2}=-\lambda M^2, i=1,\cdots,n. \qquad (87)$$

We may eliminate $\lambda M^2$ from the system (87) of $n$ equations so as to get a system of $n-1$ equations in $n$ unknowns. Such a systems represents, generically, a curve.

From now on we assume that $n=2$. We are then led to the equation

$$2-\left(\frac{U_1'''U_1}{(U_1'')^2}\right)(c_1^*)=2-\left(\frac{U_2'''U_2}{(U_2'')^2}\right)(c_2^*), \qquad (88)$$

valid whenever $(c_1^*,c_2^*)$ is a point of least concavity and $U$ is given by (15). Setting

$$2-\frac{U_i'''U_i'}{(U_i'')^2}(c_i)\equiv a_i(c_i), i=1,2, \qquad (89)$$

we see that the set $\{(c_1^*,c_2^*)\}$ of points of least concavity coincides with the set of solutions of the equation $a_1(c_1)=a_2(c_2)$, or is a subset of the set of solutions.

We distinguish now between several cases.

Case (i): The set of minimum concavity points coincides with the full set $\Omega$. This may happen if and only if $a_1=a_2=$const. A similar statement holds in the n commodity case where $a_1=...=a_i=...a_n=$const.

Case (ii): The set of minimum concavity points is contained in the curve defined by $a_1(c_1)=a_2(c_2)$.

In all cases, consider the ordinary differential equation

$$2-\frac{U'''U'}{(U'')^2}(x)\equiv a(x), \qquad (90)$$

where $a(x)$ is a given function and $U$ is the unknown one. Setting $v=U'$ we obtain the ordinary differential equation

$$\frac{v''v}{(v')^2}(x)\equiv 2-a(x)=_{def}\widetilde{a}\,(x)\,. \qquad (91)$$

The solution of (91) is well known, see Kamke ([5], item 6.187, p. 586). Using the substitution

$$\frac{v'}{v}=y,$$

the equation (91) assumes the form

$$\frac{y'}{y^2}=\widetilde{a}\,(x)-1=_{def}-b(x) \qquad (92)$$

(where $b(x) = a(x) - 1$). Integrating (92) we see that

$$\frac{1}{y} = \int b(x)dx$$

so that

$$y = \frac{1}{\int^x b(t)dt};$$

integrating once more we obtain the representation

$$v(x) = e^{\int^x \frac{dt}{\int^t b(s)ds}},$$

leading us to the representation

$$U(x) = \int e^{\int^x \frac{dt}{\int^t [a(s)-1])ds}} dx. \tag{93}$$

In the case (i) (see also Kamke [5], item 6.125, p. 573) we distinguish the following sub-cases:

a) $a(x) \equiv 1$. Then $b \equiv 0, y =$ const., and $v$ is an exponential function, so that $U$ is also exponential.

b) $a(x) \equiv 0$. Then $u$ is a logarithm.

c) $a$ is a constant, different from 0,1. Then

$$v(x) = |c_1 x + c_2|^{1-a}, U = |c_1' x + c_2'|^{\frac{1}{1-a}+1}.$$

It turns out that if the preferences are additively separable, then the Rows 1 through 5 in Table 1 cover **all** cases where every point in the choice set is a point of minimum concavity. This concludes the Proof of Proposition 1.

It should be noted that in case (ii), not covered by the Proposition, the set $\{(c_1^*, c_2^*)\}$ is contained in (if not equal to) the curve $a_1(c_1) = a_2(c_2)$, and every such curve may be realized in this fashion if we solve the equation(s) (90) for $U_1$ and $U_2$, where $a_1$ and $a_2$ are given. If $a_2^{-1} \circ a_1(c_1)$ is a linear (affine) function, then we have a line.[10]

The following extension will be needed in Kannai, Selden and Kang [9]. We can apply a similar method to the classic consumption-savings or consumption portfolio problem (where the choice space is certain×uncertain). In this setting the analog of $a(U, \mathbf{c})$ is the function $\overline{a}$ (introduced in [9], equation (2.7)). Accordingly, we now turn our attention to points where the

[10]If $a_1 \neq a_2$ everywhere then $(c_1^*, c_2^*)$ is not a finite point (but is contained in a compcatification).

second derivative of $\overline{a}$ is minimal relative to an indifference curve. Recall that by definition,

$$\overline{a} = \frac{U_2''}{(U_2')^2}.$$

In the separable case $\frac{\partial \overline{a}}{\partial c_1} = 0$, so that the Lagrange multiplier condition for minimum (0,1)-concavity points, analog to (86), reads $\lambda U_1' = 0$. But $U_1' > 0$. Hence $\lambda = 0$. Thus it is possible to get finite $(c_1^*, c_2^*)$ only if $\frac{\partial \overline{a}}{\partial c_2}$ vanishes somewhere.

In [9] a point $x^*$, such that the set of minimum (0,1)-concavity points is of the form $\{(x^*, c_2)\}$, is studied extensively (see e.g., Theorem 1 of [9]). To find when is $c_1 = x^*$, we have to solve the equation $\frac{\partial \overline{a}}{\partial c_2} \equiv 0$, or $\overline{a}$ =const= $\alpha$. Two cases are possible:

If $\alpha = 0$ then $U_2'' = 0$ so that $U_2$ is linear.

If $\alpha \neq 0$ then the relation

$$\alpha = \frac{U_2''}{(U_2')^2} = -\left(\frac{1}{U_2'}\right)'$$

may be integrated so that

$$U_2' = -\frac{1}{\alpha c_2 + \beta} \Longrightarrow U_2 = -\frac{\ln(\alpha c_2 + \beta)}{\alpha} + \gamma.$$

Moreover Rows 4 and 8 in Table 1 in the Appendix are typical (just consider $\ln U$ to make the utility function separable).

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