# UNIMODAL DOMAINS, ANTI-EXCHANGE PROPERTIES, AND COALITIONAL STRATEGY-PROOFNESS OF VOTING RULES

STEFANO VANNUCCI

ABSTRACT. It is shown that simple and coalitional strategy-proofness of a voting rule on the full unimodal domain of a convex idempotent interval space $(X, I)$ are equivalent properties if $(X, I)$ satisfies *interval anti-exchange*, a basic property also shared by a large class of convex geometries including -but not reducing to- trees and linear geometries. Therefore, *strategy-proof location problems in a vast class of networks* fall under the scope of that proposition.

It is also established that a much weaker *minimal anti-exchange* property is necessary to ensure equivalence of simple and coalitional strategy-proofness in that setting. An immediate corollary to that result is that such 'unimodal' equivalence fails to hold both in certan median interval spaces including those induced by bounded distributive lattices that are not chains, and in certain non-median interval spaces including those induced by non-trivial Hamming graphs.

Thus, anti-exchange properties of the relevant interval space provide a powerful *general* common principle that explains the varying relationship between simple and coalitional strategy-proofness of voting rules for full unimodal domains across different interval spaces, both median and non-median.

MSC 2010 Classification: 05C05, 52021, 52037

JEL Classification number : D71

## 1. Introduction

Whenever a decision of public import has to be taken, any acceptable procedure to produce it has to rely on some information concerning the evaluations, preferences and judgments of the stakeholders that is most typically distributed and private. Therefore, voting by a suitable committee is likely to be the most effective way to collect and amalgamate the relevant private, non-verifiable information.

Clearly, admissible voting rules are normally required to be reasonably 'nice', namely to respect voter-unanimity and treat both voters and outcomes in a fairly unbiased way. Moreover, since the information to be gathered by the voting process is private and non-verifiable, *simple or individual strategy-proofness* is a key property of the underlying voting rule. Indeed, strategy-proofness is required to prevent voters' attempts to manipulate the outcome by submitting false private information that may easily result in inefficient outcomes. Furthermore, if cheap communication facilities are available, simple strategy-proofness may be not enough since *coalitional* manipulations might also be attempted by voters: in that case *coalitional strategy-proofness* of the voting rule i.e. immunity from coalitional manipulations is also to be required. Notice that the outcome induced by truthful revelation of voters' most preferred alternatives under a coalitionally strategy-proof voting rule is also both a core outcome and a strong Nash equilibrium outcome. Thus, it enjoys a remarkable stability under *several* solution concepts: arguably, 'nice' coalitionally strategy-proof voting rules provide a major successful example of *robust mechanism design*.

Therefore, establishing *under what circumstances a strategy-proof decision mechanism is also coalitionally strategy-proof* is an open issue of considerable interest that has in fact attracted some attention in the recent literature (both Le Breton, Zaporozhets (2009) and Barberà, Berga, Moreno (2010) provide some results on the same issue concerning social choice functions for a wide class of domains).

It is now well-established that on certain domains (e.g. affine and quasilinear domains) some interesting anonymous, neutral and unanimity-respecting strategy-proof voting rules (such as random dictatorships and VCG mechanisms, respectively) do exist but are *not* coalitionally strategy-proof. By contrast, on other domains (e.g. the so called 'universal' domain of all total preorders over three or more outcomes) the strategy-proof voting rules are clearly *also* coalitionally strategy-proof, but that is so simply because they reduce to the meager and scarcely interesting class consisting of dictatorial rules and trivial or constant rules as essentially[1] implied by the Gibbard-Satterthwaite theorem (see Danilov, Sotskov (2002)).

[1]Actually, the Gibbard-Satterthwaite theorem implies that non-trivial non-dictatorial voting rules which are strategy-proof on the 'universal' domain *and* have a range including at least *three* distinct outcomes *do not exist*. Some care is needed to extend that argument in order to cover the case of *non-sovereign two-valued* voting rules (on that issue see Corollary 2 (iii) below, and the discussion following it).

In that connection, *unimodal (or single peaked) domains* are of special interest because it is well-known that for certain outcome spaces where a **median operation** is well-defined, some non-dictatorial median-based strategy-proof voting rules -including **simple majority** which is indeed the only anonymous, neutral and unanimity-respecting rule among them- can be defined on the corresponding domains of unimodal preference profiles. Moreover, and more to the point, it is also known that in *some* of those unimodal domains *all the strategy-proof voting rules are coalitionally strategy-proof* as well i.e. immune from coalitional manipulations (see e.g. Moulin (1980), Danilov (1994)) while *in other cases they are not, and the median rule itself is not coalitionally strategy-proof* (see e.g. Savaglio, Vannucci (2012)).

What are then the factors of success for that particular class of robust mechanism design problems? It is crystal clear that existence of a well-defined median rule on the outcome space and restriction to unimodal profiles are key to ensure success. However, in view of the results of Savaglio, Vannucci (2012) mentioned above, this cannot be the whole story.

Clearly enough, some further properties of the outcome space must play a role, but which ones and in what combinations with the other requirements?

**The present paper will show that entering explicit incidence-geometric considerations can contribute a considerable clarification to that matter, and provide some (partial) answers to the foregoing questions.** Let us then briefly outline the approach to be proposed here.

To begin with, recall that an **unimodal** -or single peaked - domain embodies two basic requirements for each admissible preference: (i) existence of **a unique optimum choice** and (ii) **consistency with a shared notion of 'compromise'** between every pair of outcomes (namely, a true 'compromise' between two more 'extremal' outcomes is never regarded by a voter as worse than *both* of its extrema). The **full** unimodal domain on a certain outcome space denotes the collection of *all* profiles of unimodal preferences of the required type (e.g. total preorders, or linear orders).

To be sure, *two* basic variants of the notion of 'compromise' in requirement (ii) of preference unimodality have been widely considered in the extant literature and should therefore be carefully distinguished, namely: (a) 'compromise' as a *qualitative and primitive* notion, and (b) 'compromise' as a *derivative notion somehow induced by -or anyway related to- an underlying metric*. Following Moulin (1980) and Danilov (1994), we shall be focussing on *the former* notion of 'compromise', namely on unimodality as a property embodying a **primitive notion of 'compromise' which is represented by the collection of 'intervals'** between pairs of outcomes. As mentioned above, under the present approach 'intervals' between pairs of outcomes include precisely the true compromises between them as certified by the common consent of voters. That common consent is in turn modelled by requirement (ii) of preference unimodality, which ensures that voter-preferences are indeed consistent with such description of compromises in that a compromise is never regarded by a voter as worse than both of its extreme outcomes. It should be noticed that such a notion of unimodality leaves a considerable degree of indeterminacy to voters' preferences and is therefore most convenient whenever the relevant voter-'types' -unlike locations and distances- consist of usually non-observable, non-verifiable private characteristics.

Thus, we are confronted with the following situation:

1) a ground-breaking paper (Moulin (1980)) provides a characterization of all strategy-proof voting rules on the *full unimodal domain in a bounded chain*, showing that (a) such rules can be represented by (iterated) medians of dictatorial and constant rules and (for a voter population of odd size) include a unique anonymous neutral and unanimity-respecting rule, namely the extended median i.e. the simple majority rule, and (b) every strategy-proof voting rule on such full unimodal domain is also coalitionally strategy-proof. Therefore, Moulin (1980) shows that -at least for an odd-sized population of voters- the extended median provides a successful robust solution to the important mechanism design problem of finding a nice coalitionally strategy-proof voting rule that works on the full unimodal domain in a bounded chain;

2) a subsequent very important paper (Danilov (1994)) shows, deploying a more general algebraic approach as applied to the median interval space induced by a bounded tree, that Moulin's positive and robust solution can be essentially[2] replicated and extended to the similar mechanism design problem that arises for *the full unimodal domain in a tree*;

3) some recent work (Savaglio, Vannucci (2012)) explores the possible extension of Moulin's solution in a *further* direction by considering the case of bounded distributive lattices, and shows that algebraic techniques as suggested by Danilov (1994) can be fruitfully applied to that latticial setting to replicate the Moulin-Danilov-characterization of strategy-proof voting rules on the full unimodal domain in a bounded distributive lattice. However, in that case some strategy-proof voting rules including the (extended) median can be shown to be *not* coalitionally strategy-proof, and *the remarkable robustness of Moulin-Danilov's solution collapses* in the general distributive-latticial setting.

Hence for some full unimodal domains simple (or individual) strategy-proofness and coalitional strategy-proofness turn out to be equivalent properties, while for others that equivalence fails. Also notice that **the outcome spaces mentioned above have well-defined medians and median-based strategy-proof voting rules: it is rather coalitional strategy-proofness of those rules that fails when moving from full unimodal domains in bounded chains or trees to full unimodal domains in general bounded distributive lattices.**

[2] I say 'essentially' because, strictly speaking, Danilov (1994) only addresses the case of *linear* preference orders. But Proposition 1 below shows that Danilov's Lemma 1 can be generalized to cover preference profiles of arbitrary unimodal total preorders.

But then, why is that so? What is the dividing line between such an equivalence and its failure i.e. under which cases does a full unimodal domain support equivalence of simple and coalitional strategy-proofness?

Previous works are in fact silent on that particular issue because they have been looking for conditions that hold in a wide class of domains, and therefore quite predictably focus on certain *combinatorial* restrictions that, as it happens, do not apply to unimodal domains as described above[3]. Of course, one might look for *other* general combinatorial properties to explain coalitional manipulability of median-based rules on the full unimodal domain of bounded distributive lattices, but that is *not* the approach to be followed here.

The present paper purports to address the foregoing open issue by introducing a suitably general **geometric** setting in order to cover -among other issues- *strategy-proof location problems in a vast class of networks*. It will be shown that some **incidence** properties of the **geometry of outcome space, namely certain anti-exchange properties of the latter (to be defined in Section 3), play a crucial role in determining whether a full unimodal domain supports equivalence of simple and coalitional strategy-proofness or not.**

Indeed, the most fitting environment to introduce the general notion of 'compromise' required by unimodality is perhaps provided by **interval spaces**. An interval space is a set $X$ endowed with a suitable *interval function* $I : X^2 \rightarrow \mathcal{P}(X)$ mapping each pair of points of $X$ into a subset of $X$ denoting their (closed) 'interval' namely the set of points located 'between' them (see e.g. Sholander (1952, 1954), Prenowitz, Jantosciak (1979), Mulder (1980), van de Vel (1993), Coppel (1998), Nebeský (2007), Mulder, Nebeský (2009), Chvátal, Rautenbach, Schäfer (2011)). Then, the available compromises between two outcomes $a, b$ consist precisely of the outcomes that belong to the interval of $a$ and $b$. **Therefore, we shall henceforth identify the interval space with the compromise-structure of the outcome space which voters have been able to agree upon.**

In particular, interval spaces are said to be *median* if for any three points $a, b, c$, the intervals of their three pairs have precisely one point in common (their median). Thus, a total preorder on a certain interval space is *unimodal* if it has a unique maximum and is such that for any $a, b, c$ of the underlying space, if $c$ lies 'between' $a$ and $b$ then its lower contour must include at least one of the latter. It is well-known that under many relevant specifications of the interval space (including the -median- interval spaces induced by finite chains, by bounded chains of the extended real line, by bounded median semilattices, by bounded undirected trees, by bounded median graphs) and,

[3]See Le Breton, Zaporozhets (2009) and Barberà, Berga, Moreno (2010) whose 'richness' and 'indirect sequential inclusion' properties (respectively), only apply to 'locally strict unimodality', a particular version of 'single peakedness'. A total preorder is 'locally strictly unimodal' if it has a unique maximum and any proper compromise between the maximum and another outcome is *strictly better* than the latter (a detailed discussion of that point is included in Appendix C).

possibly, under some slight variation on the notion of unimodality, there exist non-dictatorial strategy-proof voting rules on unimodal domains even allowing for 'many' -i.e. three or more- possible outcomes. Moreover, as already mentioned above, it has also been shown that in a few key cases namely bounded chains (see Moulin (1980)) and bounded undirected trees (see Danilov (1994), Danilov, Sotskov (2002)) *all* the strategy-proof voting rules on full unimodal domains are also *coalitionally strategy-proof*, hence simple (or individual) strategy-proofness and coalitional strategy-proofness turn out to be equivalent properties. But then**,** again, **to what extent such an equivalence between simple and coalitional strategy-proofness of voting rules on full unimodal domains can be generalized to other interval spaces?**

Unfortunately, to the best knowledge of the author, no result is available[4] concerning the relation of simple to coalitionally strategy-proofness in full unimodal domains within general, minimally 'regular' -i.e. convex and idempotent- interval spaces (an interval space is denoted here 'convex' if its intervals are convex in the obvious sense, and 'idempotent' if the degenerate interval between one point and itself reduces precisely to that point; a convex idempotent interval space may or may not be median, but it is well-known that a median interval space is both convex and idempotent: see e.g. Mulder (1980)).

The present paper addresses the issue of equivalence between simple and coalitional strategy-proofness of voting rules on full unimodal domains in such a **general setting of minimally 'regular' interval spaces.** A sufficient condition for equivalence on full unimodal domains (Theorem 1 below) is provided : it is shown that such **equivalence holds whenever the interval space satisfies a certain 'Interval Anti-Exchange' property**. Interval Anti-Exchange is a property that is sometimes used as a basic axiom to characterize linear geometries among convex geometries[5], and is shared by all trees. The argument goes as follows: (a) **strategy-proofness of a voting rule $f$ on the full unimodal domain in a convex idempotent interval space and Interval Anti-Exchange of that space jointly imply that two arbitrary profiles $x_N$,$y_N$ of voters' choices result in distinct outcomes $u = f(x_N)$, $v = f(y_N)$ only if there exists at least one voter $i$ among those that vote differently at $x_N$ and $y_N$ such that $u$ is a compromise between $v$ and her choice $x_i$ at $x_N$**; but then, (b) **if $x_i$ is the top outcome of voter $i$, unimodality implies that $v$ cannot be strictly better than $u$ for voter $i$, hence coalitional strategy-proofness of $f$ follows.**

A significant implication of that result for *location problems in networks* is quite clear: **whenever the network is a tree, a linear geometry or indeed any graph whose interval function is convex, idempotent and satisfies Interval Anti-Exchange,**

[4]The reader is addressed again to Appendix C for a proper articulation of that statement.

[5]See Section 3 below for the relevant definitions. In particular, recall that standard Euclidean convex sets amount in fact to a very special subclass of linear geometries.

**any strategy-proof voting rule for the corresponding full unimodal domain is also coalitionally strategy-proof on that domain**.

A much weaker **'Minimal Anti-Exchange' property is also shown to be a necessary condition for equivalence of simple and coalitional strategy-proofness of voting rules on full unimodal domains** (see Theorem 2 below). It follows that, as a consequence, equivalence fails to hold in any median interval space induced by a bounded distributive lattice (or indeed by a bounded median graph) that is not a chain and in a large class of non-median interval spaces, including those induced by non-trivial Hamming graphs as discussed below.

Such an equivalence failure is established by proving the existence of a non-trivial non-dictatorial strategy-proof voting rule on the relevant full unimodal domain that admits at least four distinct outcomes in its range and is not immune from coalitional manipulations. In that connection, it should also be emphasized that since constant and dictatorial rules are obviously coalitionally strategy-proof, it follows that -from a mechanism-design perspective- **equivalence failure on a certain unimodal domain has also some positive, constructive implications because it implies the existence of non-trivial non-dictatorial strategy-proof voting rules on that domain.**

Summing up, the **main contributions** of the present paper may be described as follows.

First, the characterization via interval-monotonicity of strategy-proof rules for the full unimodal domain of linear orders on the interval space of a tree due to Danilov (1994) is **extended to the full unimodal domain of total preorders in an arbitrary convex interval space**.

Second, it is shown that **for any minimally 'regular' interval space Interval Anti-Exchange (an incidence-geometric property shared by trees and linear geometries but independent of minimal 'regularity') is sufficient to ensure equivalence of simple and coalitional strategy-proofness of voting rules on the corresponding full unimodal domain, both for median and non-median interval spaces**.

Third, a considerably weaker anti-exchange property called **Minimal Anti-Exchange is shown to be necessary to ensure equivalence of simple and coalitional strategy-proofness of voting rules on the corresponding full unimodal domain***:* thus, violation of Minimal Anti-Exchange explains equivalence failure for simple and coalitional strategy-proofness of voting rules on full unimodal domains in certain outcome spaces both median and non-median such as (the interval spaces of) distributive lattices and non-trivial Hamming graphs as defined below, respectively.

Finally, the **implications of the two foregoing results for full unimodal equivalence in several interval spaces arising from outcome sets of special interest listed in Section 2 are pointed out** (and collected under Corollary 2 below).

Apparently**, an explicit consideration of incidence-geometric (as opposed to metric) properties of the underlying outcome space offers a distinctive insight on the reasons underlying the respective success and failure of coalitional strategy-proofness of nice median-based voting rules for full unimodal domains in bounded chains and trees, and in bounded distributive lattices.**

The remainder of this paper is organized as follows: Section 2 provides a list of unimodal domains of some interest, including both extensively studied and largely unexplored examples; Section 3 introduces the formal framework of the paper and presents its main results; Section 4 offers some short concluding remarks; all the proofs are collected in Appendix A.

Appendix B and Appendix C collect some supplementary material. In particular, Appendix B includes a couple of remarkable (counter) examples concerning interval spaces and convex geometries that help to clarify that the scope of the present inquiry is remarkably broad; Appendix C provides an extensive, detailed discussion of some related literature and explains its precise relationship to the present paper.

## 2. Simple and coalitional strategy-proofness on full unimodal domains: equivalent properties or not?

Let us consider a location problem on a network (or graph), or on a suitably ordered structure to be settled using some voting rule under the assumption that the voters' preferences are **unimodal with respect to a shared, non-controversial description of the set of compromises between any two alternative outcomes** as summarized by an interval function $I : X^2 \rightarrow P(X)$ on the set $X$ of feasible alternative outcomes. Broadly speaking, we say that a total preference preorder is *unimodal* if it has a **unique maximum** and **'respects' compromises** between outcomes in the following sense: **a compromise can never be worse than both of its extrema.** That is indeed the notion of 'single peakedness' that is used in a few seminal and widely quoted papers such as Moulin (1980) and Danilov (1994). The rationale for that restriction is the following. Suppose that voters -perhaps taking advantage of some interaction protocol suggested by *deliberative-democratic* best practices- have come to an *agreed comprehensive description of available outcomes including the list of all available compromises between any two of them.* Such compromises may or may not be determined with the help of some metric information, but if so, the relevant metrics are not expected to be the unique predictor of preferences, and need not be invariant across voters. Thus, a specific voter's type is private information that does not reduce to his/her best outcome: information on the outcome-spaces including available compromises (but not on a possible underlying metric) is the only relevant information that is shared by all voters, and that shared information is supposed to be significant for their choices. In particular, it is assumed that preferences are consistent with the agreed structure of compromises. Hence, a compromise should never be worse than both of its extrema, which is of course the characteristic feature of unimodality as defined in the present work that we started with. Therefore, the notion of unimodality adopted here has both a most respectable pedigree in the literature, and a solid rationale[6].

Once the outcome space and its compromise structure have been defined, it is quite natural to focus on strategy-proof voting rules on the full unimodal domain as described above. Moreover, it is also worth asking which -if any- of the available strategy-proof voting rules are also coalitionally strategy-proof i.e. immune from coalitional manipulations on that domain. In particular, 'unimodal' equivalence between simple and coalitional strategy-proofness obtains on a certain outcome space whenever all of the voting rules that are strategy-proof on the full unimodal domain of that space turn out to be coalitionally strategy-proof as well.

We shall prove that certain incidence-geometric properties of the relevant outcome space are in fact key features that make it possible to settle the equivalence-issue concerning simple and coalitional strategy-proofness on the corresponding full unimodal domain in any minimally 'regular' interval space i.e. virtually under any conceivable 'compromise-structure' of the outcome set.

The rest of this section is devoted to a detailed description of a few concrete remarkable examples of outcome spaces that are covered by the results of the present paper.

As mentioned above, the simple-coalitional strategy-proofness equivalence issue for unimodal domains has been partially explored in some specific classes of outcome spaces, including some *median* interval spaces (recall that median interval spaces are those interval spaces such that the intervals of any three points have precisely one point in common, their *median)*.

Indeed, some facts about equivalence of simple and coalitional strategy-proofness (or its failure) on full unimodal domains in some specific *median* interval spaces are well-known. That is largely due to the circumstance that the structure of strategy-proof voting rules in those spaces is now well understood: in fact, it has been established that strategy-proof voting rules on unimodal domains in median interval spaces can be represented by iterated medians of projections (i.e. dictatorial rules) and constants (see e.g. Moulin (1980), Danilov (1994), Savaglio, Vannucci (2012)). Let us then start with a quick review of the best known classes of examples:

[6]It should be recalled that a few alternative versions of that notion are available in the literature under the same label ('single peakedness' and, occasionally, even 'unimodality'). Such alternative notions tend to rely on *metric* information to define compromises, and require more or less explicitly that preferences be determined according to distances from the most preferred outcome: an outcome that is closer to the best outcome than another one should also be strictly better than the latter. We denote as '(locally) strictly unimodal' those preferences. (Appendix C discusses some results concerning locally strictly unimodal domains).

**The outcome space is a bounded chain**

If $(X, I(\leqslant))$ is the median interval space canonically induced by a bounded chain $(X, \leq)$ with $I(\leqslant)(x,y) = \{z \in X : x \leq z \leq y \text{ or } y \leq z \leq x\}$ for all $x, y \in X$, then the equivalence-issue is settled by the pioneering work of Moulin (1980), showing that (i) the strategy-proof rules for the full unimodal domain on $(X, I(\leqslant))$ are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) all such strategy-proof rules are also coalitionally strategy-proof on the same domain. Thus, simple strategy-proofness and coalitional strategy-proofness are equivalent properties here. In particular, the ordinary (extended) median rule is coalitionally strategy-proof.

**The outcome space is a bounded tree**

If $(X, I)$ is the median interval space canonically induced by a (discrete) bounded tree $G = (X, E)$ (i.e. a bounded connected graph without cycles) [7]- namely $I = I^G$ with

$I^G(x,y) = \{z \in X : z$ lies on the unique shortest path joining $x$ and $y\}$

for all $x, y \in X$-

the equivalence-issue is also settled by Danilov (1994), showing that (i) the strategy-proof rules for the full unimodal domain of linear orders on $(X, I)$ are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) all such strategy-proof rules are also coalitionally strategy-proof on the same domain. Thus, simple strategy-proofness and coalitional strategy-proofness are equivalent properties for full unimodal domains in bounded trees. In particular, the ordinary (extended) median rule is coalitionally strategy-proof.

**The outcome space is a complete graph or clique**

If $\mathcal{I} = (X, I^G)$ is the (non-median) interval space canonically induced by a complete graph or clique (namely a graph $G = (X, E)$ with $E = \{\{x, y\} : x, y \in X\}$) -hence $I^G(x,y) = \{x, y\}$ for all $x, y \in X$-

then the full unimodal domain of total preorders on $(X, I)$ amounts to the domain of all total preorders on $X$ with a unique maximum. Therefore, the restriction of the so-called 'universal domain' to profiles of total preorders with a unique maximum

[7]For any *graph* (or network) $G = (X, E)$ with vertex set $X$ and edge set $E \subseteq \{\{x, y\} : x, y \in X\}$ the interval space $\mathcal{I} = (X, I^G)$ canonically induced by $G$ is determined by defining the *distance* $d^G(x, y)$ of any outcomes/vertices $x, y$ as the length of the *geodesics* or *shortest paths* connecting them, and including in the interval of two arbitrary outcomes or vertices $x, y$ all the outcomes/vertices lying on one of such geodesics. Thus, for any $x, y \in X$ :

$I^G(x,y) = \left\{z \in X : d^G(x,y) = d^G(x,z) + d^G(z,y)\right\}.$

Hence, some *metric* information is involved in the very definition of interval spaces canonically induced by graphs. Considering *unimodal* preferences in such interval spaces is motivated by the intent to model situations where voter preferences do respect that compromise-structure but are not observable and verifiable and in any case *are not entirely determined by such metric information.*

may be regarded as *the full unimodal domain over an outcome space without a shared nonempty compromise-structure*. In particular, if $|X| \geq 3$, it follows from the Gibbard-Satterthwaite Theorem (see e.g. Danilov, Sotskov (2002)) that dictatorial rules are the only strategy-proof voting rules for the full unimodal domain on $(X, I^G)$ that admit at least *three* distinct outcomes in their range. Moreover, it can also be shown that if $\#X \geq 3$ there are no strategy-proof voting rules $f : X^N \to X$ having precisely *two* distinct outcomes (on this point, see footnote 1 and the discussion following Corollary 2 below). Hence constant rules and dictatorial rules are the only strategy-proof voting rules on the full unimodal domain of such $(X, I^G)$. Since both constant and dictatorial rules are clearly also coalitionally strategy-proof, it also follows that simple and coalitional strategy-proofness are equivalent properties here: the equivalence issue is quite easily settled for that domain.

**The outcome space is a bounded distributive lattice (or its covering graph)**

If $\mathcal{I}$ =$(X, I^m)$ is the (median) interval space canonically induced by an arbitrary bounded distributive lattice $\mathcal{X} = (X, \leqslant, 0, 1)$ that is *not* a chain ( as defined by the rule $I^m(x, y) = \{z : x \wedge y \leqslant z \leqslant x \vee y\}$, where $\wedge$ and $\vee$ denote the $\leqslant$-induced g.l.b. and l.u.b. operations[8]), the equivalence-issue is also already settled *in the negative* by Savaglio,Vannucci (2012) showing that (i) the strategy-proof rules for the full unimodal domain on $(X, I^m)$ are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) if $(X, \leqslant, 0, 1)$ is a bounded distributive lattice but is not a chain, then there are strategy-proof voting rules on that domain that are not coalitionally strategy-proof.[9] Notice that the former equivalence-failure result can be instantly extended to the interval space induced by the covering graph (or Hasse diagram) of $(X, \leqslant)$ itself: the general equivalence-issue is therefore also settled in the negative for the class of all median interval spaces induced by median graphs that are not trees (and a fortiori for the even larger class of interval spaces induced by connected graphs that are not trees).

Namely, simple strategy-proofness and coalitional strategy-proofness are *not* equivalent properties for full unimodal domains in the class of *all* median interval spaces induced by some arbitrary bounded distributive lattice, or by some arbitrary median graphs. In particular, it can be shown that the ordinary (extended) median rule retains its strategy-proofness on such domains but may be *not coalitionally strategy-proof*. To check the last point, consider for instance the following example borrowed from Bandelt, Barthélemy (1984) and Nehring, Puppe (2007 (b)), and adapted to our own full

[8]Note that $I^m = I(\leqslant)$ whenever $(X, \leqslant)$ is a chain.

[9]Nehring, Puppe (2007 (a),(b)) do not address the equivalence-issue as such, but include results implying failure of coalitional strategy-proofness of the median rule on the domain of 'locally strictly unimodal' linear orders in Boolean $k$-hypercubes with $k \geq 3$. (A detailed discussion of that point is included in Appendix 3).

unimodal domain. Take the interval space induced by the Boolean cube $\mathbf{2^3} = (\mathbf{2}^3, \leqslant)$ where

$\leqslant = \{(y,x) : x \in \mathbf{2}^3 \text{ and } y \in \{(1,1,1), x\}\} \cup$
$\cup \left\{ \begin{array}{c} (x_1,x_4),(x_1,x_5),(x_1,0),(x_2,x_4),(x_2,x_6),(x_2,0), \\ (x_3,x_5),(x_3,x_6),(x_3,0),(x_4,0),(x_5,0),(x_6,0) \end{array} \right\}$

(we also posit for convenience of notation $1 = (1,1,1)$, $0 = (0,0,0)$, $x_1 = (1,1,0)$, $x_2 = (1,0,1)$, $x_3 = (0,1,1)$, $x_4 = (1,0,0)$, $x_5 = (0,1,0)$, $x_6 = (0,0,1)$).

Notice that such a (median) interval space $\mathcal{I} = (\mathbf{2}^3, I^m)$ canonically induced by the Boolean cube

$(\mathbf{2}^3, \leqslant)$ is defined as follows:

$I^m(1,x_4) = I^m(x_1,x_2) = \{1,x_1,x_2,x_4\}$, $I^m(1,x_5) = I^m(x_1,x_3) = \{1,x_1,x_3,x_5\}$, $I^m(1,x_6) = I^m(x_2,x_3) = \{1,x_2,x_3,x_6\}$,

$I^m(x_1,0) = I^m(x_4,x_5) = \{x_1,x_4,x_5,0\}$, $I^m(x_2,0) = I^m(x_4,x_6) = \{x_2,x_4,x_6,0\}$,

$I^m(x_3,0) = I^m(x_5,x_6) = \{x_3,x_5,x_6,0\}$, $I^m(1,0) = I^m(x_1,x_6) = I^m(x_2,x_5) = I^m(x_3,x_4) = \mathbf{2}^3$, and $I^m(x,y) = \{x,y\}$ otherwise.

Let $N = \{1,2,3\}$ and consider $\mathcal{I}$-unimodal preference profile

$(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ defined as follows:

$x_1 \succ_1 x_3 \sim_1 1 \sim_1 x_5 \succ_1 x_4 \sim_1 x_2 \sim_1 x_6 \sim_1 0$ ,

$x_2 \succ_2 x_3 \sim_2 1 \sim_2 x_6 \succ_2 x_4 \sim_2 x_1 \sim_2 x_5 \sim_2 0$,

$0 \succ_3 x_3 \sim_3 x_5 \sim_3 x_6 \succ_3 x_4 \sim_3 x_1 \sim_3 x_2 \sim_3 1$.

Now, it is immediately checked that the median of the top outcomes of preference profile $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ is $\mu(x_1,x_2,0) = x_4$ because $I^m(x_1,x_2) \cap I^m(x_2,0) \cap I^m(x_1,0) = \{x_4\}$. However, observe that e.g. $\mu(x_3,x_3,z) = \mu(z,x_3,x_3) = \mu(x_3,z,x_3) = \mu(x_3,x_3,x_3) = x_3$ for any $z \in \mathbf{2}^3$. It follows that the median rule is in fact manipulable by the grand coalition and by any two-player coalition, hence it is clearly not coalitionally strategy-proof.

Let us now move on to a few interesting classes of networks/interval spaces where -to the best of the author's knowledge- very little is known about the structure of strategy-proof voting rules on the corresponding full unimodal domains.

To begin with, let us consider the class of (convex, idempotent) interval spaces as resulting from the following important class of outcome spaces.

**The outcome space is a simplex in an Euclidean convex space**

In that case $(X, I^E)$ is the (convex, idempotent) interval space canonically induced by a simplex in an Euclidean convex space in the standard manner, namely $X = \left\{x \in \mathbb{R}^m_+ : \sum_{i=1}^m x_i = 1\right\}$, and for all $x, y \in X$,

$I^E(x,y) = \{z \in X : z = \lambda x + (1-\lambda)y \text{ for some } \lambda \in [0,1]\}$.

That is clearly *not* a median interval space if $m > 1$: in fact, any nondegenerate triangle in $X$ fails to admit a median as defined above. A considerable amount of work has been devoted to the study of simple and coalitional strategy-proofness of voting rules on the domains of affine preferences on $(X, I^E)$ (see e.g. Danilov, Sotskov (2002)), and of Euclidean preferences with a 'bliss point' on $(X, I^E)$ (see Peters, van der Stael, Storcken (1992)). Notice that such Euclidean preferences on that space amount to *metric* strictly unimodal preferences, that are a proper subclass of (incidence-based) unimodal preferences as considered in the present paper. Accordingly, the notion of strategy-proofness used here is distinct from -and *much stronger* than- Euclidean metric-based strategy-proofness as employed e.g. by Peters, van der Stel, Storcken (1992): the coordinatewise median rule that they show to be strategy-proof but *not* coalitionally strategy-proof on the *metric* unimodal domain over the Euclidean plane i.e. with $m = 2$ is *not at all strategy-proof on the full unimodal domain.*

In general, the fact that *the (extended* $n$-ary, $n$ odd) *median rule and other median–based non-trivial non-dictatorial strategy-proof voting rules are only available for* $m = 1$ *has not gone unnoticed* (see e.g. Danilov, Sotskov (2002)). However, very little is apparently known about the class of all strategy-proof voting rules for the full unimodal domain on $(X, I^E)$, or the existence of non-trivial non-dictatorial strategy-proof voting rules on such domain.

**The outcome space is a partially ordered set**

Let $(X, \leqslant)$ be a partially ordered set (or poset), and $(X, I(\leqslant))$ the interval space canonically induced by $(X, \leqslant)$, namely

$I(\leqslant)(x, y) = \{x, y\} \cup \{z \in X : x \leqslant z \leqslant y \text{ or } y \leqslant z \leqslant x\}$ for all $x, y \in X$.

Ordered sets are a pervasive structure and interval spaces of that kind encompass a massive collection of cases. Very little is known - and can be said in general - about the nature of strategy-proof voting rules for the full unimodal domain on $(X, I(\leqslant))$, but the results of the present paper will enable us to predict that all of them are also coalitionally strategy-proof.

The next class of networks is also of considerable interest as models of location problems in a large collection of abstract spaces:

**The outcome space is a Hamming graph**

A Hamming graph can be regarded as a network whose vertices $a \in X^H$ denote words $a = (l_1, ..., l_k)$ of a fixed length $k$ having at each position $i = 1, ..., k$ a letter $l_i$ chosen from a finite alphabet $A^i = \left\{a^i_1, ...a^i_{h_i}\right\}$ (distinct positions may have distinct alphabets), while edges join any two vertices denoting two distinct words having distinct letters at just one position (see e.g. Mulder (1980)). Equivalently, the vertices of a Hamming graph may be regarded as points of a finite multiattribute space, with edges joining pairs of points that differ for the value of just one attribute. Therefore, the (convex, idempotent) interval space $(X^H, I^{G_H})$ induced by a Hamming graph $G_H$ is defined by the following rule:

$X^H = \Pi_{i=1}^k A^i$, and for all $x, y \in X^H$,

$I^{G_H}(x,y) = \{z \in X^H : z$ lies on one of the shortest paths joining $x$ and $y\}$.

A Hamming graph $G^H$ is denoted here as *non-trivial* if $k \geq 2$ and $h_i \geq 2$, $i = 1, ..., k$.

Notice that if $h_i \geq 3$ for some $i = 1, ..., k$, a Hamming graph is not triangle-free and, as a consequence, its interval space $\mathcal{I} = (X, I^{G_H})$ is *not* median. To the best of the author's knowledge, the structure of strategy-proof voting rules on the full unimodal domain of $(X^H, I^{G_H})$ is still essentially unknown.

Finally, let us consider a further class of networks/interval spaces where the existence of non-trivial non-dictatorial strategy-proof voting rules for the full unimodal domain has been already established (while the resulting 'unimodal' equivalence-issue between simple and coalitional strategy-proofness has not been addressed yet).

**The outcome space is the join of a clique and a chain**

Let $(X, I^G)$ be the interval space induced by a graph that can be decomposed into a clique (or complete graph) and a chain as joined through a common vertex. The latter sort of graph is a special subclass of the class of networks studied by Schummer, Vohra (2002), where it is shown that some non-trivial, non-dictatorial voting rules exist on a certain unimodal domain on $(X, I^G)$ (e.g. those rules resulting from the combination of a locally-dictatorial rule that applies whenever the top outcome of the relevant 'clique-dictator' lies on the clique, and a median rule as restricted to the subset of outcomes on the chain lying between the top outcome of the 'clique-dictator' and the outcome that is closest to the clique, that applies otherwise). However, the equivalence-issue concerning simple and coalitional strategy-proofness of voting rules on that full unimodal domain has never been addressed in the extant literature.

Thus, the foregoing list includes examples of outcome spaces where the status of available information on the issue concerning equivalence of simple and coalitional strategy-proofness of voting rules for full unimodal domains is quite diverse. In a few of them, the 'unimodal' equivalence-issue has been addressed and settled (either affirmatively, for bounded chains, bounded trees and complete graphs, or negatively, for bounded distributive lattices - and bounded median graphs - that are not chains). In other cases (e.g. Euclidean simplexes, Hamming graphs) no general results on the existence of non-trivial non-dictatorial strategy-proof voting rules for the full unimodal domain are available in earlier works. In the last case of that list (i.e. joins of one clique and one chain) it is known that non-trivial non-dictatorial strategy-proof voting rules for the full unimodal domain do exist but is not known whether equivalence of simple and coalitional strategy-proofness on that domain holds true.

*It is therefore quite remarkable that the main results of the present paper ( i.e. Theorems 1 and 2) jointly address and settle at once such 'unimodal' equivalence-issue in all of the outcome spaces considered above (and several others).* Indeed, it is easily checked that (the 'natural' interval spaces induced by) bounded chains, bounded trees, cliques, Euclidean simplexes, partially ordered sets, and joins of cliques and chains do satisfy the Interval Anti-Exchange property (to be introduced in the next Section) so that Theorem 1 applies to them to the effect of ensuring that 'unimodal' equivalence holds for strategy-proof voting rules on those spaces. Conversely, it is easily checked that (the interval spaces induced by) bounded distributive lattices and bounded median graphs which are not chains violate Minimal Anti-Exchange (as defined in the next Section), and the same observation applies to (the non-median interval spaces induced by) non-trivial Hamming graphs. Therefore, Theorem 2 below applies, establishing that *non-trivial non-dictatorial strategy-proof voting rules that are coalitionally manipulable (and admit at least four distinct outcomes) do exist on the full unimodal domains of those interval spaces*: as a consequence, 'unimodal' equivalence fails to hold for such spaces (all of the above is made precise in Corollary 2 at the end of the next Section).

Let us then eventually turn to the formal setting and the ensuing analysis.

## 3. Strategy-proofness and anti-exchange properties: model and results

Let $N = \{1, .., n\}$ denote the finite population of voters (with cardinality $|N| = n$), $X$ an arbitrary nonempty set of alternative outcomes, and $\mathcal{I} = (X, I)$ the **interval space** of $X$ , namely $I : X^2 \to \mathcal{P}(X)$ is an *interval function on* $X$ i.e. it satisfies the following conditions:

$I$-$(i)$ (**Extension**): $\{x, y\} \subseteq I(x, y)$ for all $x, y \in X$,

$I$-$(ii)$ (**Symmetry**): $I(x, y) = I(y, x)$ for all $x, y \in X$.

Notice that for any $Y \subseteq X$, an interval space $\mathcal{I} = (X, I)$ induces a natural interval space on $Y$, namely its *interval subspace* $\mathcal{I}_Y = (Y, I_Y)$ where $I_Y$ denotes the restriction of $I$ to $Y^2$.

In particular, we also assume that $n \geq 2$ in order to avoid tedious qualifications, and will be mostly concerned with *idempotent* interval spaces i.e. with interval spaces whose interval function also satisfy the following condition, namely

(**Idempotence**): $I(x, x) = \{x\}$ for all $x \in X$.

A subset $Y \subseteq X$ is *$\mathcal{I}$-convex* iff $I(x, y) \subseteq Y$ for all $x, y \in Y$. For any $Y \subseteq X$, the *$\mathcal{I}$-convex hull* of $Y$ - denoted $co_{\mathcal{I}}(Y)$- is the smallest $\mathcal{I}$-convex superset of $Y$, namely

$co_{\mathcal{I}}(Y) = \bigcap \{A \subseteq X : A \text{ is } \mathcal{I}\text{-}convex \text{ and } A \supseteq Y\}$.

An interval space $\mathcal{I} = (X, I)$ is *convex* (or *interval-monotonic*) if $I$ also satisfies

(**Convexity**): $I(x, y)$ is $\mathcal{I}$-convex for all $x, y \in X$.

Observe that Idempotence and Convexity are indeed mutually independent properties of interval spaces. To confirm that statement, consider interval spaces $\mathcal{I}_1 = (X, I_1)$, $\mathcal{I}_2 = (\{x, y, v, z\}, I_2)$ where $|X| > 1$, $|\{x, y, v, z\}| = 4$, $I_1(a, b) = X$ for all $a, b \in X$, while $I_2(x, y) = \{x, y, z\}$, $I_2(y, z) = \{y, v, z\}$, and $I_2(a, b) = \{a, b\}$ for all $a, b \in X$

such that $\{x,y\} \neq \{a,b\} \neq \{y,z\}$. It is immediately checked that $\mathcal{I}_1$ is convex but not idempotent, while $\mathcal{I}_2$ is idempotent but not convex since $\{y,z\} \subseteq I_2(x,y)$ and $v \in I_2(y,z) \setminus I_2(x,y)$.

**Remark 1** An *idempotent* interval space $(X,I)$ is said to be a **convex geometry**[10] if it also satisfies

(**Peano Convexity**) for all $x,y,v_1,v_2,z \in X$, if $y \in I(x,v_1)$ and $z \in I(y,v_2)$ then there exists $v \in I(v_1,v_2)$ such that $z \in I(x,v)$.

It can be quite easily shown that a convex geometry is in particular a convex interval space (see e.g. Coppel (1998), chpt.2, Proposition 1). The converse however does not hold: to check the latter statement, consider interval space $\mathcal{I}^* = (X, I^*)$ with $X = \{x, y_1, y_2, v_1, v_2, z\}$, $|X| = 6$, and $I^*$ defined as follows: $I^*(x,y_1) = \{x,y_1,v_1\}$, $I^*(x,y_2) = \{x,y_2,v_2\}$, $I^*(v_1,v_2) = \{v_1,v_2,z\}$, $I^*(y_1,z) = \{y_1,v_1,z,\}$, $I^*(y_2,z) = \{y_2,v_2,z,\}$, $I^*(y_2,v_1) = \{y_2,y_1,v_1\}$, and $I^*(a,b) = \{a,b\}$ otherwise. As it is easily checked, $\mathcal{I}^*$ is idempotent and convex by construction. However, it can also be shown (see e.g. Coppel (1998), chpt.2, Proposition 2) that a convex geometry also satisfes the following property:

(*) for all $x,y_1,y_2,v_1,v_2,z \in X$, if $v_1 \in I^*(x,y_1)$, $v_2 \in I^*(x,y_2)$ and $z \in I^*(v_1,v_2)$ then there exists $w \in I^*(y_1,y_2)$ such that $z \in I^*(x,w)$.

Now, it is immediately checked that no such $w$ exists in $\mathcal{I}^*$

for $x,y_1,y_2,v_1,v_2,z \in X$, since by construction $I^*(y_1,y_2) = \{y_1,y_2\}$, and $z \notin I^*(x,y_1) \cup I^*(x,y_2) = \{x,y_1,y_2,v_1,v_2\}$.

Therefore, $\mathcal{I}^*$ fails to satisfy Peano Convexity.

It follows that Peano Convexity is indeed a *strictly stronger* requirement than Convexity for an idempotent interval space as previously defined or, equivalently, a convex idempotent interval space amounts to a *generalized convex geometry. Therefore all of the results of the present paper clearly hold in particular when restricting the statements to convex geometries.*

Occasionally, *antisymmetric* interval spaces will also be considered in the sequel. Indeed, an interval space $\mathcal{I} = (X,I)$ is *antisymmetric* if $I$ satisfies

(**Antisymmetry**): for all $x,y,z \in X$, if $x \in I(y,z)$ and $y \in I(x,z)$ then $x = y$.

Observe that *Antisymmetry implies Idempotence* of an interval space $\mathcal{I} = (X,I)$: to see that, notice that since $x \in I(y,x)$ by Extension, $y \in I(x,x)$ for some $y \neq x$ entails a violation of Antisymmetry.

[10]Some authors denote as 'convex geometries' those *convexity spaces* that satisfy the *Anti-Exchange* property (see Remark 3 below for the relevant definitions). I follow here the alternative usage persuasively advocated by Coppel (1998) where it is argued that the property I denote as Peano Convexity (Coppel's Property [C]) is much more appropriate than Anti-Exchange to describe the truly fundamental features of a convex structure.

It can be shown that an important subclass of convex idempotent interval spaces do satisfy Antisymmetry[11], but in general even convex geometries may be not antisymmetric: to see this, consider e.g. interval space $(\{x,y,z\},I)$ with $x \neq y \neq z \neq x$ , $I(x,z) = I(y,z) = \{x,y,z\}$ and $I(a,b) = \{a,b\}$ otherwise, which is by construction idempotent and can be shown to satisfy Peano Convexity (thus, it is indeed a convex geometry), but is not antisymmetric since $x \neq y$ while both $x \in I(y,z)$ and $y \in I(x,z)$ hold.

Finally, we should also mention that an interval space $\mathcal{I} = (X,I)$ is said to be a **median space** if $I$ satisfies the following

(**Median Property**): for all $x,y,z \in X$, $|I(x,y) \cap I(y,z) \cap I(x,z)| = 1$.

The common point of the three intervals defined by each pair of any three points $x,y,z$ in a median interval space $(X,I)$ is said to be the *median* of those points, that therefore defines a ternary operation on $X$.[12]

It is well-known that e.g. the interval spaces induced by trees or median semilattices (including distributive lattices) are median (see Sholander (1952), (1954)), and that any median interval space is also convex (see Mulder (1980), Theorem 3.1.4) and idempotent. It should also be emphasized that all the properties of an interval space considered above *are inherited by its interval subspaces.*

Let $\succcurlyeq$ denote a total preorder i.e. a reflexive, connected and transitive binary relation on $X$ (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively; the following notation shall also be occasionally used: for any $x \in X$, $L(\succcurlyeq,x) := \{y \in X : x \succcurlyeq y\}$, $L^*(\succcurlyeq,x) := \{y \in X : x \succ y\}$).

Then, the total preorder $\succcurlyeq$ is said to be *unimodal* with respect to interval space $\mathcal{I} = (X,I)$ - or $\mathcal{I}$-**unimodal** - if and only if

$U$-$(i)$ there exists a *unique maximum* of $\succcurlyeq$ in $X$, its *top* outcome -denoted $top(\succcurlyeq)$- and

$U$-$(ii)$ for all $x,y,z \in X$, if $z \in I(x,y)$ then $\{u \in X : z \succcurlyeq u\} \cap \{x,y\} \neq \varnothing$.

We denote by $U_{\mathcal{I}}$ the set of all $\mathcal{I}$-*unimodal* total preorders on $X$. An $N$-profile of $\mathcal{I}$-unimodal total preorders is a mapping from $N$ into $U_{\mathcal{I}}$. We denote by $U_{\mathcal{I}}^N$ the **full** $I$**-unimodal domain**, namely the set of all $N$-profiles of $\mathcal{I}$-unimodal total preorders.

A **voting rule** for $(N,X)$ is a function $f : X^N \rightarrow X$ . A voting rule $f$ is (simply) **strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal $N$-profiles $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$. Moreover, a voting rule $f$ is **coalitionally strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal $N$-profiles $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$, and for all $C \subseteq N$, $(y_i)_{i\in C} \in X^C$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in$

[11]For instance, it is easily checked that any idempotent interval space that satisfies the Interval Anti-Exchange property to be defined below is also antisymmetric.

[12]An interval space $(X,I)$ is *quasi-median* if $I(x,y) \cap I(y,z) \cap I(x,z) \neq \varnothing$ for all $x,y,z \in X$.

$N$, there exists $i \in C$ with $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i)_{i\in C},(x_j)_{j\in N\smallsetminus C}))$. Clearly, *a coalitionally strategy-proof voting rule is in particular strategy-proof, while the converse may not hold*. Finally, a voting rule $f : X^N \rightarrow X$ is $\mathcal{I}$-**monotonic** *(or interval-monotonic)* iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$, $f((x_j)_{j\in N}) \in I(x_i, f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$.

Notice that by definition a voting rule $f : X^N \rightarrow X$ is (**coalitionally**) **strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $Y \subseteq X$ its $Y$-co-restricted restriction $f_Y : Y^N \rightarrow Y$ is (**coalitionally**) **strategy-proof** *on* $U_{\mathcal{I}_Y}^N$. Similarly, voting rule $f : X^N \rightarrow X$ is $\mathcal{I}$-**monotonic** iff for all $Y \subseteq X$ its $Y$-co-restricted restriction $f_Y : Y^N \rightarrow Y$ is $\mathcal{I}_Y$-**monotonic.**

We are now ready to state the main results of this paper concerning equivalence of strategy-proofness and coalitional strategy-proofness of voting rules on the domain of all unimodal profiles. Our results rely on the following proposition that establishes the equivalence between *monotonicity* with respect to an arbitrary convex idempotent interval space $\mathcal{I}$ and *strategy-proofness on the corresponding (full) unimodal domain* $U_{\mathcal{I}}^N$ .

**Proposition 1.** *Let $\mathcal{I} = (X, I)$ be a convex interval space. Then, a voting rule $f : X^N \rightarrow X$ is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$ iff it is $\mathcal{I}$-monotonic.*

**Remark 2** Proposition 1 above provides a considerable generalization of Lemma 1 in Danilov (1994). It transpires from the proof of that proposition that in fact $\mathcal{I}$-monotonicity of a voting rule $f : X^N \longrightarrow X$ implies its strategy-proofness on $U_{\mathcal{I}}^N$ for *any* interval space $\mathcal{I}$, *whether convex or not*. However, in order to prove that strategy-proofness of $f$ on $U_{\mathcal{I}}^N$ invariably implies $\mathcal{I}$-monotonicity of $f$, convexity of $\mathcal{I}$ cannot be dispensed with. To see this, consider the following simple example with $N = \{1, 2\}$ and $\mathcal{I} = (X = \{x, y, u, v, z\}, I)$ such that $|X| = 5$, $I(x, y) = \{x, y, u, v\}$, and $I(a, b) = X$, $I(a, a) = \{a\}$ for all $a, b \in X$ such that $a \neq b$ and $\{a, b\} \neq \{x, y\}$. Clearly, $\mathcal{I}$ is by construction *not* convex since e.g. $u, v \in I(x, y)$ but $z \in I(u, v) \smallsetminus I(x, y)$. Now, consider a voting rule $f : X \times X \longrightarrow X$ such that $f(a, b) = f(a, c)$ for all $a, b, c \in X$, $f(x, x) = z$ and $f(y, x) = y$. By construction, $f(x, x) \notin I(x, f(y, x))$ hence $f$ is *not* $\mathcal{I}$-monotonic. Now, notice that voter 2 is a dummy hence $f$ -if it is not strategy-proof on $U_{\mathcal{I}}^N$ - can only be manipulated by voter 1. Let us now check that in fact voter 1 *cannot manipulate* $f$ and therefore $f$ is *strategy-proof* on $U_{\mathcal{I}}^N$. Indeed, suppose to the contrary that there exists $\succcurlyeq \in U_{\mathcal{I}}$ such that $x = top(\succcurlyeq)$ and $x \succ y \succ z$. We may distinguish two possible cases: (i) there exists $a \in X \smallsetminus \{x, y\}$ such that $a \succcurlyeq y$, and (ii) $y \succ a$ for all $a \in X \smallsetminus \{x, y\}$. If (i) holds then $z \in X = I(y, a)$ whence unimodality is violated i.e. $\succcurlyeq \notin U_{\mathcal{I}}$, a contradiction. If (ii) holds, then $x \succ y \succ a \in I(x, y)$ hence again $\succcurlyeq \notin U_{\mathcal{I}}$, a contradiction, and strategy-proofness of $f$ is therefore established.

The following property will play a key role in the ensuing analysis

(**Interval Anti-Exchange (IAE)**): for all $x, y, v, z \in X$ such that $x \neq y$, if $x \in I(y, v)$ and $y \in I(x, z)$ then $x \in I(v, z)$.

It should also be noticed here that Interval Anti-Exchange, Idempotence and Convexity are mutually independent properties of an interval space[13].

Since one of the main results of the ensuing analysis will concern convex idempotent interval spaces that satisfy Interval Anti-Exchange, it is also worth mentioning that the class of such spaces is *strictly larger than the class of convex geometries that satisfy Interval Anti-Exchange.*[14]

**Remark 3** An explanation concerning our terminology is in order here. Recall that a *convexity space* (or *aligned space,* or *convex closure system*) is a pair $(X, \mathcal{C})$ where $X$ is any set and $\mathcal{C}$ is a *convexity* (or *alignment*) on $X$ i.e. a family of subsets of $X$ such that: (i) $\{\varnothing, X\} \subseteq \mathcal{C}$; (ii) $\cap\mathcal{D} \in \mathcal{C}$ for any nonempty $\mathcal{D} \subseteq \mathcal{C}$ ; (iii) $\cup\mathcal{D} \in \mathcal{C}$ for any nonempty $\mathcal{D} \subseteq \mathcal{C}$ which is *nested* i.e. totally ordered by inclusion. The subsets in $\mathcal{C}$ are by definition the *convex sets* of convexity space $(X, \mathcal{C})$, while for any $Y \subseteq X$ its *convex hull* $co_{\mathcal{C}}(Y)$ in $(X, \mathcal{C})$ is the smallest superset of $Y$ that belongs to $\mathcal{C}$ (observe that $co_{\mathcal{C}}(Y)$ is well-defined for any $Y \subseteq X$ thanks to properties (i) and (ii) of $\mathcal{C}$). It is quite easy to check that the set $\mathcal{C}_{\mathcal{I}}$ of $\mathcal{I}$-convex sets of any interval space $\mathcal{I} = (X, I)$ as defined above provides a particular instance of a *convexity* on $X$, but generally speaking a convexity on $X$ need not arise in that way (see e.g. van de Vel (1993)).

Indeed, Anti-Exchange is a commonly used label denoting the following property of a convexity space $(X, \mathcal{C})$:

(**Anti-Exchange** (AE)): for all $x, y \in X$ and $Y \subseteq X$, if $x \neq y$, $x \in co_{\mathcal{C}}(Y \cup \{y\})$ and $x \notin co_{\mathcal{C}}(Y)$ then $y \notin co_{\mathcal{C}}(Y \cup \{x\})$.

Clearly enough, Anti-Exchange can be in particular regarded as a possible property of any interval space $\mathcal{I} = (X, I)$ by taking $\mathcal{C} = \mathcal{C}_{\mathcal{I}}$, and in that case - by construction - $co_{\mathcal{C}_{\mathcal{I}}}(Y) = co_{\mathcal{I}}(Y)$ as defined above, for all $Y \subseteq X$.

Now, observe that Interval Anti-Exchange can be formulated in an equivalent way as follows: for all $x, y, v, z \in X$ such that $x \neq y$, if $x \in I(y, v)$ and $x \notin I(v, z)$ then $y \notin I(x, z)$, and it can be shown that any convex geometry $\mathcal{I} = (X, I)$ that satisfies IAE does also satisfy AE while any interval space that satisfies AE must also satisfy IAE (see e.g. Coppel (1998), where IAE is denoted as 'axiom L2'). Thus, AE as applied to

[13]To check that statement, consider the following interval spaces: (i)$(X = \{x, y, u, v\}, I)$ with $I(x, y) = \{x, u, y\}$, $I(u, y) = \{u, v, y\}$ and $I(a, b) = \{a, b\}$ for all $\{a, b\} \notin \{\{x, y\}, \{u, y\}\}$, which is by construction idempotent and can be easily shown to satisfy IAE, but is clearly not convex; (ii) $(X = \{x, y\}, I)$ with $I(x, x) = I(x, y) = \{x, y\}$, $I(y, y) = \{y\}$: that interval space is not idempotent but -as it is easily seen- it satisfies IAE and is obviously convex; (iii)$(X = \{x, y, z\}, I)$ with $I(x, z) = I(y, z) = \{x, y, z\}$, and $I(a, b) = \{a, b\}$ for all $\{a, b\} \notin \{\{x, z\}, \{y, z\}\}$, which is by construction idempotent and convex but fails to satisfy IAE since $x \in I(y, z)$, $y \in I(x, z)$ and $x \notin I(z, z)$.

[14] See Example 1 in Appendix B to confirm that claim.

the convex sets of an interval space amounts in a sense to IAE plus Idempotence and Peano Convexity.

It should also be noticed here that if an interval space $\mathcal{I} = (X, I)$ satisfies AE and $X$ is *finite* then the set of *complements* of its $\mathcal{I}$-convex sets (i.e. the set of its $\mathcal{I}$-*concave* sets) is the set of 'feasible sets' of an *antimatroid*. Accordingly, let us denote as *co-antimatroidal* a finite interval space $\mathcal{I} = (X, I)$ that satisfies AE. Thus, clearly, Theorem 1 below also applies in particular to the class of all *co-antimatroidal convex and idempotent interval spaces.*[15] More generally, the class of such co-antimatroidal interval spaces includes *the class of all interval spaces* $(X, I(\leqslant))$ *where* $(X, \leqslant)$ *is a finite partially ordered set and*

$I(\leqslant)(x, y) = \{x, y\} \cup \{z \in X : x \leqslant z \leqslant y \text{ or } y \leqslant z \leqslant x\}$ for all $x, y \in X$.

**Remark 4** A **linear geometry** is a convex geometry that satisfies **interval anti-exchange** and three further properties called **additivity** (i.e. for all $x, y, z \in X$, if $z \in I(x, y)$ then $I(x, y) = I(x, z) \cup I(z, y)$), **no-branchpoint** (i.e. for all $x, y, z \in X$, if $z \notin I(x, y)$ and $y \notin I(x, z)$ then $I(x, y) \cap I(x, z) = \{x\}$), and the **Pasch-Peano condition** (i.e. for all $x, y, z, u, v \in X$, if $y \in I(x, u)$ and $z \in I(x, v)$ then $I(y, v) \cap I(z, u) \neq \varnothing$: see e.g. Coppel (1998)).

Finally, (undirected) **trees** (i.e. connected graphs without cycles) are also convex geometries if the interval $I(x, y)$ of each pair $x, y$ of vertices is defined as the set of vertices that lie on the (unique) shortest path joining $x$ and $y$. Characterizations of the interval functions of trees thus defined have been provided by Sholander (1952, 1954) and, most recently, by Chvátal, Rautenbach, Schäfer (2011) that focuses on the finite case. It should be noticed that trees do satisfy interval anti-exchange (as defined above) and additivity, but may or may not satisfy the no-branchpoint property and the Pasch-Peano condition (namely, some trees satisfy both of those properties, while others violate both). Conversely, linear geometries may or may not have cycles (we shall introduce below some examples of linear geometries with cycles). Thus, linear geometries and trees are subclasses of convex geometries that partially overlap: there are convex geometries that are both linear geometries and trees (namely, trees with no branchpoints), while other convex geometries are trees but not linear geometries (e.g. trees with branchpoints), or linear geometries but not trees (e.g. cliques i.e. complete graphs), or neither linear geometries nor trees.[16]

The next condition is a considerably weakened version of IAE:

[15] It is easily checked that the finite convex idempotent interval space defined in Example 1 (see Appendix B) does also satisfy AE and is therefore a representative of that subclass of co-antimatroidal interval spaces.

[16] An example of a convex geometry that is neither a linear geometry nor a tree is the (canonical) interval space of the quasi-complete graph presented under Example 2 in Appendix B.

(**Minimal Anti-Exchange** *(MAE)*): for all $x, y, v, z \in X$ such that $x \neq y$, and $v \neq z$ at least one of the following clauses is satisfied: (i) $I(y, v) \cap \{x, z\} \neq \{x, z\}$, (ii) $I(x, z) \cap \{y, v\} \neq \{y, v\}$, (iii) $I(v, z) \cap \{x, y\} \neq \varnothing$, (iv) $I(y, z) \cap \{x, v\} \neq \varnothing$.

**Remark 5** It is easily checked that, for an arbitrary interval space $\mathcal{I} = (X, I)$, IAE does indeed entail MAE, while the reverse does not generally hold. To see this, observe that by definition IAE amounts to requiring that for all $x, y, v, z \in X$ such that $x \neq y$, at least one of the following three clauses is satisfied: (i) $x \notin I(y, v)$, (ii) $y \notin I(x, z)$, (iii') $x \in I(v, z)$. Clearly, (iii') entails (iii) whence MAE holds true whenever IAE does. On the other hand, consider interval space $\mathcal{I} = (X, I)$ with $X = \{x, y, v, z\}$, $|X| = 4$, and $I$ as defined by the following rule: $I(x, z) = \{x, y, z\}$, $I(y, v) = \{x, y, v\}$, and $I(a, b) = \{a, b\}$ otherwise. Notice that, by construction, $\mathcal{I}$ is convex and idempotent. Moreover, $I(x, z) \cap \{y, v\} = \{y\} \neq \{y, v\}$ hence $\mathcal{I}$ satisfies MAE. However, $x \in I(y, v)$, $y \in I(x, z)$, and $x \notin I(v, z)$: therefore $\mathcal{I}$ fails to satisfy IAE. Interval space $\mathcal{I}^* = (X, I^*)$ as defined in Remark 1 is another simple example of a convex idempotent interval space that satisfies MAE but fails to satisfy IAE. Indeed, it is easily checked that $\mathcal{I}^*$ is antisymmetric: hence, for any $Y \subseteq X$, its restriction to $Y$, denoted $\mathcal{I}^*_Y = (Y, I^*_Y)$, is also antisymmetric. It follows that (in view of the proof of Corollary 1

below, establishing that antisymmetric idempotent spaces of cardinality not larger than three must satisfy IAE) $\mathcal{I}^*_Y$ satisfies IAE (hence a fortiori MAE) whenever $|Y| \leq 3$. Thus, it only remains to check for MAE with respect to four *distinct* $x, y, v, z \in X$: but then, since by construction $|I^*(a, b)| \leq 3$ for all $a, b \in X$, *both* clauses MAE(i) and MAE(ii) (that amount precisely to requiring that $|I^*(y, v)| < 4$ and $|I^*(x, z)| < 4$) are clearly satisfied, whence MAE holds. On the other hand, recall that by definition $c_1 \in I^*(a, b_1) = I^*(b_1, a)$, $b_1 \in I^*(c_1, b_2)$, and $c_1 \notin I^*(a, b_2)$: therefore $\mathcal{I}^*$ fails to satisfy IAE.

The next proposition provides a remarkable property of $\mathcal{I}$-monotonic voting rules when $\mathcal{I}$ satisfies Interval Anti-Exchange:

**Proposition 2.** *Let $\mathcal{I} = (X, I)$ be an interval space that satisfies Interval Anti-Exchange, and $f : X^N \rightarrow X$ an $\mathcal{I}$-monotonic voting rule. Then, for all $x_N, y_N \in X^N$ , $f(x_N) \neq f(y_N)$ entails that $f(x_N) \in I(x_i, y_i)$ for some $i \in N$.*

As suggested by the crucial role it plays in the proof of the foregoing proposition, Interval Anti-Exchange is definitely required to ensure that the property of $\mathcal{I}$-monotonic voting rules identified by Proposition 2 does indeed hold. To confirm that, consider again the median interval space $\mathcal{I} =(\mathbf{2}^3, I^m)$ induced by the Boolean cube introduced in Section 2, and the *ternary* median operation $\mu$ as defined on $\mathbf{2}^3$. It is easily checked that $\mathcal{I} =(\mathbf{2}^3, I^m)$ does *not* satisfy Interval Anti-Exchange: e.g. $x_1 \in I^m(x_5, 1)$,

$x_5 \in I^m(x_1, x_3)$ but $x_1 \notin I^m(1, x_3) = \{1, x_3\}$. It can also be shown that $\mu$ is $\mathcal{I}$-monotonic, because projections and constants are obviously $\mathcal{I}$-monotonic, and the median preserves $\mathcal{I}$-monotonicity and can be represented as an iterated median of projections and constants (see Danilov, Sotskov (2002) and Savaglio, Vannucci (2012) for details). Next, take a $\mathcal{I}$-unimodal preference profile $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ such that $top(\succcurlyeq_1) = x_1$, $top(\succcurlyeq_2) = x_2$, $top(\succcurlyeq_3) = 0$, and $x_3 \succ_i x_4$ for all $i \in \{1, 2, 3\}$ as previously considered in Section 2, and notice that $\mu(x_1, x_2, 0) = x_4$, while $\mu(x_3, x_3, 0) = x_3$. However, $I^m(x_1, x_3) = \{1, x_1, x_3, x_5\}$, $I^m(x_2, x_3) = \{1, x_2, x_3, x_6\}$, $I^m(0, 0) = \{0\}$ hence $x_4 \notin I^m(x_1, x_3) \cup I^m(x_2, x_3) \cup I^m(0, 0)$: thus, the thesis of Proposition 2 fails to hold for $\mu$ (for that choice of $\mathcal{I}$).

The next Theorem establishes that in convex idempotent interval spaces Interval Anti-Exchange ensures that simple (or individual) strategy-proofness and coalitional strategy-proofness of a voting rule on the full unimodal domain are equivalent properties.

**Theorem 1.** *Let $\mathcal{I} = (X, I)$ be a convex idempotent interval space that satisfies Interval Anti-Exchange (IAE), and $f : X^N \to X$ a voting rule that is strategy-proof on the full unimodal domain $U^N_{\mathcal{I}}$. Then, $f$ is also coalitionally strategy-proof on $U^N_{\mathcal{I}}$.*

Observe that, as mentioned in the Introduction, the argument underlying Theorem 1 may be summarized as follows: (a) *strategy-proofness of a voting rule $f$ on the full unimodal domain in a convex idempotent interval space and Interval Anti-Exchange of that space jointly imply that two arbitrary profiles $x_N$,$y_N$ of votes result in distinct outcomes $u = f(x_N)$, $v = f(y_N)$ only if -for at least one voter $i$ with $x_i \neq y_i$- $u$ is a compromise between $v$ and $i$'s choice $x_i$ at $x_N$*; but then, (b) *if $x_i$ is the top outcome of voter $i$, unimodality implies that $v$ cannot be strictly better than $u$ for voter $i$, whence coalitional strategy-proofness of $f$ follows*.

It should be noticed here that Propositions 1 and 2 and Theorem 1 extend and generalize some properties of the standard interval spaces induced by trees that are pointed out and exploited by Danilov (1994). Moreover, it turns out that *Theorem 1 implies at once that simple/individual and coalitional strategy-proofness on the full $\mathcal{I}$-unimodal domain are equivalent if $\mathcal{I} = (X, I)$ is an antisymmetric interval space with at most three points,* as made precise by the following Corollary[17]:

[17]To be sure, the present Corollary also follows from a result due to Barberà, Berga, Moreno (2010) establishing that coalitional strategy-proofness holds for any strategy-proof social choice function with an arbitrary domain of profiles of total preorders over an outcome set with at most three outcomes. The proof of the latter result, however, does not rely on the geometry of the outcome space, but rather on an 'intrinsic' property - called 'sequential inclusion'- of the relevant preference profiles. Also notice that, while any profile of total preorders over an outcome space of size three or less satisfies 'sequential inclusion', unimodal preference profiles over larger outcome spaces generally do not. Thus we 'almost'

**Corollary 1.** *Let $\mathcal{I} = (X, I)$ be an antisymmetric interval space such that $|X| \leq 3$, and $f : X^N \to X$ a voting rule that is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$. Then, $f$ is also coalitionally strategy-proof on $U_{\mathcal{I}}^N$.*

It should also be emphasized that Theorem 1 above amounts to a considerable generalization of the previous results on equivalence of simple and coalitional strategy-proofness due to Moulin (1980) and Danilov (1994), concerning (bounded) chains and trees, respectively. Clearly, Theorem 1 applies to all trees and linear geometries. However, its scope is much wider than trees or linear geometries: it clearly includes interval spaces that are induced by the geodesics (i.e. paths of minimum length joining two points) of some graphs with cycles, but are not linear geometries.[18]

Concerning linear geometries, that fall entirely under the scope of Theorem 1, it should be emphasized that they cover a wealth of interesting structures. To begin with, it should be recalled here that Euclidean convex sets can be shown to reduce to linear geometries with three special properties namely *denseness, unendingness,* and *completeness* (see Coppel (1998)). Moreover, both chains and trees with the no-branchpoint property are special instances of linear geometries. But, as a matter of fact, the class of linear geometries is much wider than that. To mention just a pair of very simple interesting examples, consider the interval space $\mathcal{I}' = (X, I')$ induced by the clique or complete graph on $X$ (i.e. with $I'(x, y) = \{x, y\}$ for all $x, y \in X$), and the interval space $\mathcal{I}'' = (Y, I'')$ induced by the join or linear sum of a clique and a chain (see Section 2 above): it is readily checked that both of them are indeed linear geometries (and neither of them is a median interval space). Interval space $\mathcal{I}''$ is particularly interesting in the present connection, since the results provided by Schummer, Vohra (2002) imply the existence of nontrivial nondictatorial voting rules on the full unimodal domain $U_{\mathcal{I}''}^N$(e.g. those resulting from the combination of a clique-related, locally-dictatorial rule that applies whenever the top outcome of the appointed 'clique-dictator' lies on the clique, and a median rule as restricted to the subset of outcomes lying between the top outcome of the 'clique-dictator' and the outcome that is closest to the clique among those located on the line, that applies otherwise).

Then, Theorem 1 above does indeed imply at once that even all such nontrivial nondictatorial strategy-proof voting rules are also coalitionally strategy-proof on $U_{\mathcal{I}''}^N$.

---

prove by incidence-geometric arguments the (stronger) proposition previously obtained by Barberà-Berga-Moreno through purely combinatorial arguments. We report here our Corollary 1 and its proof precisely to highlight this point (see Appendix A). Notice that unfortunately we cannot ameliorate our result by IAE-based incidence-geometric arguments: to check that, just consider the three-point interval space

$(X = \{x, y, z\}, I)$ with $I(y, z) = I(z, y) = I(x, z) = I(z, x) = X$ and
$I(a, b) = \{a, b\}$ for any other pair $\{a, b\}$.
Clearly, $(X, I)$ is not antisymmetric and violates IAE.

[18]To check the latter statement the reader is referred to Example 2 in Appendix B.

We conclude with a partial converse result. Namely, a convex idempotent interval space $\mathcal{I}$ ensures equivalence of simple and coalitional strategy-proofness on the full unimodal domain only if it also satisfies Minimal Anti-Exchange, as established by the following:

**Theorem 2.** *Let $\mathcal{I} = (X, I)$ be a convex and idempotent interval space such that every voting rule $f : X^N \to X$ which is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$ is also coalitionally strategy-proof on $U_{\mathcal{I}}^N$. Then, $\mathcal{I} = (X, I)$ satisfies Minimal Anti-Exchange (MAE).*

It should be emphasized that the foregoing Theorem holds for both median and nonmedian interval spaces. One of the simplest examples of a convex idempotent space that fails to satisfy MAE is the median interval space $(X, I^m)$ induced by the Boolean lattice $\mathbf{2^2} = (\{0, 1, x, y\}, \vee, \wedge)$ by taking $X = \{0, 1, x, y\}$ and defining $I^m$ by the rule $I^m(a, b) = \{c \in X : a \wedge b \leqslant c \leqslant a \vee b\}$ where $u \leqslant v$ if and only if $u = u \wedge v$. Indeed, the results of Savaglio, Vannucci (2012) imply equivalence failure in such an interval space (and, more generally, in any interval space induced by a bounded distributive lattice that is not a chain). Moreover, notice that a simple adaptation of the foregoing proof also shows that the median $\mu : \{0, 1, x, y\}^3 \longrightarrow \{0, 1, x, y\}$ as defined by the rule $\mu(z_1, z_2, z_3) = (z_1 \wedge z_2) \vee (z_1 \wedge z_3) \vee (z_2 \wedge z_3)$ is not coalitionally strategy-proof on $U_{\mathcal{I}}^N$ with $\mathcal{I} = (\{0, 1, x, y\}, I^m)$: to see this, just consider a third total preorder $\succcurlyeq^\circ$such that $x \succ^\circ y \sim^\circ v \sim^\circ z$, and observe that $\succcurlyeq^\circ \in U_{\mathcal{I}}$, $\mu(v, y, x) = x$ and $\mu(z, z, x) = z$, hence coalition $\{1, 2\}$ can successfully manipulate the 'sincere' median outcome at unimodal preference profile $(\succcurlyeq^*, \succcurlyeq', \succcurlyeq^\circ) \in U_{\mathcal{I}}^3$ as defined above in the proof of Theorem 2.

Now, as it is well-known, the interval spaces thus induced by distributive lattices are another prominent class of median interval spaces (along with the interval spaces induced by trees). Notice however that since both non-trivial Hamming graphs typically include cubes, their (typically non-median) interval spaces also violate MAE and therefore -precisely as the interval space of a Boolean distributive lattice $\mathbf{2^K}$ with $\mathbf{K} > 1$- admit nontrivial nondictatorial strategy-proof voting rules (such as rule $f$ as defined in the proof of Theorem 2) that are *not* coalitionally strategy-proof. Indeed, in view of the proof of Theorem 2 (and as also suggested by Corollary 1 above), if a given convex idempotent interval space fails to satisfy MAE then there exists a non-trivial non-dictatorial strategy-proof voting rule on the full unimodal domain of that space that admits at least *four* distinct outcomes, and is manipulable by some coalitions. Therefore Theorem 2 confirms that, generally speaking, 'unimodal' equivalence of simple and coalitional strategy-proofness fails to hold in several important classes of interval spaces, both median and non-median.

In particular, relying on Theorems 1 and 2 we are now ready to provide a definite answer to the question concerning equivalence of simple and coalitional strategy-proofness of voting rules for full unimodal domains in the outcomes spaces considered in Section 2 above. That answer - actually, an eight-item list of answers - is detailed in the next Corollary where *'full unimodal equivalence'* is to be read as a shorthand for *equivalence of simple and coalitional strategy-proofness of voting rules for full unimodal domains* (in the interval space under consideration), and the inserted bibliography items single out results previously established -or implied- through alternative ad hoc arguments by other Authors (and address the reader to the original sources or Authors).

**Corollary 2.** *Let $\mathcal{I}$ be a convex and idempotent interval space. Then*

*(i) (Moulin (1980)) if $\mathcal{I} = (X, I(\leqslant))$ is the (median) interval space canonically induced by a bounded chain $\mathcal{X} = (X, \leqslant)$ then full unimodal equivalence holds;*

*(ii) (Danilov (1994)) if $\mathcal{I} = (X, I^G)$ is the (median) interval space canonically induced by a bounded tree $G = (X, E)$ (see Section 2 above, footnote 7) then full unimodal equivalence holds;*

*(iii) (Gibbard, Satterthwaite and others) if $\mathcal{I} = (X, I^G)$ is the interval space canonically induced by a clique $G = (X, E)$ then full unimodal equivalence holds;*

*(iv) (Savaglio, Vannucci (2012)) if $\mathcal{I} = (X, I^m)$ is the (median) interval space canonically induced by a bounded distributive lattice $\mathcal{X} = (X, \leqslant, 0, 1)$ that is not a chain then full unimodal equivalence fails;*

*(v) if $\mathcal{I} = (X, I^E)$ is the (non-median) interval space canonically induced by a simplex in an Euclidean convex space then full unimodal equivalence holds;*

*(vi) if $\mathcal{I} = (X, I(\leqslant))$ is the interval space canonically induced by a partially ordered set $\mathcal{X} = (X, \leqslant)$ then full unimodal equivalence holds;*

*(vii) if $\mathcal{I} = (X, I^G)$ is the interval space canonically induced by a non-trivial Hamming graph $G^H$ then full unimodal equivalence fails;*

*(viii) if $\mathcal{I} = (X, I^G)$ is the interval space canonically induced by the graph $G$ resulting from the join of a clique and a chain then full unimodal equivalence holds.*

Notice that point (i) is established by Moulin (1980) [19] through specific arguments based upon properties of medians in chains, and an explicit proof is only given for a subclass of anonymous strategy-proof social choice functions. Point (ii) is established by Danilov (1994) for the subdomain of unimodal *linear orders* in a tree by means of an argument that *implicitly* relies on Interval Anti-Exchange. Point (iii) can also be regarded as a corollary to well-known results which can be established without any reference to Interval Anti-Exchange, but some care is needed to articulate a proper argument which substantiates that claim. Indeed, what follows from the Gibbard-Satterthwaite theorem for the 'universal' domain of linear orders is that if $|X| \geq 3$ then

[19] To be sure, Moulin proves a version of Corollary (ii) for a *restricted* unimodal domain where voters are not allowed to have the maximum or the minimum of the chain (or lattice) as their unique optimum. But Moulin's proof can be adapted to the full unimodal domain.

the only strategy-proof voting rules for the full unimodal domain are the constant rules, the dictatorial rules (which are both also coalitionally strategy-proof), and *possibly some other non-sovereign (i.e. non-surjective) voting rules with a two-valued range.* Moreover, if $|X| \leqslant 3$ then the combinatorial argument recently provided by Barberà, Berga, Moreno (2010) implies that any voting rule on $X$ is strategy-proof on a arbitrary domain of total preorders if and only if it is also coalitionally strategy-proof on that domain. Finally, Vannucci (2013) shows that no two-valued non-sovereign voting rule for a full unimodal domain is strategy-proof whence point (iii) eventually follows without any appeal to anti-exchange properties.

Point (iv) concerning failure of full unimodal equivalence in bounded distributive lattices was recently established by Savaglio, Vannucci (2012) by a direct argument that *implicitly* invokes necessity of Minimal Anti-Exchange for full unimodal equivalence.

The proofs of points (i), (ii), (iii), (iv) of Corollary 2 proposed here offer an alternative general argument to establish those points by means of Interval Anti-Exchange and/or Minimal Anti-Exchange.

Points (v)-(viii) of Corollary 2 are -to the best of the author's knowledge- entirely novel results.[20]

Apparently, Interval Anti-Exchange properties (through Theorems 1 and 2) provide a common unifying approach to 'full unimodal equivalence' issues that offers both a new proof of several important well-known theorems and some quite interesting novel results. Some comments on the practical significance of those results when combined with information on the median vs non-median character of the relevant interval space are in order here.

We have already observed that when the interval space is median and full unimodal equivalence obtains (see cases (i),(ii), (vi) of Corollary 2), it follows that nice voting rules such as the (extended) median rule and its variants are both well-defined and coalitionally strategy-proof hence qualify as robust successful solutions to an important mechanism design problem in voting.

If the relevant interval space is median and full unimodal equivalence fails (see case (iv) of Corollary 2) then the (extended) median rule is well defined and strategy proof but is not coalitionally strategy-proof: that fact puts in jeopardy the availability of reasonably nice and coalitionally strategy-proof voting rules, and acceptable solutions (if any) have to embody trade-offs between unbiasedness and coalitional strategy-proofness.

When full unimodal equivalence obtains and the interval space is not median (e.g. cases (iii), (v), (viii) of Corollary 2) then the median rule and median-related rules

[20]Moreover, many other examples might be considered including median semilattices with no infinite chains, multi-crosses (a 'basic' class of irreducible linear geometries: see Coppel (1998)), and partial cubes (or mediatic graphs) that are not trees (see Eppstein, Falmagne, Ovchinnikov (2008)). Multi-crosses of course satisfy Interval Anti-Exchange, while median semilattices (with no infinite chains) that are not chains, and partial cubes that are not trees violate even Minimal Anti-Exchange.

are not-well defined. Thus, strategy-proof voting rules may either reduce to dictatorial or constant rules (see case (iii), that is the situation represented by the Gibbard-Satterthwaite impossibility result) or include some minimally unbiased rule (the range of available depending perhaps on further properties of the interval space such as being quasi-median or not). The last statement applies e.g. to case (viii) as suggested by some findings of Schummer, Vohra (2002), and as far as we know, might also apply to case (v). But in any case, of course, there is nothing to be gained by relaxing the requirement of coalitional strategy-proofness to simple strategy-proofness.

When the interval space is not median and full unimodal equivalence fails (case (vii) of Corollary 2) the (extended) median rule is not well-defined and that is likely to imply that perfectly unbiased strategy-proof voting rules are not available. On the positive side, however, one may conclude that there certainly exist nondictatorial nontrivial strategy-proof voting rules (that are not, however, coalitionally strategy-proof).

## 4. Concluding remarks

It should be emphasized again that the sufficient condition for equivalence of simple and coalitional strategy-proofness of voting rules on full unimodal domains that has been established in the present paper is in fact quite general. As repeatedly mentioned above, *Interval Anti-Exchange* (*IAE*) is shared by all trees and indeed by all linear geometries but is characteristic of a much larger class of convex idempotent interval spaces. Therefore, our results provide significant information concerning problems of *strategy-proof location in a vast class of networks.* The case of Euclidean outcome spaces is quite remarkable in that respect, because frequently encountered in many models and applications: it is now well-appreciated that all the strategy-proof voting rules that are available on the Euclidean real line are also coalitionally strategy-proof, and that the median rule and several median-based voting rules that are nontrivial nondictatorial and coalitionally strategy-proof on the Euclidean real line are not available on a convex Euclidean space in $\mathbb{R}^m$ for $m \geq 2$. Theorem 1 above predicts that, when moving to convex Euclidean spaces of higher dimensions, coalitional strategy-proofness of *all* strategy-proof voting rules as defined on the full unimodal domains of those spaces (including any nontrivial nondictatorial one that might possibly exist amongst them) is still warranted due to the fact that Interval Anti-Exchange is satisfied by *every* convex Euclidean space. Moreover, it can be shown that Theorem 1 also implies that the 'median voter theorem' for committee-decisions - establishing that the extended $n$-ary median over an 'odd' unimodal domain invariably selects a Condorcet majority winner - holds whenever the underlying median interval space satisfies Interval Anti-Exchange (the details will be spelled out elsewhere).

We have also shown that any convex idempotent interval space where the foregoing 'unimodal' equivalence obtains must satisfy *Minimal Anti-Exchange (MAE),* which in turn implies that such equivalence fails to hold in certain convex idempotent interval spaces, both median and non-median (and that such spaces admit the existence of non-trivial non-dictatorial strategy-proof voting rules with at least four distinct outcomes on their full unimodal domains). It remains to be seen whether or not some convex, idempotent interval spaces that satisfy MAE while violating IAE do also support such an equivalence of simple and coalitional strategy-proofness of voting rules on full unimodal domains. For any such interval space (both median and non-median), and for all convex, idempotent interval spaces that satisfy IAE but are not median the search for reasonably unbiased and coalitionally strategy-proof voting rules provides a somewhat intriguing and possibly challenging open problem for future research.

## 5. Appendix A. Proofs

*Proof of Proposition 1.* Let us assume that $f : X^N \rightarrow X$ is *not* $\mathcal{I}$-monotonic: thus, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i \in N} \in X^N$ such that $f(x_N) \notin I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$. To begin with, observe that for any $x \in X$, if there exists $\succcurlyeq \in U_{\mathcal{I}}$ such that $x = top(\succcurlyeq)$ then $I(x,x) = \{x\}$. Indeed, suppose that there exists $y \in I(x,x) \smallsetminus \{x\}$: then -by definition of $U_{\mathcal{I}}$- $\succcurlyeq \notin U_{\mathcal{I}}$, a contradiction.

Next, consider the total preorder $\succcurlyeq^*$ on $X$ defined as follows: $x_i = top(\succcurlyeq^*)$ and for all $y, z \in X \smallsetminus \{x_i\}$ , $y \succcurlyeq^* z$ iff

(i) $\{y, z\} \subseteq I(x_i, f(x'_i, x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ or (ii) $y \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ and $z \notin I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ or (iii) $y \notin I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ and $z \notin I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$. Clearly, by construction $\succcurlyeq^*$ consists of three indifference classes with $\{x_i\}$,

$I(x_i, f(x'_i, x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ and $X \smallsetminus I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ as top, medium and bottom indifference classes, respectively.

Now, observe that $\succcurlyeq^* \in U_{\mathcal{I}}$. To check that statement, take any $y, z, v \in X$ such that $v \in I(y,z)$. If $y = z = v$ there is in fact nothing to prove. If $y = z \neq v$ then by the first observation above, $y$ cannot be a top outcome of any $\succcurlyeq \in U_{\mathcal{I}}$ hence in particular $y \neq x_i$. It follows that either $y = z \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ or $y = z \in X \smallsetminus I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$: in the latter case there is again nothing to prove. If $y = z \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ then, by Convexity of $\mathcal{I}$, $v \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ whence $v \succcurlyeq^* y$.

If $y \neq z$ then clearly $\{y, z\} \neq \{x_i\}$: let us then suppose without loss of generality that $y \neq x_i$. If, moreover, $\{y, z\} \subseteq I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ then, by Convexity of $\mathcal{I}$ again, $v \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$. Hence, $v \succcurlyeq^* y$ by definition of $\succcurlyeq^*$. If on the contrary $\{y, z\} \cap (X \smallsetminus I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))) \neq \varnothing$ then take $w \in \{y, z\} \cap (X \smallsetminus I(x_i, f(x'_i, x_{N \smallsetminus \{i\}})))$.

Clearly, by definition of $\succcurlyeq^*$, $v \succcurlyeq^* w$. Since $w \in \{y, z\}$, it follows that the unimodality condition is satisfied again and therefore $\succcurlyeq^* \in U_{\mathcal{I}}$ as claimed.

Also, by assumption $f(x_N) \in X \smallsetminus I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ while

$f(x'_i, x_{N \smallsetminus \{i\}}) \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ by Extension, whence by construction $f(x'_i, x_{N \smallsetminus \{i\}}) \succ^* f(x_N)$. Thus, posit $(\succcurlyeq_j)_{j \in N} \in U_{\mathcal{I}}^N$ such that $x_j = top(\succcurlyeq_j)$ for all $j \in N$ and $\succcurlyeq_i = \succcurlyeq^*$: then, $f$ is *not* strategy-proof on $U_{\mathcal{I}}^N$ .

Conversely, let $f$ be monotonic with respect to $\mathcal{I}$. Now, consider any $\succcurlyeq = (\succcurlyeq_j)_{j \in N} \in U_{\mathcal{I}}^N$ and any $i \in N$. By definition of $\mathcal{I}$-monotonicity $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \in I(top(\succcurlyeq_i), f(x_i, x_{N \smallsetminus \{i\}}))$ for all $x_{N \smallsetminus \{i\}} \in X^{N \smallsetminus \{i\}}$ and $x_i \in X$. But then, since clearly by definition $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}})$, either $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \succcurlyeq_i f(x_i, x_{N \smallsetminus \{i\}})$ by unimodality of $\succcurlyeq_i$.

Hence, $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \succcurlyeq_i f(x_i, x_{N \smallsetminus \{i\}})$ in any case. It follows that $f$ is indeed strategy-proof on $U_{\mathcal{I}}^N$.

*Proof of Proposition 2.* Let $x_N, y_N \in X^N$ , and $f(x_N) \neq f(y_N)$. Then, by $\mathcal{I}$-monotonicity of $f$,

$f(x_N) \in I(x_i, f(y_i, x_{N \smallsetminus \{i\}}))$ and $f(y_i, x_{N \smallsetminus \{i\}}) \in I(y_i, f(x_N))$ for each $i \in N$.

Now, take $i = 1$. If $f(x_N) \neq f(y_1, x_{N \smallsetminus \{1\}})$ then, thanks to Symmetry of $I$, Interval Anti-Exchange applies, whence $f(x_N) \in I(x_1, y_1)$, and the thesis immediately follows. Let us then suppose that, on the contrary, $f(x_N) = f(y_1, x_{N \smallsetminus \{1\}})$. Next, consider $f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$.

By $\mathcal{I}$-monotonicity of $f$, $f(y_1, x_{N \smallsetminus \{1\}}) \in I(x_2, f(y_1, y_2, x_{N \smallsetminus \{1,2\}}))$ and $f(y_1, y_2, x_{N \smallsetminus \{1,2\}}) \in I(y_2, f(y_1, x_{N \smallsetminus \{1\}}))$.

If $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) \neq f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$ then again, by Interval Anti-Exchange of $\mathcal{I}$, it follows that $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) \in I(x_2, y_2)$ as required by the thesis. Thus, assume again that on the contrary $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) = f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$. A suitable iteration of the previous argument allows us to establish that either $f(x_N) \in I(x_i, y_i)$ for some $i \in \{1, ..., n-1\}$ or $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n)$. In the former case the thesis holds. In the latter case, by $\mathcal{I}$-monotonicity of $f$, $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n) \in I(x_n, f(y_N))$ and $f(y_N) \in I(y_n, f(y_{N \smallsetminus \{n\}}, x_n))$. Since by hypothesis $f(x_N) \neq f(y_N)$ it follows, by Interval Anti-Exchange of $\mathcal{I}$, that $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n) \in I(x_n, y_n)$ and the thesis is therefore established.

*Proof of Theorem 1.* Indeed, suppose that $f$ is not coalitionally strategy-proof on $U_{\mathcal{I}}^N$. Then, there exist $S \subseteq N$, $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$, $x_N \in X^N$ and $x'_S \in X^S$ such that for all $i \in S$, $top(\succcurlyeq_i) = x_i$ and $f(x'_S, x_{N \smallsetminus S}) \succ_i f(x_N)$ (where $\succ_i$denotes the asymmetric component of $\succcurlyeq_i$).

Notice that it may be assumed without loss of generality that $S = N$: to see this, consider $f_{x_{N \smallsetminus S}} : X^S \to X$ as defined by the rule $f_{x_{N \smallsetminus S}}(y_S) = f(y_S, x_{N \smallsetminus S})$ for all $y_S \in X^S$ and observe that, by construction, $f_{x_{N \smallsetminus S}}$ is both strategy-proof and *not* coalitionally strategy-proof. Let us then posit $f(x_N) = f(x_S) = u$, and $f(x'_N) = f(x'_S) = v$: by construction, $v \succ_i u$ for all $i \in N$. By Proposition 1 above, $f$ is $\mathcal{I}$-monotonic: therefore, $f(v, x'_{N \smallsetminus \{1\}}) \in I(v, f(x'_N)) = I(v, v)$, whence $f(v, x'_{N \smallsetminus \{1\}}) = v$, by idempotence of $\mathcal{I}$.

Similarly, by $\mathcal{I}$-monotonicity of $f$ again, $f(v,v,x'_{N\smallsetminus\{1,2\}}) \in I(v, f(v,x'_2,x'_{N\smallsetminus\{1,2\}})) = I(v,v)$: thus, by idempotence of $\mathcal{I}$ again,

$f(v,v,x'_{N\smallsetminus\{1,2\}}) = v$. A suitable iteration of the same argument establishes that $f(v,v,...,v) = f(x'_N) = v$.

Now, suppose that there exists $i \in N$, such that $f(x_N) = u \in I(x_i, v)$: since $x_i = top(\succcurlyeq_i)$ and $v \succ_i u$ by assumption, then $\succcurlyeq_i \notin U_{\mathcal{I}}$, a contradiction. Therefore, $f(x_N) \notin I(x_i,v)$ for each $i \in N$. By Proposition 2 above it follows that $f(x_N) = f(v,...,v) = f(x'_N)$, a contradiction again, whence the thesis is established.

*Proof of Corollary 1.*To begin with, notice that if $|X| \leq 3$, then any interval space $(X,I)$ is convex: indeed, recall that in order to be *not* convex an interval space has to include at least two points $x,y$ and two points $u,v$ such that $\{u,v\} \subseteq I(x,y)$ but $I(u,v) \nsubseteq I(x,y)$ whence at least *four* points are needed. It is also immediately checked that *every* antisymmetric interval space $\mathcal{I} = (X,I)$ with $|X| \leq 3$ does satisfy Interval Anti-Exchange: to see that, take $X = \{x,y,z\}$ and assume that on the contrary there exist $a,b,c,d \in X$ such that $a \neq b$, $a \in I(b,c)$, $b \in I(a,d)$, and $a \notin I(c,d)$. Now, $a \notin I(c,d)$ implies $a \notin \{c,d\}$ hence either $c = d$ or $c = b$ or else $d = b$. If $c = d$ then by antisymmetry $a = b$, a contradiction. If $c = b$ then $a \in I(b,b)$ hence by idempotence $a = b$, a contradiction again (recall that, as already observed above, an antisymmetric interval space is also idempotent). Then, it must be the case that $d = b$ whence $a \in I(d,c) = I(c,d)$, a contradiction again. But then, Theorem 1 applies and the proof is complete.

*Proof of Theorem 2.* Indeed, suppose $\mathcal{I}$ does not satisfy MAE. Then, there exist $x,y,v,z \in X$ such that $x \neq y$, $v \neq z$, $x \in I(y,v)$, $y \in I(x,z)$, $v \in I(x,z)$, $z \in I(y,v)$, $I(v,z) \cap \{x,y\} = \varnothing$ and $I(y,z) \cap \{x,v\} = \varnothing$ (notice that by definition of $I$ it follows at once that $|X| \geq 4$). But then, consider the following total preorder $\succcurlyeq^*$ on $Y = \{x,y,v,z\}$:

$\succcurlyeq^* = \{(v,z),(v,x),(v,y),(z,x),(z,y),(x,y),(y,x),(x,x),(y,y),(v,v),(z,z)\}$,

namely $v \succ^* z \succ^* x \sim^* y$.

Notice that we can assume without loss of generality that $X = Y = \{x,y,v,z\}$: (otherwise, we might apply the following proof to subspace $\mathcal{I}_Y = (Y, I_Y)$, to the same effect).

To begin with, observe that by construction $\mathcal{I}$-unimodality of $\succcurlyeq^*$ only requires that both $x \notin I(v,z)$ and $y \notin I(v,z)$. Thus, $\succcurlyeq^*$ is $\mathcal{I}$-unimodal.

Next, consider another total preorder $\succcurlyeq'$on $\{x,y,v,z\}$:

$\succcurlyeq' = \{(y,z),(y,x),(y,v),(z,x),(z,v),(x,v),(v,x),(x,x),(y,y),(v,v),(z,z)\}$,

namely $y \succ' z \succ' x \sim' v$. Clearly, $\mathcal{I}$-unimodality of $\succcurlyeq'$ only requires that $x \notin I(y,z)$ and $v \notin I(y,z)$. Thus, $\succcurlyeq'$is also $\mathcal{I}$-unimodal.

Then, consider the class of all voting rules $f' : X^N \to X$ such that for all $\mathbf{u} = u_{N\smallsetminus\{1,2\}} \in X^{N\smallsetminus\{1,2\}}$

$f'(v, y, \mathbf{u}) = x$, and $f'(z, z, \mathbf{u}) = z$.

Let us now show that there exists a voting rule $f$ in that class which is $\mathcal{I}$-monotonic. To see that, observe that $\mathcal{I}$-monotonicity of $f$ amounts precisely to conditions (a)-(l) as listed below: for all $\mathbf{u} \in X^{N \setminus \{1,2\}}$,

(a) $f(x, x, \mathbf{u}) \in I(x, f(y, x, \mathbf{u})) \cap I(x, f(v, x, \mathbf{u})) \cap I(x, f(z, x, \mathbf{u})) \cap$
$\cap I(x, f(x, y, \mathbf{u})) \cap I(x, f(x, v, \mathbf{u})) \cap I(x, f(x, z, \mathbf{u}))$ hence
positing $f(x, x, \mathbf{u}) = x$ is clearly consistent with (a);

(b) $f(y, y, \mathbf{u}) \in I(y, f(x, y, \mathbf{u})) \cap I(y, f(v, y, \mathbf{u})) \cap I(y, f(z, y, \mathbf{u})) \cap$
$\cap\ I(y, f(y, x, \mathbf{u})) \cap I(y, f(y, v, \mathbf{u})) \cap I(y, f(z, y, \mathbf{u}))$ hence positing $f(y, y, \mathbf{u}) = y$ is clearly consistent with (b) (and (a));

(c) $f(v, v, \mathbf{u}) \in I(v, f(x, v, \mathbf{u})) \cap I(v, f(y, v, \mathbf{u})) \cap I(v, f(z, v, \mathbf{u})) \cap$
$\cap\ I(v, f(v, x, \mathbf{u})) \cap I(v, f(y, v, \mathbf{u})) \cap I(y, f(z, y, \mathbf{u}))$ hence positing $f(v, v, \mathbf{u}) = v$ is similarly consistent with the whole of (a),(b) and (c);

(d) $f(x, y, \mathbf{u}) \in I(x, f(y, y, \mathbf{u})) \cap I(x, f(v, y, \mathbf{u})) \cap I(x, f(z, y, \mathbf{u})) \cap$
$\cap\ I(y, f(x, x, \mathbf{u})) \cap I(y, f(x, v, \mathbf{u})) \cap I(y, f(x, z, \mathbf{u}))$ hence it must be the case that $f(x, y, \mathbf{u}) = x$ since by construction $I(x, f(v, y, \mathbf{u})) = I(x, x) = \{x\}$ (also notice that since by construction $x \in I(y, v)$ that value is certainly consistent with the whole of (a),(b),(c),(d) if $\{f(x, v, \mathbf{u}), f(x, z, \mathbf{u})\} \subseteq \{x, v\}$: so let us assume the latter inclusion as well);

(e) $f(v, x, \mathbf{u}) \in I(v, f(x, x, \mathbf{u})) \cap I(v, f(y, x, \mathbf{u})) \cap I(v, f(z, x, \mathbf{u})) \cap$
$\cap\, I(x, f(v, y, \mathbf{u})) \cap I(x, f(v, v, \mathbf{u})) \cap I(x, f(v, z, \mathbf{u}))$ hence $f(v, x, \mathbf{u}) = x$ since $I(x, f(v, y, \mathbf{u})) = I(x, x) = \{x\}$ (notice that that value is certainly consistent with the whole of (a),(b),(c)(d),(e) if

$\{f(y, x, \mathbf{u}), f(z, x, \mathbf{u})\} \subseteq \{x, y\}$ as well: then, let us also assume that inclusion);

(f) $f(y, v, \mathbf{u}) \in I(y, f(x, v, \mathbf{u})) \cap I(y, f(v, v, \mathbf{u})) \cap I(y, f(z, v, \mathbf{u})) \cap$
$\cap\ I(v, f(y, x, \mathbf{u})) \cap I(v, f(y, y, \mathbf{u})) \cap I(v, f(y, z, \mathbf{u}))$ (notice that, therefore, positing $f(y, v, \mathbf{u}) = f(x, v, \mathbf{u}) = f(z, v, \mathbf{u}) = v$ is consistent with (a),(b),(c),(d),(e),(f) as introduced above);

(g) $f(y, z, \mathbf{u}) \in I(y, f(x, z, \mathbf{u})) \cap I(y, f(v, z, \mathbf{u})) \cap I(y, f(z, z, \mathbf{u})) \cap$
$\cap\, I(z, f(y, x, \mathbf{u})) \cap I(z, f(y, y, \mathbf{u})) \cap I(z, f(y, v, \mathbf{u}))$ hence, $f(y, z, \mathbf{u}) = z$ and $f(x, z, \mathbf{u}) = v$ are jointly consistent with (a),(b),(c),(d),(e),(f),(g) since by assumption $z \in I(y, v)$.

(h) $f(v, z, \mathbf{u}) \in I(v, f(x, z, \mathbf{u})) \cap I(v, f(y, z, \mathbf{u})) \cap I(v, f(z, z, \mathbf{u})) \cap$
$\cap\ I(z, f(v, x, \mathbf{u})) \cap I(z, f(v, y, \mathbf{u})) \cap I(z, f(v, v, \mathbf{u}))$: observe that, since $v \in I(x, z)$, $f(v, z, \mathbf{u}) = v$ is indeed consistent with (a),(b),(c),(d),(e),(f),
(g),(h) as introduced above;

(i) $f(z, y, \mathbf{u}) \in I(z, f(x, y, \mathbf{u})) \cap I(z, f(y, y, \mathbf{u})) \cap I(z, f(v, y, \mathbf{u})) \cap$
$\cap\ I(y, f(z, x, \mathbf{u})) \cap I(y, f(z, v, \mathbf{u})) \cap I(y, f(z, z, \mathbf{u}))$ hence $f(z, y, \mathbf{u}) = y$ is consistent with (a),(b),(c),(d),(e),(f),(g),(h),(i)
since $y \in I(x, z) = I(z, f(x, y, \mathbf{u})) = I(z, f(v, y, \mathbf{u}))$;

(l) $f(z, v, \mathbf{u}) \in I(z, f(x, v, \mathbf{u})) \cap I(z, f(y, v, \mathbf{u})) \cap I(z, f(v, v, \mathbf{u})) \cap$

$\cap\, I(v, f(z,x,\mathbf{u})) \cap I(v, f(z,y,\mathbf{u})) \cap I(v, f(z,z,\mathbf{u}))$ hence in view of (e) $f(z,v,\mathbf{u}) = z$ and $f(z,x,\mathbf{u}) = y$ are jointly consistent with (a),(b),(c),(d),(e),

(f),(g),(h),(i),(l) since $z \in I(y,v) = I(v, f(z,x,\mathbf{u})) = I(v, f(z,y,\mathbf{u}))$;

Thus, we have just shown that there indeed exists a voting rule $f$ that satisfies all of the requirements (a)-(l) above, and is therefore $\mathcal{I}$-monotonic, while at the same time being such that for all $\mathbf{u} = u_{N\smallsetminus\{1,2\}} \in X^{N\smallsetminus\{1,2\}}$

$f(v,y,\mathbf{u}) = x$, and $f(z,z,\mathbf{u}) = z$.

Now, take any profile $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$ of $\mathcal{I}$-unimodal total preorders on $X$ such that $\succcurlyeq_1 = \succcurlyeq^*$and $\succcurlyeq_2 = \succcurlyeq'$.

Then, by construction, $top(\succcurlyeq_1) = v$, $top(\succcurlyeq_2) = y$, $z \succ_1 x$, $z \succ_2 x$, $f(v, y, (top(\succcurlyeq_i)_{i\in N\smallsetminus\{1,2\}})) = x$, and $f(z, z, (top(\succcurlyeq_i)_{i\in N\smallsetminus\{1,2\}})) = z$. It follows that $f$ is *not* coalitionally strategy-proof, yet in view of Proposition 1 $f$ is (individually) strategy-proof, a statement that contradicts our general hypothesis.

*Proof of Corollary 2.* (i) It follows immediately from Theorem 1 and the proof of point (vi) of the present Corollary as reported below;

(ii) Recall that for any graph $G = (X, E)$, and any $x, y, z \in X$, $x \in I^G(y,z)$ if and only if

$$d^G(y,z) = d^G(y,x) + d^G(x,z).$$

Now, suppose that $x \neq y$, $d^G(y,z) = d^G(y,x) + d^G(x,z)$ and $d^G(x,v) = d^G(x,y) + d^G(y,v)$.

Since $x \in I^G(y,z)$ and $y \in I^G(x,v)$, $x$ lies on the unique path joining $y$ and $z$, and $y$ lies on the unique path joining $x$ and $v$. Hence there exists a path joining $z$ first to $x$, then $x$ to $y$, and finally $y$ to $v$. Since $G$ is a tree, that is in fact the unique path joining $z$ to $v$. Thus, by construction,

$d^G(z,v) = d^G(z,x) + d^G(x,v)$, and $x \in I^G(z,v)$ i.e. Interval Anti-Exchange holds, and the thesis follows immediately from Theorem 1.

(iii) Indeed, suppose that $x \neq y$, $x \in I^G(y,z)$ and $y \in I^G(x,v)$. Then, since $G$ is a clique, it follows that $x = z$ and $y = v$. But then, $I^G(z,v) = I^G(x,y)$ hence clearly $x \in I^G(z,v)$ and Interval Anti-Exchange holds. Therefore, the thesis follows immediately from Theorem 1.

(iv) Let $\mathcal{X} = (X, \leqslant, 0, 1)$ be a bounded distributive lattice that is not a chain. Thus, there exist $x, y \in X$ such that $x \nleqslant y$ and $y \nleqslant x$. Hence, there also exist $x \wedge y \notin \{x,y\}$, $x \vee y \notin \{x,y\}$: clearly, by construction, $x \wedge y < x \vee y$. Now, consider $Y = \{x, y, x\wedge y, x \vee y\}$, $I^m(a,b)$ with $a, b \in Y$: indeed, by construction, $I^m(x,y) \cap I^m(x\wedge y, x\vee y) \supseteq Y$ while $I^m(a,b) \cap Y = \{a,b\}$ for any $a, b \in Y$ such that $\{a,b\} \notin \{\{x,y\}, \{x\wedge y, x \vee y\}\}$. Then, $I^m(x,y) \cap \{x\wedge y, x\vee y\} = \{x \wedge y, x \vee y\}$, $I^m(x\wedge y, x \vee y) \cap \{x,y\} = \{x,y\}$, $I^m(y, x\wedge y) \cap \{x \vee y, x\} = I^m(x, x\wedge y) \cap \{x\vee y, y\} = \varnothing$ whence Minimal Anti-Exchange does not hold. Therefore, the thesis follows immediately from Theorem 2.

(v) It is well-known that Euclidean convex spaces are a special subclass of linear geometries as defined above in the text, and therefore satisfy Interval Anti-Exchange (see Coppel (1998)) . A direct proof of that claim is provided here just for the sake of completeness. Let us then suppose without loss of generality that $x \neq y$, $x \in I^E(y,v)$, and $y \in I^E(x,z)$ i.e. there exist real numbers $\lambda_1, \lambda_2 \in (0,1)$ such that $x = \lambda_1 y + (1-\lambda_1)v$ and $y = \lambda_2 x + (1-\lambda_2)z$ (if $\{\lambda_1,\lambda_2\} \cap \{0,1\} \neq \varnothing$ the thesis follows immediately). Then, $x = \lambda_1(\lambda_2 x + (1-\lambda_2)z) + (1-\lambda_1)v$ i.e.

$x = \frac{\lambda_1(1-\lambda_2)}{1-\lambda_1\lambda_2} z + \frac{1-\lambda_1}{1-\lambda_1\lambda_2} v$

where $1 \geq \frac{\lambda_1(1-\lambda_2)}{1-\lambda_1\lambda_2} \geq 0$, $1 \geq \frac{1-\lambda_1}{1-\lambda_1\lambda_2} \geq 0$ and $\frac{\lambda_1(1-\lambda_2)}{1-\lambda_1\lambda_2} + \frac{1-\lambda_1}{1-\lambda_1\lambda_2} = 1$ hence $x \in I^E(v,z)$ and Interval Anti-Exchange holds. Then, the thesis follows immediately from Theorem 1.

(vi) Let $x \neq y$, $x \in I(\leqslant)(y,v)$, $y \in I(\leqslant)(x,z)$. Hence, from $x \neq y$, $x \in I(\leqslant)(y,v)$ it follows that either $x = v$, or $y < x < v$, or else $v < x < y$; moreover, from $x \neq y$, $y \in I(\leqslant)(x,z)$ it follows that either $y = z$, or $x < y < z$, or else $z < y < x$. If $x = v$ or $y = z$ then clearly $x \in I(\leqslant)(v,z)$, while neither ($y < x < v$ and $x < y < z$) nor ($v < x < y$ and $z < y < x$) can possibly hold. If both $y < x < v$ and $z < y < x$ hold then clearly $z < y < x < v$ hence $x \in I(\leqslant)(v,z)$, and if both $v < x < y$ and $x < y < z$ hold clearly $v < x < y < z$ hence $x \in I(\leqslant)(v,z)$. Thus, in any case $x \in I(\leqslant)(v,z)$, Interval Anti-Exchange holds, and the thesis follows immediately from Theorem 1.

(vii) It follows from the observation that a non-trivial Hamming graph must include a 'square' $\{x,y,v,z\}$ with $I^{G_H}(x,z) = I^{G_H}(y,v) \supseteq \{x,y,v,z\}$ and $I^{G_H}(a,b) \cap \{c,d\} = \varnothing$ for any $a,b,c,d$ such that

$\{a,b,c,d\} = \{x,y,v,z\}$ and $\{a,b\} \notin \{\{x,z\},\{y,v\}\}$: to such a 'square' the same argument provided in the proof of point (iv) above applies, whence the thesis follows from Theorem 2.

(viii) First, let $X = Y \cup Z$ where $Y$ denotes a chain and $Z$ denote the vertex set of a clique, $G = (X,E)$

with $E = \{\{u,v\} : (u,v) \in Y^2 \cup Z^2$ or $\{u,v\} = \{y^*,z^*\}\}$, $z^* \in Z$ is the (unique) 'hub' that lies on any path connecting an arbitrary element of $Y$ to an arbitrary element of $Z$, and $y^*$ is the unique element of $Y$ which is adjacent to $z^*$. Now, observe that by construction there is precisely one shortest path joining any two vertices/outcomes (and $z^*$,$y^*$ both lie on the path joining one vertex in $Y$ to one vertex in $Z$). Therefore the same argument previously used under point (ii) for trees applies, Interval Anti-Exchange holds, and the thesis follows from Theorem 1.

## References


[1] Bandelt H.J., J.P. Barthélemy (1984): Medians in median graphs, *Discrete Applied Mathematics* 8, 131-142.

[2] Barberà S., D. Berga, B. Moreno (2010): Individual versus group strategy-proofness: When do they coincide?, *Journal of Economic Theory* 145, 1648-1674.

[3] Chvátal V., D. Rautenbach, P.M. Schäfer (2011): Finite Sholander trees, trees, and their betweenness, arXiv:1101.2957v1 [math.CO]

[4] Coppel W.A. (1998): *Foundations of convex geometry.* Cambridge University Press, Cambridge UK.

[5] Danilov V.I. (1994): The structure of non-manipulable social choice rules on a tree, *Mathematical Social Sciences* 27, 123-131.

[6] Danilov V.I., A.I. Sotskov (2002): *Social choice mechanisms.* Springer, Berlin.

[7] Eppstein D., J.-C. Falmagne, S. Ovchinnikov (2008): *Media theory.* Springer, Berlin Heidelberg.

[8] Le Breton M., V. Zaporozhets (2009): On the equivalence of coalitional and individual strategy-proofness properties, *Social Choice and Welfare* 33, 287-309.

[9] Moulin H. (1980): On strategy-proofness and single peakedness, *Public Choice* 35, 437-455.

[10] Mulder H.M. (1980): *The interval function of a graph.* Mathematical Centre Tracts 132, Amsterdam.

[11] Mulder H.M., L. Nebeský (2009): Axiomatic characterization of the interval function of a graph, *European Journal of Combinatorics* 30, 1172-1185.

[12] Nebeský L. (2007): The interval function of a connected graph and road systems, *Discrete Mathematics* 307, 2067-2073.

[13] Nehring K., C. Puppe (2007 (a)): The structure of strategy-proof social choice - Part I: General characterization and possibility results on median spaces, *Journal of Economic Theory* 135, 269-305.

[14] Nehring K., C. Puppe (2007 (b)): Efficient and strategy-proof voting rules: a characterization, *Games and Economic Behavior,* 59, 132-153.

[15] Peters H., H. van der Stel, T. Storcken (1992): Pareto optimality, anonymity, and strategy-proofness in location problems, *International Journal of Game Theory* 21, 221-235.

[16] Prenowitz W., J. Jantosciak (1979): *Join geometries. A theory of convex sets and linear geometry.* Springer, New York.

[17] Savaglio E., S. Vannucci (2012): Strategy-proofness and unimodality in bounded distributive lattices, Working Paper DEPS 642, June 2012, Siena.

[18] Schummer J., R.V. Vohra (2002): Strategy-proof location on a network, *Journal of Economic Theory* 104, 405-428.

[19] Sholander M. (1952): Trees, lattices, order, and betweenness, *Proceedings of the American Mathematical Society* 3, 369-381.

[20] Sholander M. (1954): Medians and betweenness, *Proceedings of the American Mathematical Society* 5, 801-807.

[21] van de Vel M.L.J. (1993): *Theory of convex structures.* North Holland, Amsterdam.

[22] Vannucci S. (2013): On two-valued nonsovereign strategy-proof voting rules, Working Paper DEPS 672, April 2013, Siena.

## 6. Appendix B. Examples on interval spaces

**Example 1. A convex idempotent interval space that satisfies Interval Anti-Exchange but is not a convex geometry**

Take $\mathcal{I} = (X, I)$ with $X = \{x, u, v, y, z\}$, $\#X = 5$, and $I$ as defined by the following rule: $I(x, u) = \{x, u, y\}$, $I(y, v) = \{y, v, z\}$, and $I(a, b) = \{a, b\}$ otherwise. Observe that, by construction, $\mathcal{I}$ is a convex and idempotent interval space. It is also quickly established that $\mathcal{I}$ does satisfy Interval Anti-Exchange: indeed, it is immediately seen that $a \in I(b, c)$ and $b \in I(a, d)$ with $a \neq b$ only hold in $\mathcal{I}$ if one of the following clauses is satisfied: (1) $a = c$, (2) $a = y$,$\{b, c\} = \{x, u\}$, and $b = d$, (3) $a = z$ , $\{b, c\} = \{y, v\}$, and $b = d$. Now, if (1) holds then $a \in I(c, d)$ by Extension of $I$ (i.e. by definition of interval space). If (2) holds, $\{c, d\} = \{x, u\}$ hence $a = y \in I(x, u) = I(c, d)$ by construction. If (3) holds, then $\{c, d\} = \{b, c\}$ hence $a \in I(b, c) = I(c, d)$ by hypothesis. Therefore, $\mathcal{I}$ satisfies Interval Anti-Exchange as claimed. However, it can be quite easily shown that $\mathcal{I}$ fails to satisfy Peano Convexity as defined above (see e.g. Coppel (1998), chpt.2), hence it is *not a convex geometry.* It should also be noticed that $\mathcal{I}$ is *not median* (since e.g. $I(x, y) \cap I(y, z) \cap I(x, z) = \varnothing$).

**Example 2. A convex geometry that satisfies Interval Anti-Exchange but is not a linear geometry**

Take an idempotent interval space $\mathcal{I} = (X, I)$ with $X = \{x, y, v, z\}$, $\#X = 4$, and $I$ as defined by the following rule: $I(x, z) = I(z, x) = X$ and $I(a, b) = \{a, b\}$ for any $a, b \in X$ such that $\{a, b_1\} \neq \{x, z\}$ (see Coppel (1998)). Indeed, $\mathcal{I}$ is the interval space induced by the *quasi-complete graph* with vertex set $X$ obtained by removing edge $xz$ from the complete graph on $X$. It is readily checked that interval space $\mathcal{I}$ is convex (and, in fact, a convex geometry: to confirm the latter statement, suppose that $c \in I(a, b_1)$, $d \in I(c, b_2)$; if $\{a, b_1\} \neq \{x, z\}$ then $c \in \{a, b_1\}$, whence $c \in I(a, b_1)$; if $\{a, b_1\} = \{x, z\}$ then $c \in I(a, b_1) = X$ hence in any case Peano Convexity holds). Interval space $\mathcal{I}$ also satisfies the no-branchpoint property: if $c \notin I(a, b)$ and $b \notin I(a, c)$ then clearly $b \neq c$ and $\{a, b\} \neq \{x, z\} \neq \{a, c\}$ hence $I(a, b) \cap I(a, c) = \{a\}$). However, $\mathcal{I}$ cannot possibly be induced by any linear geometry or tree since $I$ does not satisfy the additivity property: e.g. $y \in I(x, z)$, while $I(x, y) \cup I(y, z) = \{x, y, z\} \neq X = I(x, z)$. Also, $\mathcal{I}$ is *not* a *median interval space (notice that e.g.* $I(x, y) \cap I(x, v) \cap I(y, v) = \varnothing$). On the other hand, it is straightforward to verify that $\mathcal{I}$ satisfies IAE: suppose $a \neq b$, $a \in I(b, c)$, $b \in I(a, d)$. If $\{a, d\} \neq \{x, z\}$ then $a \in I(b, c)$ entails $a = c$: therefore $a \in I(a, d) = I(c, d)$. If $\{b, c\} \neq \{x, z\}$, then $b \in I(a, d)$ entails $b = d$ hence $a \in I(b, c) = I(c, d)$. If on the contrary $\{a, d\} = \{x, z\} = \{b, c\}$ then it must be the case that $a = c$ and $b = d$ whence $a \in I(a, b) = I(c, d)$. In any case $a \in I(c, d)$ and IAE is therefore satisfied.

## 7. Appendix C. Related literature

As repeatedly mentioned above, Moulin (1980) provides the first characterization of all strategy-proof voting rules (and of a large subclass of strategy-proof social choice functions) on a full unimodal domain addressing the case of bounded chains on the extended real line, and proves equivalence between simple and coalitional strategy-proofness in that setting. Given the seminal nature of that work and its quite specific focus, its proof techniques, quite predictably, take full advantage of several properties of medians of finite sets of points on a chain that cannot be exported to more general domains (see Savaglio, Vannucci (2012) for a discussion of that issue).

On the contrary, the extension of Moulin's characterization to full unimodal domains of linear orders on trees due to Danilov (1994), while of course relying on several specific properties of trees and their interval spaces, adopts proof techniques that are more easily adapted to other settings. Accordingly, Savaglio, Vannucci (2012) proceeds to extend some results of Danilov (1994) -including the application of some of its proof techniques- to bounded distributive lattices, showing however that equivalence of simple and coalitional strategy-proofness of voting rules for full unimodal domains fails in that setting. As mentioned above, the present work itself relies to a significant extent on extensions of both results and proof techniques of Danilov (1994) to more general interval spaces.

It should be emphasized that the results of the present paper clarify an important point concerning the different role respectively played by Median Property and Interval Anti-Exchange in both Moulin's and Danilov's theorems, namely: while the median property of the underlying interval space is obviously required for their median-based characterizations of the class of strategy-proof voting rules on full unimodal domains, *it is precisely Interval Anti-Exchange that guarantees the equivalence of simple and coalitional strategy-proofness on those domains.*

Some valuable works have been recently devoted to the study of strategy-proof voting rules and social choice functions on 'single peaked' domains in finite interval spaces induced by certain 'property spaces' ( see e.g. Nehring, Puppe (2007 (a), 2007 (b))) and to general conditions for equivalence of simple and coalitional strategy-proofness of voting rules and social choice functions (see Le Breton, Zaporozhets (2009), Barberà, Berga, Moreno (2010)): the results provided by those works, however, turn out to have virtually no bearing on the main issues addressed in the present paper concerning voting rules for *full unimodal domains.*

In order to explain and motivate the latter statement, a quite extensive discussion including a long detour through somewhat technical details is unfortunately required.

Indeed, Nehring, Puppe (2007 (b)) provides a characterization of efficient strategy-proof voting rules on *'rich'* domains of '*generalized single peaked*' profiles of *linear* preference orders on *finite* outcome sets as classified according to a shared *property space.* A *property space* is a *list of (extensionally defined) basic properties* $\mathbf{P} = \{P_1, P_2, ..., P_k\} \subseteq$

$\mathcal{P}(X)$ that is only required to be *contradiction-free* (the empty set is not a basic property), *closed with respect to negation* or complementation (a property $P$ is on the list if and only if its negation or complement $P^c = X \setminus P$ is also on the list), and *outcome-separating* (for each pair of distinct outcomes there exists a basic property that is satisfied by one and only one of them). Two basic properties $P_1, P_2 \subseteq X$ are said to be *independent* if and only if each one of the pairs $(P_1, P_2)$, $(P_1, P_2^c)$, $(P_2, P_1^c)$, $(P_1^c, P_2^c)$ is contradiction-free (i.e. $\varnothing \notin \{P_1 \cap P_2, P_1 \cap P_2^c, P_2 \cap P_1^c, P_1^c \cap P_2^c\}$). The *dimension* of a property space is the maximum size of a set of pairwise independent properties of that space. A ternary *betweenness relation* on outcomes $B_{\mathbf{P}} \subseteq X^3$ is defined by means of the list of basic properties: outcome $z$ lies *between* outcomes $x$ and $y$, written $(x, z, y) \in B_{\mathbf{P}}$, if and only if $z$ satisfies each basic property that is shared by $x$ and $y$. Subsets of outcomes are $\mathbf{P}$-*convex* if they are the intersection of a set of basic properties, the $\mathbf{P}$-*convex hull* of a subset of outcomes is of course its smallest convex superset, and the $\mathbf{P}$-induced *interval* (or *segment*) $I_{\mathbf{P}}(x, y)$ of any two outcomes $x, y \in X$ is the $\mathbf{P}$-convex hull of $\{x, y\}$. Hence, as it is easily checked, for any $x, y, z \in X$ , $z \in I_{\mathbf{P}}(x, y)$ if and only if $(x, z, y) \in B_{\mathbf{P}}$: clearly, by construction, $\{x, y\} \subseteq I_{\mathbf{P}}(x, y)$ and $I_{\mathbf{P}}(x, y) = I_{\mathbf{P}}(y, x)$ for all $x, y \in X$.

Therefore, under that approach any set of basic properties induces an *interval space* $(X, I_{\mathbf{P}})$ on the outcome set (it is easily checked that such an interval space is indeed convex and idempotent[21]): *voters are assumed to agree not directly on the compromise-structure of outcomes as such, but rather on a finite set of basic properties or classification criteria for outcomes*, that in turn induce a compromise-structure consisting in a ternary betweenness relation or equivalently in an interval space on the outcome set. *'Generalized single peaked'* linear orders on the outcome set $X$ are defined in the foregoing setting as follows: a linear order i.e. an antisymmetric total preorder $\succcurlyeq$ on $X$ is said by Nehring and Puppe to be *generalized single peaked* (or *locally strictly unimodal* according to the label we would rather suggest in order to ease a careful comparison with the notion of unimodality as defined in the present paper) if and only if the following two conditions hold:

$U-(i)$ : there exists a *unique maximum* of $\succcurlyeq$ in $X$, its *top* outcome -denoted $top(\succcurlyeq)$- and

$U^*-(ii)$ : for all $x, y, z \in X$, if $x = top(\succcurlyeq)$, $y \neq z$ and $y \in I_{\mathbf{P}}(x, z)$ (or equivalently $(x, y, z) \in B_{\mathbf{P}}$) then $y \succ z$ (i.e. $y \succcurlyeq z$ and not $z \succcurlyeq y$) or, equivalently, $\{u \in X : y \succ u\} \cap \{x, z\} \neq \varnothing$.

[21] And conversely: it is easily checked that every finite convex and idempotent interval space gives rise to a property space with basic properties 'lying in the interval of $x$ and $y$' (for any $x, y \in X$). Therefore, for *finite* outcome sets, interval space-representations and property space-representations of the compromise-structure shared by voters are *essentially equivalent*. For infinite outcome sets, however, Nehring-Puppe's property spaces are too weak to induce a well-defined interval function hence a well-defined interval space, and *should therefore be supplemented with an intersection-closure requirement* to ensure that convex hulls - hence in particular intervals- be always well-defined.

Notice that the same definition applies to arbitrary i.e. possibly not antisymmetric total preorders. The collection of all $N$-profiles of locally strictly unimodal total preorders (linear orders) on $X$ is denoted $(U^*_{\mathbf{P}})^N$ ($(U^*_{\mathbf{P},L})^N$, respectively). It should also be emphasized that local strict unimodality and unimodality embody two remarkably different attitudes towards the compromise-structure of the outcome set as represented by its interval function. Indeed, while unimodality requires a global agreement on what constitutes an effective compromise between any two outcomes with no pretence to extend the agreement to evaluations concerning degrees of proximity of compromises to their extrema, local strict unimodality on the contrary imposes a local role for metric/proximity considerations in the neighbourhood of the top outcome while at the same time depriving the compromise-structure of any impact on voter preferences away from their top outcomes.

The main result of Nehring, Puppe (2007 (b)) concerning the characterization of efficient strategy-proof social choice functions on locally strictly unimodal domains implies in particular that if the outcome space includes the Boolean cube and the property space $\mathbf{P}$ consists of the properties 'lying to the left (right) of point $x$ on dimension $h$' (for each outcome $x$ and dimension $h$) then the median voting rule is *inefficient* on any *rich* subdomain of the full locally strictly unimodal (or generalized single peaked) domain $(U^*_{\mathbf{P},L})^N$ [22] .

Moreover, in view of Nehring, Puppe (2007 (a)) , the (extended) median voting rule is strategy-proof on $(U^*_{\mathbf{P},L})^N$. It immediately follows that -being both inefficient and sovereign or surjective- the (extended) median voting rule is strategy-proof but *not* coalitionally strategy-proof on $(U^*_{\mathbf{P},L})^N$. Hence, it also follows that equivalence of simple and coalitional strategy-proofness fails *on the full locally strictly unimodal* (or generalized single peaked) *domain of linear orders* $(U^*_{\mathbf{P},L})^N$.

That result is an early counterpart of a similar result subsequently established by Savaglio, Vannucci (2012) for *full unimodal* domains. Notice however that the Nehring-Puppe proposition only covers *finite* Boolean hypercubes of dimension $k \geq 3$, while the latter result concerning full unimodal domains also applies both to the *Boolean square* and to *infinite* Boolean lattices. Moreover, it turns out that those two results are in fact *independent even for finite* Boolean hypercubes of dimension $k \geq 3$. That is so because in general a locally strictly unimodal linear order need not be unimodal and conversely, an unimodal total preorder need not be locally strictly unimodal. To check that point, consider e.g. the Boolean cube $2^3 = \{(x, y, z) : x, y, z \in \{0, 1\}\}$ with its canonical interval space $(X, I^m)$, and linear order

$$(1,1,1) \succ (1,1,0) \succ (1,0,1) \succ (0,1,1) \succ (1,0,0) \succ (0,1,0) \succ (0,0,1) \succ (0,0,0)$$

[22] A locally strictly unimodal domain of linear orders $D \subseteq U^*_{\mathbf{P},L}$ is said to be *rich* in Nehring, Puppe (2007 (b)) if (i) for any two outcomes $x, y$ with no other outcomes lying between them there exists a linear order in $D$ having $x$ and $y$ as its first and second best outcomes, and (ii) for any triple of outcomes $x, y, z$ such that $y$ does not lie between $x$ and $z$ there exists a linear order $\succcurlyeq$ in $D$ having $x$ as its top outcome and such that $z \succ y$.)

which is clearly locally strictly unimodal but not unimodal because e.g. $(0,1,0) \in I^m((1,1,0),(0,1,1))$ but $(1,1,0) \succ (0,1,1) \succ (0,1,0)$),

and total preorder

$(1,1,1) \succ (1,1,0) \sim (1,0,1) \sim (0,1,1) \sim (1,0,0) \sim (0,1,0) \sim (0,0,1) \sim (0,0,0)$

which is clearly unimodal but not locally strictly unimodal because e.g. $(1,1,0)$ lies between $(1,1,1)$ and $(1,0,0)$ but $(1,1,0) \sim (1,0,0)$.

Total preorders $(1,1) \succ (1,0) \succ (0,1) \succ (0,0)$ and $(1,1) \succ (1,0) \sim (0,1) \sim (0,0)$ establish the same point for the Boolean square.

Moreover, it can be easily shown that *whenever the outcome space includes the Boolean square no linear order is unimodal* (because for any choice of the unique bottom outcome $x$ on the Boolean square there will be $y, v \in 2^2 \smallsetminus \{x\}$ such that $x \in I^m(y,v)$, which contradicts unimodality) . Thus, the subdomain consisting of unimodal linear orders that are also locally strictly unimodal is *empty* hence in particular *is not rich.*[23] It follows that the Nehring-Puppe proposition concerning inefficiency of the (extended) median voting rule discussed above *does not apply at all to domains of unimodal total preorders* and *cannot therefore be used to infer coalitional manipulability of the (extended) median rule on the full unimodal domain in Boolean k-hypercubes* with $k \geq 2$ including of course the *Boolean square.*

Let us now turn to contributions concerning conditions that ensure *equivalence of simple and coalitional strategy-proofness of voting rules* and social choice functions on *general domains* of preference profiles.

Le Breton, Zaporozhets (2009) identifies a general *richness* condition on a set $D$ of total preorders on an outcome set $X$ which ensures that any social choice function $f : D^N \to X$ that is *regular* (namely such that *every outcome in $f(D^N)$ is the unique top outcome of some total preorder in $D$*) *is strategy-proof if and only if it is also coalitionally strategy-proof.*[24] That richness condition is defined as follows: a set $D$ of total preorders on $X$ is said to be *rich* with respect to *strictly monotonic transformations* (or *SMT-rich)* if for every total preorder $\succcurlyeq$ in $D$ and every pair of outcomes $x, y \in X$ such that $y \succ x$ and $y = top(\succcurlyeq')$ for some total preorder $\succcurlyeq'$ in $D$ there exists a a total preorder $\succcurlyeq^*$ in $D$ such that $y = top(\succcurlyeq^*)$ and $x \succ^* z$ for all $z \neq x$ with $x \succcurlyeq z$.

---

[23]In fact, no locally strictly unimodal total preorder defined on the Boolean square $\mathbf{2}^2$ is unimodal. That is so because it can be easily checked that any locally strictly unimodal total preorder $\succcurlyeq$ on $\mathbf{2}^2$ that is not a linear order must be of the following type:

$a \succ b \sim c \succ d$

where $\{a,d\}$ and $\{b,c\}$ denote pairs of 'opposite points' on the square.

But then, $\succcurlyeq$ is not unimodal since $d \in I^m(b,c)$.

[24]Note that social choice functions whose outcomes only depend on profiles of top outcomes essentially amount to voting rules which inherit the strategy-proofness properties of the former: thus results of this type concerning social choice functions are *also relevant to voting rules.*

Le Breton, Zaporozhets (2009) only briefly addresses the case of 'single peaked' domains, claiming essentially that 'single peaked domains' large enough to support surjective social choice functions are SMT-rich: but that paper clearly refers to (what we denoted as) locally strictly unimodal preference profiles in a chain. When moving to general (full) unimodal domains, the verdict concerning SMT-richness is in fact much more blurred.

To begin with, observe that the social choice functions associated to voting rules on *full* unimodal domains clearly satisfy by definition the general 'regularity' clause of the Le Breton-Zaporozhets proposition. However, the collection $U_{\mathcal{I}}$ of all unimodal total preorders of a convex and idempotent (or even median) interval space $\mathcal{I} = (X, I)$ may be or may be *not* SMT-rich, depending on the choice of $\mathcal{I}$. For instance, if $\mathcal{I} = (X, I)$ is the interval space of a clique (which, recall, amounts to the 'universal' domain of all total preorders with a unique maximum) then $U_{\mathcal{I}}$ is clearly SMT-rich. On the contrary, if $\mathcal{I} = (X, I)$ is the interval space of a Boolean hypercube then $U_{\mathcal{I}}$ is *not* SMT-rich. To see that, consider for the sake of simplicity the Boolean square $2^2 = \{(x, y) : x, y \in \{0, 1\}\}$ and its canonical interval space $\mathcal{I}$ , unimodal total preorder $\succcurlyeq$ such that $(1, 0) \succ (1, 1) \sim (0, 1) \sim (0, 0)$, and outcomes $(1, 0)$, $(0, 1)$. SMT-richness of $U_{\mathcal{I}}$ would imply existence of a unimodal total preorder $\succcurlyeq'$in $U_{\mathcal{I}}$ such that $(1, 0) \succ' (0, 1) \succ' (1, 1)$ and $(1, 0) \succ' (0, 1) \succ' (0, 0)$. Since by construction $(1, 0)$, $(0, 1)$ is a pair of 'opposite points' $I((1, 0), (0, 1)) = 2^2$ : it follows that $\succcurlyeq'$ is not unimodal since e.g. $L(\succcurlyeq', (1, 1)) \cap \{(1, 0), (0, 1)\} = \varnothing$ whence SMT-richness fails here. Concerning SMT-richness of (full) unimodal domains in chains or trees, additional clarifications may be obtained by the discussion of Barberà, Berga, Moreno (2010) to follow.

Indeed, Barberà, Berga, Moreno (2010) also studies general sufficient conditions of a combinatorial nature ensuring equivalence of simple and coalitional strategy-proofness of social choice functions and voting rules on arbitrary preference domains. In particular, two such sufficient conditions for single profiles are identified and denoted as *Sequential Inclusion,* a requirement that applies to single preference profiles (i.e. an intraprofile condition), and *Indirect Sequential Inclusion* (an interprofile existence or 'richness' condition based upon Sequential Inclusion). Specifically, for each preference profile $(\succcurlyeq_i)_{i \in N}$ *Sequential Inclusion* relies on a family of binary relations $\succcurlyeq (S((\succcurlyeq_i)_{i \in N}, y, z))$ as parameterized by ordered pairs $(y, z)$ of outcomes and defined on $S((\succcurlyeq_i)_{i \in N}, y, z)$, the set of voters that strictly prefer $y$ to $z$ at $(\succcurlyeq_i)_{i \in N}$: in particular, voter pair $(i, j)$ is in $\succcurlyeq (S((\succcurlyeq_i)_{i \in N}, y, z))$ if and only if $i$ and $j$ are in $S((\succcurlyeq_i)_{i \in N}, y, z)$ and $L(z, \succcurlyeq_i) \subseteq L^*(y, \succcurlyeq_j)$. Of course any such $\succcurlyeq (S((\succcurlyeq_i)_{i \in N}, y, z))$ is reflexive: Sequential Inclusion requires that all of them be also *connected and acyclic. Indirect Sequential Inclusion* is satisfied by a profile $(\succcurlyeq_i)_{i \in N}$ if *either* $(\succcurlyeq_i)_{i \in N}$ itself satisfy Sequential Inclusion *or* for each pair $(y, z)$ of outcomes there exists a profile $(\succcurlyeq'_i : i \in S((\succcurlyeq_i)_{i \in N}, y, z))$ such that: (i) $y \succ'_i z$ for each $i \in S((\succcurlyeq_i)_{i \in N}, y, z)$, (ii) $z \succ'_i x$ for each $i \in S((\succcurlyeq_i)_{i \in N}, y, z)$ and each outcome $x \neq z$ such that $z \succcurlyeq_i x$, and (iii) $\succcurlyeq (S((\succcurlyeq'_i : i \in S((\succcurlyeq_i)_{i \in N}, y, z)))$ is connected and acyclic.

A preference domain is then said to satisfy *Sequential Inclusion (Indirect Sequential Inclusion)* if each preference profile in that domain does satisfy it. Moreover, in Barberà, Berga, Moreno (2010) it is also shown that *if* the outcome set and the preference domain $D^N$ of a social choice function meet a mild *consistency clause* (i.e. they may also be the outcome set and preference domain of a surjective and *regular* social choice function according to the definition by Le Breton and Zaporozhets as reported above) *then* $D^N$ is SMT-rich only if it also satisfies Indirect Sequential Inclusion. Now, observe that our full unimodal domains obviously satisfy that consistency clause. Hence, when it comes to full unimodal domains $U_{\mathcal{I}}^N$ as discussed in the present paper, Indirect Sequential Inclusion qualifies as a more general sufficient condition than SMT-richness for equivalence of simple and coalitional strategy-proofness.

Under the label 'single peaked domains' Barberà, Berga, Moreno (2010) addresses the case of locally strictly unimodal domains in a chain[25], and points out that such domains satisfy Sequential Inclusion and therefore support equivalence of simple and coalitional strategy-proofness.

Thus, (Indirect) Sequential Inclusion and related properties seem to work quite well to assess equivalence of simple and coalitional strategy-proofness in (at least some) locally strictly unimodal domains.

However, unimodal profiles of total preorders in a chain may well violate Sequential Inclusion, as established by the following example.

Let $\leqslant$ be a linear order on an outcome set $X = \{x, y, z, w\}$ such that $x < y < z < w$, and $N = \{1, 2\}$. Next, consider total preorders $\succcurlyeq_1$ and $\succcurlyeq_2$ on $X$ defined as follows (using a most conventional notation, with $\succ_i$ and $\sim_i$ denoting of course the asymmetric and symmetric components of $\succcurlyeq_i$, $i = 1, 2$):

$x \succ_1 y \succ_1 w \sim_1 z$ and $z \succ_2 y \succ_2 w \sim_2 x$.

Notice that *unimodality* of $\succcurlyeq_1$ on $(X, I(\leqslant))$ holds because $I(\leqslant)(x, y) = \{x, y\}$ hence $I(\leqslant)(x, y) \cap \{w, z\} = \varnothing$ (and the second unimodality clause is also trivially satisfied), and *unimodality* of $\succcurlyeq_2$ on $(X, \leqslant)$ holds because $I(\leqslant)(z, y) = \{z, y\}$ hence $I(\leqslant)(z, y) \cap \{w, x\} = \varnothing$ (and the second unimodality clause is trivially satisfied, again).

Now, consider $S((R_1, R_2), (y, w))$ and $\succeq (S((R_1, R_2), (y, w)))$ as defined above. Clearly, $S((R_1, R_2), (y, w)) := \{i \in \{1, 2\} : w \in L^*(R_i, y)\} = \{1, 2\}$.

Moreover, $z \in L(R_1, w) \setminus L^*(R_2, y)$, and $x \in L(R_2, w) \setminus L^*(R_1, y)$.

Hence, by definition, neither $1 \succeq (S((R_1, R_2), (y, w)))2$

---

[25]Notice that if the underlying interval space is the interval space of a chain then locally strict unimodality of a total preorder $\succcurlyeq$ is equivalent to *strict unimodality* defined as follows:

$U - (i)$ there exists a *unique maximum* of $\succcurlyeq$ in $X$, its *top* outcome -denoted $top(\succcurlyeq)$- and

$U^{**} - (ii)$ : for all $x, y, z \in X$, if $x \neq y \neq z$ and $y \in I(x, z)$ then $\{u \in X : y \succ u\} \cap \{x, z\} \neq \varnothing$.

Thus, locally strict unimodality (or, equivalently, strict unimodality) in a chain is essentially related to a path-length-based *metric* that is implicitly defined on the outcome set.

nor $2 \succeq (S((R_1, R_2), (y, w)))1$. Therefore, $\succeq (S((R_1, R_2), (y, w)))$ is *not connected* : it follows that profile $R_N = (R_1, R_2)$ -while being unimodal on chain $(X, \leqslant)$ - *does not satisfy Sequential Inclusion*.

It can also be easily checked by the reader that the foregoing profile *does not satisfy Indirect Sequential Inclusion either*, because the required strictly monotonic transformation $\succcurlyeq_1'$of unimodal total preorder $\succcurlyeq_1$ should be such that $y \succ_1' w \succ_1' z$ hence *not* unimodal on $(X, I(\leqslant))$ since $z \in I(\leqslant)(y, w)$. Furthermore, consider the same profile with $X = 2^2$, $x = (1, 1)$, $y = (1, 0)$, $w = (0, 1)$, $z = (0, 0)$ i.e. the Boolean square as endowed with its canonical interval space $(X, I^m)$ that has been repeatedly considered in the present paper. It can be shown (an exercise left to the reader) that the given profile is unimodal in the Boolean square but does not satisfy Indirect Sequential Inclusion because e.g. the required strictly monotonic transformations of $\succcurlyeq_1$ would necessarily end up again in a total preorder $\succcurlyeq_1'$ such that $y \succ_1' w \succ_1' z$ while $z \in I^m(y, w)$ hence *not* unimodal. It follows that, as previously claimed in the Introduction, the results included in Barberà, Berga, Moreno (2010) are in fact quite inconclusive about equivalence of simple and coalitional strategy-proofness on full unimodal domains in Boolean hypercubes and even in chains (where equivalence is in fact well-established thanks to Moulin (1980)).

Therefore, Indirect Sequential Inclusion and related properties definitely fail in the assessment of simple and coalitional strategy-proofness properties for voting rules on *full unimodal domains* as considered in the present paper. That apparent weakness of Indirect Sequential Inclusion should be contrasted with the remarkable effectiveness of Interval Anti-Exchange in the analysis of coalitional strategy-proofness properties on full unimodal domains.

Indeed, a general comment on SMT-richness, Indirect Sequential Inclusion and Interval Anti-Exchange as alternative sufficient conditions for equivalence of simple and coalitional strategy-proofness of a voting rule on a full unimodal domain is in order here. SMT-richness and the even more general Indirect Sequential Inclusion property are conditions that apply to *virtually arbitrary preference domains* and are therefore *combinatorial* in nature, while Interval Anti-Exchange takes full advantage of the *incidence-geometric structure embodied in the interval space underlying the relevant unimodal domain*. As a consequence, it is indeed not so difficult to devise *necessary conditions* for equivalence of simple and coalitional strategy-proofness on full unimodal domains *that are similar to (in fact a considerable weakening of)* Interval Anti-Exchange, as testified by Theorem 2 of the present paper. By contrast, identifying necessary conditions for such an equivalence by just relying on some weakening of SMT-richness or Indirect Sequential Inclusion seems to be quite an hard task, that is very unlikely to be accomplished without introducing novel, specific restrictions more or less explicitly related to the relevant interval-space-theoretic structure. As a matter of fact, Le Breton, Zaporozhets (2009) does not address at all the issue of necessary conditions for equivalence, while Barberà, Berga, Moreno (2010) does include a result on necessary conditions for equivalence of simple and coalitional strategy-proofness that relies on Indirect Sequential Inclusion. However, quite remarkably, the latter result combines Indirect Sequential Inclusion and closure of the preference domain with respect to preference inversions, a condition blatantly violated by both full unimodal domains and full locally strictly unimodal domains of total preorders, and therefore hardly helpful in the analysis of such domains (see Barberà, Berga, Moreno (2010), Theorem 4). In any case, it still remains to be explored the exact relationship of Interval Anti-Exchange properties to simple and coalitional strategy-proofness of voting rules on the *full locally strictly unimodal* domain of a convex and idempotent interval space.

Department of Economics and Statistics, University of Siena, Piazza S.Francesco 7, 53100 Siena, Italy