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DISCUSSION PAPER

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Abstract

We model problems of allocating disputed properties as generalized exchange economies in which agents have preferences and claims over multiple goods, and the social endowment of each good may not be sufficient to satisfy all individual claims. In this context, we investigate procedural and end-state principles of fairness, their implications and relations. To do so, we explore "procedural" allocation rules represented by a composition of a rights-assignment mechanism (to assign each profile of claims individual property rights over the endowment) and Walrasian, or other individually rational, exchange rule. Using variants of fairness based on no-envy as end-state principles, we provide axiomatic characterizations of the three focal egalitarian mechanisms, known in the literature on rationing problems as constrained equal awards, constrained equal losses, and proportional mechanisms. Our results are connected to focal contributions in political philosophy, and also provide rationale for market-based environ- mental policy instruments (such as cap-and-trade schemes and personal carbon trading) and moral foundation for the three proposals to allocate GHG emission rights known as the equal per capita sharing, the polluter pays principle and the equal burden sharing (the victims pay principle).

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1 Introduction

Fairness and distributive justice is the primary concern in practical procedures for property rights disputes. Very often used in the final allocation of rights are proportional division, equal division and equal sacrifice. Examples can be found in numerous institutional setups including laws, social and religious norms, and agreed conventions, such as bankruptcy laws, divorce laws, court settlements for accident damage, international conventions on environmental problems, domestic or international resolutions on contested natural resources,[1] etc. It is important, especially in the case of agreed conventions, for the enforceability of the resolution whether the division rule adopted is perceived as being fair by the disputers. Indeed, this issue of fairness is at the heart of the recent failure to agree on a new international framework dealing with Greenhouse Gas (GHG) emissions after the Kyoto protocol. The above three division rules form the three prominent proposals for the allocation of GHG emission rights, domestic or international.[2] Countries in different stages of economic development have different perceptions of fairness and support different rules. Making the gap closer and reaching a final resolution is a political economy problem. However, the core issue is an ethical one and investigating it from the perspective of normative science will help facilitating the political resolution.[3]

The normative foundation of allocation schemes, such as the above three division rules, has been a key subject in the literature of fair allocation. Nevertheless, most studies in this literature either focus on the issue of allocating a single good (money), dismissing the issue of fair initial allocation and its influence on the final allocation after the subsequent interactions among claimants (e.g., O'Neill, 1982;

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[1]See Ansink and Weikard (2009) for the case of contested water rights and Ansink (2011) for the case of sharing arctic resources.

[2]The proposal to allocate on an "equal per capita basis" (see Carley et al., 1991, Neumayer, 2000, Saltzman, 2010) corresponds to equal division, the polluter pays principle (paraphrased as "you broke it, you fix it"; e.g., Singer 2002, 2006) to proportional division, and the principle of equal burden sharing (Posner and Weisbach, 2010) to equal sacrifice. Equal per capita allocation is also advocated by the so-called contraction and convergence proposal; see Meyer (2000) and Starkey (2008).

[3]There are quite a few authors who have pursued normative investigation on GHG emission reduction and the allocation of emission rights. Gardiner (2004) and Starkey (2008) provide an extensive outline. See also Margalioth (2012) and Posner and Weisbach (2010). The ethics of intergenerational distribution is an important topic, which is not considered in our current investigation. See, for instance, Llavador, Roemer, Silvestre (2010) or Roemer (2011) for extensive treatments on this topic.

Young, 1988; Moulin, 2000; Thomson, 2003, 2013), or assume a fixed initial distribution of property rights, without dispute, and investigate end-state fairness (e.g., Pazner and Schmeidler, 1974, 1978; Thomson and Varian, 1985; Roemer, 1986; Thomson, 2011).[4] Therefore, they are somewhat limited to apply for the investigation of fairness (both end-state and procedural) in some environments with conflicting claims, or property rights dispute.

Our objectives are, first, to construct a comprehensive framework where one can investigate fairness in the initial allocation of rights on disputed properties, fairness in the transaction of rights, and fairness of the end-state allocation, and, second, to investigate implications of the three categories of fairness and their relations. Our model dealing with disputed properties is an extended model of exchange economies, the extension allowing for conflicts over properties. More precisely, agents have preferences over the consumption space with a finite number of goods, and they also have initial claims on those goods. Furthermore, the available endowment of each good in the economy may not be sufficient to satisfy all claims; i.e., the sum of individual claims either exceeds or equals the endowment. In other words, we consider rationing problems with "multiple" goods when agents have preferences over those goods.

The end-state fairness will be formalized in this context as three variants of *no-envy*, probably the concept with longest tradition in the theory of fair allocation (Foley, 1967; Kolm, 1972; Varian, 1974), used also by a moral philosopher, Ronald Dworkin (1981, p.285) as a basic test for resource egalitarian allocations. No-envy is satisfied if no agent prefers the consumption by anyone else to her own. The same comparative notion of fairness, defined through interpersonal comparisons of (absolute) sacrifices gives rise to the notion of *sacrifice-no-envy*. Likewise, we define *relative no-envy* through interpersonal comparison of relative sacrifices (or reward rates). No-envy conceptualizes the impartial spectator's point of view, à la Adam Smith, by necessitating the equal standard of fairness through the senses of all individuals equally. A different, yet related, conceptualization of the impartial spectator is the contractarian construct of veil of ignorance by John Harsanyi (1953, 1955) and John Rawls (1971), which enforces the decision maker to evaluate the outcome through the individual standards of well-being.[5]

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[4]There exist generalizations of the rationing model introduced by O'Neill (1982) to a multidimensional setting, but therein, even though claims refer to multiple types of assets, the endowment to be allocated is still unidimensional (e.g., Ju, Miyagawa, and Sakai, 2007).

[5]While no-envy provides a specific standard of fair allocations, the contractarian theories only provide the environment of impartial decision making and leave it up to the "rational" decision maker to come up with the exact standard of fair allocations, namely, the utilitarian allocation for

However, unlike the contractarian theories, no-envy does not rely on "cardinal" preferences; it is based purely on "ordinal" preferences.[6]

Our main results characterize procedural rules that lead to fair (and efficient) end-state allocations. We shall focus on the family of market-based rules that solve the problems of allocating disputed properties with two successive procedures: the first procedure, the adjudication of conflicting claims, determines an initial allocation of property rights , and the second procedure of Walrasian or other individually rational exchange from the initial allocation determines a final allocation. Our main results show that the first procedures taking the forms of the three focal rationing mechanisms, known as *constrained equal awards*, *constrained equal losses*, and *proportional*, are the unique *solidaristic* ones that lead to fair end-state allocations.[7] The three mechanisms have a long tradition of use to solve (standard) rationing problems, which can be traced back to Aristotle and Maimonides (e.g., Thomson, 2003). Although they assign property rights in quite different ways, they all achieve equality with different perspectives; namely, equality of the absolute amounts of properties, losses from claims, or relative amounts. Our characterization results give a new rationale for the three egalitarian mechanisms based on end-state notions of fairness.

1.1 Placing our contribution

Practical handling of disputed properties often assign individual rights over the properties relying on claims, and leave individuals some freedom of exchanging their rights, which leads to a final allocation. This is, for instance, the rationale underlying cap-and-trade systems for permit allocations, as well as some other popular market-based environmental policies (e.g., Brennan and Scoccimarro 1999; Stavins, 2003). The end-state is determined through the combination of two procedures: the assignment of rights and the subsequent exchange. The first procedure treats all cases with identical claims and resource constraints equally; preferences are not considered, which makes the procedure informationally simple, as well as impartial. Although this procedure is represented by a reduced form of an (rights-)assignment mechanism (a function mapping each claims problem

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John Harsanyi and the maximin principle for John Rawls.

[6]More general, and possibly individualized notions of fairness criteria are studied by Corchón and Iturbe-Ormaetxe (2001).

[7]Solidarity is modeled here by the combination of two axioms, known as resource monotonicity and consistency, as previously suggested by Moreno-Ternero and Roemer (2006, 2012), among others.

into a single allocation), we do not rule out decentralized decision procedures, e.g., non-cooperative games. It is required, though, that the decentralized procedures have a unique prediction, or a unique "equilibrium" outcome as well as satisfying our basic solidarity axioms. Examples can be found in Dagan, Serrano and Volij (1997, 1999) among others. The second exchange procedure is based on decentralized individual decisions within the boundaries of the socioeconomic institution the agents belong to; Walrasian market exchange is the best known example, although many others exist. It is again described in a reduced form of social choice rules, for simplicity. The procedure can take a form of decentralized decision procedures such as non-cooperative games as long as the predicted equilibrium outcomes meet the basic individual rationality; uniqueness is not required in the second procedure. Examples of such procedures abound in the literature of market games and implementation theory.[8] The combination of the two procedures provides the advantage of simple information processing and decentralized decision making. This motivates our focus on procedural allocation rules.

Conflicts in property ownership are at the center of our resource allocation problems. The liberal theories of private ownership, most notably developed by John Locke (e.g., Locke, 1976) and Robert Nozick (e.g., Nozick, 1974), are too restrictive (in the case of Locke) or too lenient (in the case of Nozick) to provide a useful guideline in our framework. In particular, Nozick's extension of Locke's theory gives a green signal to any resolution satisfying some minimal respect of the claims when they are representative of rights coming from the thesis of "self-ownership". It can admit even an extremely biased resolution of conflicting claims.[9] Our investigation may be viewed as an alternative way of extending Locke's theory in a highly stylistic framework of claims problems. We take advantage of the extensive literature on the claims-problems model that has served as a simple, yet rich, environment for studying allocative fairness during the last thirty years.

In a similar line of investigation, Roemer (1987, 1988, 1989), Moulin (1987, 1990), and Roemer and Silvestre (1989, 1993) propose generalizations of Locke's theory in the framework of common resources under a decreasing-returns-to-scale technology, which gives rise to the so-called *tragedy of the commons*. The allocation rules proposed in these works respect Locke's thesis based on self-ownership:

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[8]The most powerful notion of decentralizability in this literature is based on dominant strategy equilibrium and is known as strategy-proofness. Examples of strategy-proof exchange rules can be found in Barberà and Jackson (1995) among others.

[9]See Otsuka (2003, pp. 23-24) for the same line of criticism against Nozick, with regard to the acquisition of the outside world.

that is, they coincide with the unlimited appropriation outcome in the case of a constant returns to scale technology, the case satisfying the Lockean proviso. Nevertheless unlike Nozick's radical generalization, they all have egalitarian features. It turns out that the so-called "Proportional" and "Nash dominator" mechanisms (e.g., Roemer, 1989) singled-out in this literature are similar to our proportional and equal award mechanisms, respectively.

Somewhat related, Gibbard (1976) and Grunebaum (1987) propose "equal rights" or "public (or joint) ownership" of unowned properties to be the baseline upon which the appropriation should be judged.[10] Moulin and Roemer (1989), in a production economy model, investigate implications of the baseline of public ownership without denying the thesis of self-ownership. Their axiomatic approach shows that the axioms for public ownership and self-ownership, together with other standard axioms, imply a unique welfare-egalitarian outcome. This result, which has a similar flavor to our results, suggests that a fair assignment of property ownership, implicitly assumed through the requirement of their axioms, implies egalitarianism.[11] Interestingly, their model assumes a single representative utility function and, due to this feature, their axiom of self-ownership, which is essentially an order preservation property, coincides with no-envy. All their axioms are for end-state rules and they do not deal with the issue of procedural assignment of ownership rights.

In our model, claims can be viewed as representing the rights based on the thesis of self-ownership. In our procedural approach, assuming the basic condition of claims boundedness captures minimally the thesis of self-ownership.[12] The axiom of self-ownership, proposed by Moulin and Roemer (1989), is similar to an order-preservation property of rights-assignment mechanisms, which we do not impose from the outset because it is implied by other basic axioms. Our resource monotonicity axiom (for rights-assignment mechanisms) may be compared to their axiom of public ownership, called "technology monotonicity". However,

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[10]Nozick sets the baseline to be the state where the unowned properties are unowned; their appropriation, according to Nozick, gives the appropriator the entitlement to the properties as long as no one is harmed relative to the baseline. Roemer (1996, Chapter 6) gives a comprehensive overview of the related literature.

[11]Ownership rights in their paper are assumed to be respected when a mechanism satisfies certain axioms.

[12]When claims represent self-ownership rights, it is natural to require that no person should get strictly more than her claims unless the claims of anyone else are fully compensated, that is, compensating each person's claims has the priority over any further compensation above claims. In the framework of our claims problems (with no excess endowment over the total claims), such a requirement is equivalent to claims-boundedness.

our axioms are merely procedural requirements in the rights-assignment procedure. They are not requirements for end-state rules as in Moulin and Roemer (1989). Hence our axioms are far weaker than theirs; in fact, they are extremely mild allowing for a rich spectrum of mechanisms, which means, in our procedural approach, the baseline of public ownership and the thesis of self-ownership can be met jointly without putting too much restriction on the choice of possible mechanisms. To pin down a unique egalitarian mechanism in our main results, end-state fairness plays a critical role.

Although our procedural approach was taken for fairly practical reasons, as explained earlier, it is somewhat comparable to Nozick's entitlement theory of justice, which is proposed by Nozick (1973, 1974) as an objection to theories of end-state justice. Interestingly, our main results elicit the complementarity between his procedural approach and the end-state approach. To wit, suppose that individual claims in our framework represent the degrees of rightful ownership resulting from the thesis of self-ownership. Under the case of property rights dispute in our model, what would Nozick's principle of just acquisition suggest? The minimal implication of the thesis of self-ownership, a basis of Nozick's theory, in our first procedure is that respecting claims should get the highest priority so that no one can get more properties than her claims unless everyone's claims are fully satisfied. In our first procedure, the total resources are in short of satisfying all claims and so this minimal condition is equivalent to the claims-boundedness assumption (everyone gets no more than her claims). Other than this minimal condition, Nozick's theory does not provide any further guideline for the resolution of disputed property rights. Our results pin down a unique assignment mechanism that embodies the principle of just acquisition, through the application of Nozick's principle of just transfer, namely an individually rational exchange rule in the second procedure, and our end-state fairness axioms. On the other hand, our results show that Nozick's procedural approach can be useful for the "realization" of end-state fairness. In particular, in the case of no-envy and sacrifice no-envy, the procedural rules we characterize lead to fair and efficient allocations when Walrasian exchange rule is used in the second procedure. In the case of relative no-envy, we show that there is no procedural rule satisfying both fairness and efficiency.

Therefore, our investigation provides an instance where a principle of end-state fairness can facilitate the search of appropriate procedural principles of justice, in particular, principles of just acquisition, which constitute Nozick's procedural (or historical) theory of justice and, conversely, Nozick's theory can be used to implement a principle of end-state fairness through informationally simple

and decentralized procedures. This is why we claim that Nozick's procedural approach, at least in our framework, is complementary with the end-state approach.

The rest of the paper is organized as follows. We set up the model in Section 2. We provide our main results (implications of fairness axioms and characterizations of rationing mechanisms) in Section 3. We conclude in Section 4. For a smooth passage, we defer some proofs and provide them in the appendix.

2   The model and definitions

Let $\mathbb{N} \equiv \{1, 2, \ldots\}$ be the set of potential agents and $\mathscr{N} \equiv \{N \subset \mathbb{N} : 2 \leq |N| = n < \infty\}$ the family of finite non-empty subsets with at least two agents. There are ℓ privately appropriable and infinitely divisible goods. Each agent has a preference relation R_i defined on $\mathbb{R}_+^\ell$, which satisfies the classical conditions of rationality (completeness and transitivity), continuity (lower and upper contour sets are closed), strong monotonicity, and convexity (upper contour sets are convex).[13] Let $\mathscr{R}$ denote the domain of such admissible preferences.[14]

A collective resource, or (social) *endowment*, $\Omega \in \mathbb{R}_{++}^\ell$ is an ℓ-vector indicating the available amounts of each good. Each agent has a *claim* (vector) $c_i \in \mathbb{R}_+^\ell$ reflecting the entitlement from the endowment. We shall consider problems in which the endowment is not sufficient to fully cover the claims of the agents. Formally, an *economy* $e \equiv (N, \Omega, c, R)$ is composed by a set of agents $N \in \mathscr{N}$, a social endowment $\Omega \in \mathbb{R}_{++}^\ell$, a profile of individual claims $c \equiv (c_i)_{i \in N} \in \mathbb{R}_{++}^{n\ell}$, such that $\Omega \leqq \sum_{i \in N} c_i$, and a profile of preference relations $R \equiv (R_i)_{i \in N} \in \mathscr{R}^N$.[15] Note that if $\Omega = \sum_{i \in N} c_i$, then e would be a standard (pure) *exchange economy* in which c would correspond to the profile of *individual endowments*. Similarly, if $\ell = 1$ and $\Omega < \sum_{i \in N} c_i$, then e would be a standard *bankruptcy problem* (e.g., O'Neill, 1982; Thomson, 2003, 2013).

Let $\mathscr{E}$ denote the set of all economies, $\mathscr{E}(N)$ the set of economies with population $N \in \mathscr{N}$, $\mathscr{E}(\Omega)$ the set of economies with endowment Ω, and $\mathscr{E}(N, \Omega) \equiv$

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[13]As usual, we denote by P_i the strict preference relation associated with R_i and the corresponding indifference relation by I_i.

[14]As preferences are continuous, we can represent them by continuous real-valued functions, and it is sometimes convenient to do so. For each $i \in N$, let $U_i : \mathbb{R}_+^\ell \to \mathbb{R}$ be such a representation of agent i's preferences, and let $U \equiv (U_i)_{i \in N}$. These representations will not have cardinal significance.

[15]Our mathematical notation $x \leqq y$ to relate vectors $x, y \in \mathbb{R}_{++}^\ell$ means that $x_i \leq y_i$ for each $i = 1, \ldots \ell$, with at least one strict inequality. Thus, we are implicitly saying that no commodity exhausts the corresponding aggregate claim.

$\mathscr{E}(N) \cap \mathscr{E}(\Omega)$. Let $\bar{\mathscr{E}} \subset \mathscr{E}$ be the set of exchange economies. We shall often denote the claims profile for an exchange economy (the endowments profile) by $\omega \equiv (\omega_i)_{i \in N}$ instead of $c \equiv (c_i)_{i \in N}$, and will dismiss Ω in the description of the economy.[16]

A *feasible allocation* for an economy $e \in \mathscr{E}(N, \Omega)$ is a profile of individual consumption bundles $z \equiv (z_i)_{i \in N} \in \mathbb{R}_+^{n\ell}$ such that $\sum_{i \in N} z_i = \Omega$. Let $Z(N, \Omega) \equiv \{z \in \mathbb{R}_+^{n\ell} : \sum_{i \in N} z_i = \Omega\}$ be the set of all feasible allocations for economies in $\mathscr{E}(N, \Omega)$. Let $Z \equiv \cup_{N \in \mathscr{N}} \cup_{\Omega \in \mathbb{R}_{++}^{\ell}} Z(N, \Omega)$. Given an economy $e \equiv (N, \Omega, c, R)$, a feasible allocation $z \in Z(N, \Omega)$ is (Pareto) *efficient* if there is no other feasible allocation z' that makes a person better off without making anyone else worse off, that is, for some $i \in N$, $z_i' P_i z_i$ and for each $j \in N \setminus \{i\}$, $z_j' R_j z_j$. A *social choice rule* $S \colon \mathscr{E} \to Z$ associates with each economy $e \equiv (N, \Omega, c, R)$ a non-empty set of feasible allocations, i.e., a non-empty subset of $Z(N, \Omega)$. Finally, for reasons that will become clear later in the text, we denote by $[z]_+$ the truncated consumption bundle after replacing the negative amounts of $z \equiv (z_i)_{i \in N} \in \mathbb{R}^{n\ell}$ by zero, i.e., $[z]_+ \equiv (\max\{0, z_i\})_{i \in N}$.

We shall be mostly interested in social choice rules that are characterized by the following two consecutive procedures: First, an *assignment procedure* mapping the non-preference information (N, Ω, c) of each economy $e \equiv (N, \Omega, c, R)$ into a profile of individual endowments $\omega \equiv (\omega_i)_{i \in N}$, and second, an *exchange procedure* determining final allocations for the exchange economy (N, ω, R) obtained in the first procedure. In doing so, we shall be able to scrutinize the relationship between principles of procedural justice (imposed in each of these two procedures) and principles of end-state justice (imposed on the final allocations determined by social choice rules).

2.1 Claims adjudication

A problem of adjudicating claims, briefly a *claims problem*, is defined by a set of agents, a social endowment, and a profile of individual conflicting claims. Formally, it is a triple (N, Ω, c) such that $\Omega \leq \sum_{i \in N} c_i$. Let $\mathscr{C}$ denote the set of all claims problems so defined.[17] We use the notation $\mathscr{C}(N)$, $\mathscr{C}(\Omega)$, and $\mathscr{C}(N, \Omega)$ in the same manner as used for $\mathscr{E}$. An *assignment mechanism* $\varphi \colon \mathscr{C} \to \cup_{N \in \mathscr{N}} \mathbb{R}_+^{n\ell}$ associates with each claims problem $(N, \Omega, c) \in \mathscr{C}$, "individual property rights"

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[16]We denote $z_M \equiv (z_i)_{i \in N}$, for each $N \in \mathscr{N}$, $M \subset N$, and $z \in \mathbb{R}_+^{n\ell}$. Furthermore, for ease of notation, if $M = N \setminus \{i\}$, for some $i \in N$, we let $z_{-i} \equiv z_M$.

[17]Note that $\mathscr{C} \equiv \{(N, \Omega, c) : (N, \Omega, c, R) \in \mathscr{E} \text{ for some } R \equiv (R_i)_{i \in N} \in \mathscr{R}^N\}$.

over the social endowment; that is, a feasible allocation $\varphi(N,\Omega,c) \in Z(N,\Omega)$ with $\varphi_i(c) \leq c_i$ for each $i \in N$. This last inequality condition is called *claims-boundedness*, and it requires that individual property rights do not exceed the initial entitlements specified by individual claims.[18] We often refer to $\varphi(N,\Omega,c)$ as the individual endowment set by mechanism φ for the claims problem (N,Ω,c). An assignment mechanism φ then converts each economy $e \equiv (N,\Omega,c,R)$ into an exchange economy $e' \equiv (N,\varphi(N,\Omega,c),R)$.

We now define the three focal assignment mechanisms derived from the literature on claims problems.

The *constrained equal awards mechanism* φ^{CEA} maps each problem $(N,\Omega,c) \in \mathscr{C}$ into the allocation $\varphi^{CEA}(N,\Omega,c) \in Z(N,\Omega)$ such that, for each $i \in N$, and $l \in \{1,\ldots,\ell\}$,

$$\varphi_{il}^{CEA}(N,\Omega,c) = \min\{c_{il},\mu_l\},$$

where $\mu_l > 0$ is chosen so that $\sum_{i \in N} \min\{c_{il},\mu_l\} = \Omega_l$. Clearly, if all claims are above the per capita endowment, i.e., for each $i \in N$ and $l = 1,\ldots\ell$, $c_{il} \geq \frac{\Omega_l}{n}$, then

$$\varphi_i^{CEA}(N,\Omega,c) = \frac{\Omega}{n}.$$

The *constrained equal losses mechanism* φ^{CEL}, which is somewhat polar to the previous one, maps each problem $(N,\Omega,c) \in \mathscr{C}$ into the allocation $\varphi^{CEL}(N,\Omega,c) \in Z(N,\Omega)$ such that, for each $i \in N$, and $l \in \{1,\ldots,\ell\}$,

$$\varphi_{il}^{CEL}(N,\Omega,c) = \max\{0,c_{il} - \lambda_l\},$$

where $\lambda_l > 0$ is chosen so that $\sum_{i \in N} \max\{0,c_{il} - \lambda_l\} = \Omega_l$. Clearly, if all claims are above the per capita loss, i.e., for each $i \in N$ and $l = 1,\ldots\ell$, $c_{il} \geq \frac{\sum_{j \in N} c_{jl} - \Omega_l}{n}$, then

$$\varphi_i^{CEL}(N,\Omega,c) \equiv c_i - \frac{\Omega - \sum_{j \in N} c_j}{n}. \tag{2.1}$$

In particular, as we shall show later, in the two-agent case, where the problem can be depicted in an Edgeworth box, if both claims are below the endowment, then φ^{CEL} selects the midpoint of the segment joining both claims vectors in the Edgeworth box.

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[18]Note that there is also an implicit (lower) bound condition in the range of the assignment mechanism, which precludes agents from obtaining negative amounts.

The *proportional mechanism* φ^{PRO} maps each problem $(N, \Omega, c) \in \mathscr{C}$ into the allocation $\varphi^{PRO}(N, \Omega, c) \in Z(N, \Omega)$ such that, for each $i \in N$, and $l \in \{1, \ldots, \ell\}$,

$$\varphi_{il}^{PRO}(N, \Omega, c) = \frac{\Omega_l}{\sum_{j \in N} c_{jl}} c_{il}.$$

We now present a list of axioms for assignment mechanisms. The first axiom says that when there is more to be divided, other things being equal, nobody should lose.[19] Formally,

Resource Monotonicity. For each $(N, c, \Omega) \in \mathscr{C}(N)$ and $\Omega' \geq \Omega$,

$$\varphi(N, c, \Omega') \geq \varphi(N, c, \Omega).$$

An implication of resource monotonicity is that when the social endowment of a certain good does not change, ceteris paribus, the allocation of that good should remain unaffected by resource increases in other goods.

A widely applied principle, known as *consistency*, in the axiomatic literature (see, e.g., Thomson (2007, 2012) and the literature cited therein) relates the allocation by a rule for a given problem to the solutions of the subproblems that a subgroup of agents face with the total amount they received at the original problem. It requires that the application of the rule to each subproblem produces precisely the allocation that the subgroup obtained in the original problem. That is, the original resolution should be reinforced with the reassessment by any subgroup. Here we consider two weaker principles, proposed by Thomson (2006), pertaining to the exclusion of either agents with zero awards or agents with full awards.

The agents who receive zero in a given dimension of the original problem have nothing to contribute in the ensuing subproblem. Thus, they may not be of interest to other agents in the process of reassessment of the original resolution. The next axiom requires that reassessment after excluding such agents should not alter how much others get.

Zero-Award-Out-Consistency.[20] For each $(N, \Omega, c) \in \mathscr{C}(N)$, $M \subset N$, and $l \in \{1, \ldots, \ell\}$, if $\varphi_{il}(N, c, \Omega) = 0$, for each $i \in M$, then, for each $j \in N \backslash M$,

$$\varphi_{jl}(N \backslash M, c_{N \backslash M}, \sum_{k \in N \backslash M} \varphi_k(N, c, \Omega)) = \varphi_{jl}(N, c, \Omega).$$

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[19]This axiom reflects a solidarity principle whose formalization in axiomatic work can be traced back to Roemer (1986).

[20]In the bankruptcy model ($\ell = 1$), the same axiom is called "null compensations consistency" by Thomson (2006).

Similarly, the agents whose claims were fully respected will not be interested in any further reassessment. The next axiom requires that dismissing a person fully awarded should not affect how much others get.

Full-Award-Out-Consistency.[21] For each $(N, \Omega, c) \in \mathscr{C}(N)$, $M \subset N$, and $l \in \{1, \ldots, \ell\}$, if $\varphi_{il}(N, c, \Omega) = c_{il}$, for each $i \in M$, then, for each $j \in N \backslash M$,

$$\varphi_{jl}(N \backslash M, c_{N \backslash M}, \sum_{k \in N \backslash M} \varphi_k(N, c, \Omega)) = \varphi_{jl}(N, c, \Omega).$$

2.2 Exchange

An *exchange rule, $F \colon \bar{\mathscr{E}} \to Z$*, associates with each exchange economy $e \equiv (N, \Omega, \omega, R) \in \bar{\mathscr{E}}$ a non-empty set of feasible allocations, $F(e) \subseteq Z(N, \Omega)$. Exchange rules are studied extensively in the literature. The best known one is the *Walrasian (exchange) rule*, F^W, which associates with each exchange economy $e = (N, \omega, R)$ the set of Walrasian equilibrium allocations. Formally, for each vector of market prices $p \in \mathbb{R}_+^\ell$, define the individual budget, delineated by the initial endowment ω_i, as $B(\omega_i, p) = \{z_i \in \mathbb{R}_+^\ell : p \cdot z_i \leq p \cdot \omega_i\}$. An allocation z is a *Walrasian equilibrium* allocation if there exists $p \in \mathbb{R}_+^\ell$ such that for each $i \in N$ and each $z_i' \in B(\omega_i, p)$, $z_i \in B(\omega_i, p)$ and $z_i R_i z_i'$. We shall also consider other rules that are not Walrasian, yet satisfy the following basic condition for "voluntary" exchange:

Individual Rationality. For each $(N, \omega, R) \in \bar{\mathscr{E}}$, $z \in F(N, \omega, R)$, and $i \in N$,

$$z_i R_i \omega_i.$$

The composition of an assignment mechanism φ and an exchange rule F gives rise to a social choice rule $S(\cdot) \equiv F \circ \varphi(\cdot)$ such that, for each $e \equiv (N, \Omega, c, R) \in \mathscr{E}$,

$$S(e) \equiv F \circ \varphi(e) = F(N, \varphi(N, \Omega, c), R).$$

We shall be mostly interested in social choice rules arising from the composition of an assignment mechanism and the Walrasian rule, or other individually rational exchange rules.

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[21] In the bankruptcy model ($\ell = 1$), the same axiom is called "full compensations consistency" by Thomson (2006).

2.3 End-state fairness

We now move to some classical fairness axioms of social choice rules. One of the fundamental notions in the theory of fair allocation is envy-freeness, which although can be traced back to Foley (1967), was not formally introduced till Kolm (1972) and Pazner and Schmeidler (1974, 1978).[22] It is widely accepted that such a notion represents an adequate test of fairness (e.g., Fleurbaey and Maniquet, 2011; Thomson, 2011). An allocation satisfies *no-envy*, or is said to be *envy-free*, if there is no pair of agents in which one prefers the allocation of the other. Formally,

No-Envy. For each $e \equiv (N, \Omega, c, R) \in \mathscr{E}$ and $z \in S(e)$, there is no pair $i, j \in N$ such that $z_j P_i z_i$.

The above notion does not make use of any information on claims to establish envy comparisons. The following ones fill that gap. Given an allocation $z \in Z$ and an agent $i \in N$, call $c_i - z_i$ the *sacrifice* agent i makes at z.[23] An allocation satisfies *sacrifice-no-envy*, or is said to be *sacrifice-envy-free*, if no agent prefers making the sacrifice of anyone else to making his own sacrifice. Note that, as imposing the sacrifice of another agent might generate negative consumption at some dimensions, and preferences are only defined for non-negative consumptions, we consider truncated bundles to perform these comparisons in terms of sacrifices. Formally,

Sacrifice-No-Envy. For each $e \equiv (N, \Omega, c, R) \in \mathscr{E}$ and $z \in S(e)$, there is no pair, $i, j \in N$, such that $\left[c_i - (c_j - z_j) \right]_+ P_i z_i$.

This notion is similar to *fair net trades* (e.g., Schmeidler and Vind, 1972) for exchange economies. Given an allocation z, the net trade of agent i with endowment ω_i is the vector $z_i - \omega_i$. A net trade is said to be *fair* if, for any agent i, his net trade is at least as good for him as the net trade of any other agent.

Now, instead of measuring sacrifices in absolute terms, we could also measure them in relative terms. Formally, given an allocation $z \in Z$ and an agent $i \in N$, let $c_i/z_i \equiv (c_{i1}/z_{i1}, \ldots, c_{il}/z_{il})$ be the *relative sacrifice* agent i makes at z, whenever it is well-defined, including the case when $c_{il} > 0$ and $c_{il}/z_{il} = c_{il}/0 \equiv \infty$.[24] A dual

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[22]See also Varian (1974).

[23]Velez and Thomson (2012) have recently used the same word to describe a different, but somewhat related, notion in the "classical" problem of fair allocation (without claims). More precisely, they measure an agent's "sacrifice" at an allocation by the size of the set of feasible bundles that the agent prefers to her consumption.

[24]The relative sacrifice is not well-defined in the case that, for some l, $c_{il} = 0$ and $z_{il} = 0$.

concept is *relative compensation* $z_i/c_i \equiv (z_{i1}/c_{i1}, \ldots, c_{i\ell}/z_{i\ell})$. Interpersonal comparison of relative-sacrifice or relative-compensation will provide the same notion of no-envy. Here we consider relative-compensation. When $z_{il} = 0 = c_{il}$, relative compensation is not well-defined. In this case, we will adopt *the convention that when $z_{il} = 0 = c_{il}$, i is considered to be getting zero relative compensation* with regard to good l.[25] We say that allocation z satisfies *relative-no-envy*, or is said to be *relative-envy-free*, at economy $(N, \Omega, c, R) \in \mathscr{E}$, if no agent prefers the relative compensation of anyone else to his own; that is, there do not exist $i, j \in N$ such that $c_i \times (z_j/c_j) P_i z_i$, where the product $\times$ is to be defined as the coordinate-wise multiplication.[26] Then this axiom can be stated as follows.

Relative-No-Envy. For each $e \equiv (N, \Omega, c, R) \in \mathscr{E}$, and $z \in S(e)$, there is no pair, $i, j \in N$, such that $c_i \times (z_j/c_j) P_i z_i$, where for all l, the l-th component of z_j/c_j equals zero when $z_{jl} = c_{jl} = 0$.

3 The results

As we show in this section, using the three notions of fairness in Section 2.3, we provide axiomatic characterizations of the three central assignment mechanisms, the constrained equal awards mechanism, the constrained equal losses mechanism and the proportional mechanism.

3.1 No-envy and the constrained equal awards mechanism

A trivial way of guaranteeing no-envy is to divide equally. For each $(N, \Omega, c) \in \mathscr{C}$, let $\omega^{ed}(N, \Omega, c)$ be the equal division allocation in which each agent i receives Ω/n. We sometimes denote $\omega^{ed}(N, \Omega, c)$ by $\omega^{ed}(N, \Omega)$. Note that $\omega^{ed}(N, \Omega, c)$ may not necessarily satisfy claims-boundedness. In particular, when claims are distributed very unequally (one agent with extremely large claims and another with small claims), claims-boundedness restricts the choice of initial endowments which are so unequal that any individual rational exchange from these endowments will make the poor agent (the one with small claims) envy the rich agent (the one with extremely large claims). Therefore, for these economies, no-envy

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[25]If we do not follow this convention, the results we show in Section 3.3 may be more complicated and we may need an additional consistency axiom (null-consistency) to get a similar result to the main result we provide therein.

[26]In other words, if $x = (x_1, \ldots, x_\ell), y = (y_1, \ldots, y_\ell) \in \mathbb{R}_+^\ell$, then $x \times y = (x_1 y_1, \ldots, x_\ell y_\ell)$.

is too strong a requirement to satisfy. Thus, we will require no-envy only for the economies with somewhat even distribution of claims across agents.

Let $\mathscr{E}^0 \equiv \{(N,\Omega,c,R) \in \mathscr{E} : 0 \leq \omega^{ed}(N,\Omega) \leq c\}$ be the set of economies where equal division satisfies claims-boundedness. Let $\mathscr{C}^0 \equiv \{(N,\Omega,c) \in \mathscr{C} : 0 \leq \omega^{ed}(N,\Omega) \leq c\}$ be the set of the corresponding claims problems. There are infinitely many envy-free and efficient allocations at typical economies in $\mathscr{E}^0$. Thus, there are infinitely many selections from such allocations, namely social choice rules satisfying efficiency and no-envy on $\mathscr{E}^0$. Nevertheless, when one focuses on the social choice rules that are represented by a combination of an assignment mechanism and an individually rational exchange rule, all but one social choice rules are ruled out, as we show in this section. In order to do that, we need first some preliminary results.

Lemma 1. *Let F be an exchange rule satisfying individual rationality. Let φ be an assignment mechanism that, when combined with F, generates a social choice rule satisfying no-envy. Then, for each $(N,\Omega,c) \in \mathscr{C}$, $\varphi(N,\Omega,c) = \omega^{ed}(N,\Omega,c)$.*

Proof. Let F be an individually rational exchange rule and φ be an assignment mechanism that, when combined with F, generates a social choice rule satisfying no-envy. Suppose, by contradiction, that there exists $(N,\Omega,c) \in \mathscr{C}$ such that $\omega \equiv \varphi(N,\Omega,c) \neq \omega^{ed}(N,\Omega,c)$. Then, as $\sum_{k \in N} \omega_k = \Omega$, there exist $i,j \in N$, and $p \in \mathbb{R}_{++}^{\ell}$ such that $p \cdot \omega_i < p \cdot \frac{\Omega}{n} < p \cdot \omega_j$. Let $e = (N,\Omega,\omega,R) \in \mathscr{E}$ be such that R_i is represented by $U_i(x) \equiv p \cdot x$, R_h is strictly convex for some $h \in N$, and ω is Pareto efficient at R. Then ω is the only feasible allocation that satisfies individual rationality.[27] Hence, $\omega = F(N,\omega,R)$. As $p \cdot \omega_j > p \cdot \omega_i$, agent i prefers j's bundle to his own, contradicting no-envy. $\square$

Proposition 1. *The claims domain $\mathscr{C}^0$ is the maximal domain on which an assignment mechanism and an individually rational exchange rule combined together can generate envy-free allocations. Moreover, on this maximal claims domain, individual rationality, no-envy, and efficiency are compatible.*

Proof. Let $(N,\Omega,c) \in \mathscr{C}\backslash\mathscr{C}^0$. Let φ be an assignment mechanism and F an individually rational exchange rule such that, when combined together, can generate envy-free allocations. Then, by Lemma 1, $\varphi(N,\Omega,c) = \omega^{ed}(N,\Omega,c)$ but as $(N,\Omega,c) \notin \mathscr{C}^0$, this would contradict the claims boundedness assumption of assignment mechanisms.

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[27]This is because if there is any other individually rational allocation z, a convex combination of z and ω will be a Pareto improvement of ω, contradicting Pareto efficiency of ω.

On the other hand, using the constrained equal awards mechanism and the Walrasian exchange rule, the second statement follows from the First Fundamental Theorem of Welfare Economics, which states that all Walrasian equilibrium allocations are efficient. □

Proposition 2. *Let F be an exchange rule satisfying individual rationality. If an assignment mechanism φ satisfies resource monotonicity and, when combined with F, leads to a social choice rule satisfying no-envy on $\mathscr{E}^0$, then, $\varphi(N,\Omega,c) = \varphi^{CEA}(N,\Omega,c)$, for each $(N,\Omega,c) \in \mathscr{C}$, with $|N| = 2$.*

Proof. Let F be an exchange rule satisfying individual rationality, and φ an assignment mechanism satisfying resource monotonicity such that, when combined with F, leads to a social choice rule satisfying no-envy on $\mathscr{E}^0$. Furthermore, let $(N,\Omega,c) \in \mathscr{C}$, with $|N| = 2$. Without loss of generality, assume that $N \equiv \{1,2\}$. For each $l = 0,\ldots,\ell$, let $\mathscr{C}(N,l) \equiv \{(N,\Omega,c) \in \mathscr{C}(N) : \forall k \geq l+1, \forall i \in N, \Omega_k/2 \leq c_{ik}\}$. Then, $\mathscr{C}(N,0) \equiv \mathscr{C}^0(N)$, and $\mathscr{C}(N,\ell) = \mathscr{C}(N)$. We show that φ coincides with φ^{CEA} on $\mathscr{C}(N,k)$ for each $k = 0,1,\ldots,\ell$, using mathematical induction.

By Lemma 1, φ coincides with φ^{CEA} on $\mathscr{C}(N,0)$. Let $l \in \{1,\ldots,\ell\}$, Suppose, by induction, that φ coincides with φ^{CEA} on $\mathscr{C}(N,k)$ for each $k \leq l-1$. We now prove that φ coincides with φ^{CEA} on $\mathscr{C}(N,l)$.

Let $(N,\Omega,c) \in \mathscr{C}(N,l) \backslash \mathscr{C}(N,l-1)$. Then $c_{1l} < \Omega_l/2 \leq c_{2l}$ and, for each $k \geq l+1$ and all $i \in N$, $\Omega_k/2 \leq c_{ik}$.[28] Thus, $\varphi_l^{CEA}(N,c,\Omega) = (c_{1l}, \Omega_l - c_{1l})$. Let $\omega \equiv \varphi(N,\Omega,c)$. Let Ω' be such that $\Omega'_l \equiv 2c_{1l}$ and for each $k \neq l$, $\Omega'_k \equiv \Omega_k$. Then $(N,c,\Omega') \in \mathscr{C}(N,l-1)$ and, by the induction hypothesis,

$$\varphi(N,\Omega',c) = \varphi^{CEA}(N,\Omega',c). \tag{3.1}$$

In particular, $\varphi_l(N,\Omega',c) = (\Omega'_l/2, \Omega'_l/2) = (c_{1l}, c_{1l})$. As $\Omega' \leq \Omega$, then, by resource monotonicity, $\omega = \varphi(N,\Omega,c) \geq \varphi(N,\Omega',c)$. By claims-boundedness, $\omega_{1l} = c_{1l}$. Then $\omega_{2l} = \Omega_l - c_{1l}$. Therefore, $\varphi_l(N,\Omega,c) = \varphi_l^{CEA}(N,\Omega,c)$. As $\Omega'_k = \Omega_k$ for each $k \neq l$, then applying resource monotonicity to both φ and φ^{CEA}, $\varphi_k(N,\Omega',c) = \varphi_k(N,\Omega,c)$ and $\varphi_k^{CEA}(N,\Omega',c) = \varphi_k^{CEA}(N,\Omega,c)$. Hence, using (3.1), for each $k \neq l$, we conclude the proof. □

When the assignment mechanism satisfies an additional consistency axiom, this result for two-person economies can be extended to general economies as stated next in the main result of this section.

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[28] We have assumed, for ease of exposition, that $c_{1l} \leq c_{2l}$.

Theorem 1. *An assignment mechanism satisfies resource monotonicity and full-award-out-consistency and, when combined with an individually rational exchange rule, leads to a social choice rule satisfying no-envy on $\mathcal{E}^0$ if and only if it is the constrained equal awards mechanism.*

Proof. It is straightforward to see that the constrained equal awards mechanism satisfies resource monotonicity and full-award-out-consistency and, when combined with an individually rational exchange rule (e.g., the Walrasian rule), leads to a social choice rule satisfying no-envy on $\mathcal{E}^0$. We focus on the converse implication, which we prove for the case $\ell = 1$.[29] Let φ be an assignment mechanism satisfying resource monotonicity and full-award-out-consistency that, when combined with an individually rational exchange rule, leads to a social choice rule satisfying no-envy on $\mathcal{E}^0$. By Lemma 1, we only have to show that, for each $(N, \Omega, c) \in \mathscr{C} \backslash \mathscr{C}^0$, $\varphi(N, \Omega, c) = \varphi^{CEA}(N, \Omega, c)$. The proof is by induction. More precisely, let $(N, \Omega, c) \in \mathscr{C} \backslash \mathscr{C}^0$. For ease of exposition, and without loss of generality, we assume that $N = \{1, \ldots, n\}$ and that $c_1 \leq c_2 \leq \cdots \leq c_n$. Now, for each $m \in \{1, 2, \ldots, n\}$ let[30]

$$\mathscr{C}^m(N) = \{(N, \Omega, c) \in \mathscr{C} : \sum_{i=1}^{m-1} c_i + (n - m + 1)c_m < \Omega \leq \sum_{i=1}^{m} c_i + (n - m)c_{m+1}\}.$$

It is straightforward to see that $\mathscr{C}(N) \backslash \mathscr{C}^0(N) = \bigcup_{m=1}^{n} \mathscr{C}^m(N)$. We now show, by induction, that, for each $m \in \{1, 2, \ldots, n\}$, and each $(N, \Omega, c) \in \mathscr{C}^m(N)$, $\varphi(N, \Omega, c) = \varphi^{CEA}(N, \Omega, c)$.

Case $m = 1$. Let $(N, \Omega, c) \in \mathscr{C}^1(N)$. Then, $\varphi^{CEA}(N, \Omega, c) = (c_1, \lambda, \ldots, \lambda)$, where $\lambda = (\Omega - c_1)/(n - 1)$. Let $\Omega' \equiv nc_1$. Then $(N, \Omega', c) \in \mathscr{C}^0$ and, by Lemma 1, $\varphi(N, \Omega', c) = (c_1, c_1, \ldots, c_1) = \varphi^{CEA}(N, \Omega', c)$. By resource monotonicity, $\varphi_1(N, \Omega, c) \geq c_1$. Thus, by claims-boundedness, $\varphi_1(N, \Omega, c) = c_1$.

Now, as $(N \backslash \{1\}, c_{-1}, \Omega - c_1) \in \mathscr{C}^0$, it follows, by Lemma 1, that

$$\varphi(N \backslash \{1\}, c_{-1}, \Omega - c_1) = \varphi^{CEA}(N \backslash \{1\}, c_{-1}, \Omega - c_1) = \left(\frac{\Omega - c_1}{n - 1}, \ldots, \frac{\Omega - c_1}{n - 1} \right). \tag{3.2}$$

By full-award-out-consistency,

$$\begin{aligned} \varphi_{N \backslash \{1\}}(N, \Omega, c) &= \varphi(N \backslash \{1\}, c_{-1}, \Omega - c_1); \\ \varphi_{N \backslash \{1\}}^{CEA}(N, \Omega, c) &= \varphi^{CEA}(N \backslash \{1\}, c_{-1}, \Omega - c_1). \end{aligned} \tag{3.3}$$

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[29] The general proof uses an induction argument similar to the one in the proof of Proposition 2 and is provided in the appendix.

[30] We also use the notational convention $\sum_{i=1}^{0} c_i = 0$.

Combining (3.2) and (3.3), $\varphi_{N\setminus\{1\}}(N,c,\Omega) = \varphi_{N\setminus\{1\}}^{CEA}(N,c,\Omega)$, which completes the proof of this case.

Case $m \to m+1$. Suppose then that, for each $m \in \{1,2,\ldots,n-1\}$, $\varphi(N,c,\Omega) = \varphi^{CEA}(N,c,\Omega)$, for each $(N,c,\Omega) \in \mathscr{C}^m(N)$. Let $(N,c,\Omega) \in \mathscr{C}^{m+1}(N)$. We aim to show that $\varphi(N,c,\Omega) = \varphi^{CEA}(N,c,\Omega) = (c_1,\ldots,c_{m-1},\lambda,\ldots,\lambda)$, where $\lambda = [\Omega - \sum_{k=1}^{m-1} c_k]/(n-m+1)$. Let $\Omega' \equiv \sum_{k=1}^{m-1} c_k + (n-m+1)c_{m-1}$. Then, $(N,c,\Omega') \in \mathscr{C}^m(N)$, and, by the induction hypothesis, we obtain that $\varphi(N,c,\Omega') = (c_1,\ldots,c_{m-1},\lambda',\ldots,\lambda')$, where $\lambda' = [\Omega' - \sum_{k=1}^{m-1} c_k]/(n-m+1)$. Thus, in particular, $\varphi_k(N,c,\Omega') = c_k$, for each $k \leq m-1$. Then, by resource monotonicity and claims boundedness, $\varphi_k(N,c,\Omega) = c_k$, for each $k \leq m-1$. On the other hand, by full-award-out-consistency,

$$\varphi_{N\setminus\{1,\ldots,m-1\}}(N,c,\Omega) = \varphi(\{m,\ldots,n\},(c_m,\ldots,c_n),\Omega - \sum_{k=1}^{m-1} c_k). \qquad (3.4)$$

As $(\{m,\ldots,n\},(c_m,\ldots,c_n),\Omega - \sum_{k=1}^{m-1} c_k) \in \mathscr{C}^0$, it follows, by Lemma 1, that

$$\varphi(\{m,\ldots,n\},(c_m,\ldots,c_n),\Omega - \sum_{k=1}^{m-1} c_k) = (\lambda,\ldots,\lambda). \qquad (3.5)$$

Therefore, by (3.4) and (3.5), $\varphi_{N\setminus\{1,\ldots,m-1\}}(N,c,\Omega) = (\lambda,\ldots,\lambda)$, which completes the proof. $\qquad\square$

Remark 1. Full-award-out-consistency can be strengthened in the statement of the theorem to consistency. In this case, the "only if" part can be proven using Proposition 2 and the so-called Elevator Lemma (e.g., Thomson, 2007).

3.2 Sacrifice-no-envy and the constrained equal losses mechanism

A trivial way of guaranteeing sacrifice-no-envy is to allocate equally the total loss or sacrifice $\sum_{i\in N} c_i - \Omega$ the society has to bear. For each $(N,\Omega,c) \in \mathscr{C}$, let $es(N,\Omega,c) \equiv (\sum_{i\in N} c_i - \Omega)/n$ be the equal sacrifice at (N,Ω,c). Let $\omega^{es}(N,\Omega,c) \equiv (c_i - es(N,\Omega,c))_{i\in N}$ be the equal sacrifice allocation. Note that $\omega^{es}(N,\Omega,c)$ may contain some negative consumptions, in which case it is not a well-defined allocation. In particular, when agents have quite disparate claims (an agent with extremely large claims, even larger than Ω, and the others with extremely small claims), it may not be possible to satisfy sacrifice-no-envy at all. Thus, we first

restrict our attention to less extreme cases where equal sacrifice is feasible. Formally, let $\mathscr{C}^* \equiv \{(N,\Omega,c) \in \mathscr{C} : \omega^{es}(N,\Omega,c) \in Z(N,\Omega)\}$ and $\mathscr{E}^* \equiv \{e = (N,\Omega,c,R) \in \mathscr{E} : (N,\Omega,c) \in \mathscr{C}^*\}$. It is not difficult to show that there are infinite social choice rules that satisfy efficiency and sacrifice-no-envy, when restricted to $\mathscr{E}^*$. This can be demonstrated by the infinite cardinality of the set of sacrifice-envy-free and efficient allocations at typical economies in $\mathscr{E}^*$. However, when one focuses on the type of social choice rules that are a combination of an assignment mechanism and an individually rational exchange rule, all but one social choice rules are ruled out, as we show in this section. In order to do that, we need first some preliminary results.

Lemma 2. *Let F be an exchange rule satisfying individual rationality. Let φ be an assignment mechanism that, when combined with F, generates a social choice rule satisfying sacrifice-no-envy. Then, for each $(N,\Omega,c) \in \mathscr{C}$, $\varphi(N,\Omega,c) = \omega^{es}(N,\Omega,c)$.*

Proof. Let F be an individually rational exchange rule and φ be an assignment mechanism that, when combined with F, generates a social choice rule satisfying sacrifice-no-envy. Suppose, by contradiction, that there exists $(N,\Omega,c) \in \mathscr{C}$ such that $\omega \equiv \varphi(N,\Omega,c) \neq \omega^{es}(N,\Omega,c)$. Then, as $\sum_{k \in N} \omega_k = \Omega = \sum_{k \in N} \omega_k^{es}$, there exist $i,j \in N$ and $p \in \mathbb{R}_{++}^{\ell}$, such that $p \cdot \omega_i^{es} > p \cdot \omega_i$ and $p \cdot \omega_j^{es} < p \cdot \omega_j$. Thus, $p \cdot \omega_i < p \cdot \omega_i^{es} < p \cdot [c_i + (\omega_j - c_j)]_+$.[31] Let $e = (N,\Omega,\omega,R)$ be such that R_i is represented by $U_i(x) \equiv p \cdot x$, R_j is strictly convex, and ω is Pareto efficient at R. Then, ω is the only feasible allocation that satisfies individual rationality. Hence, $\omega = F(N,\omega,R)$. As $p \cdot \omega_i < p \cdot [c_i + (\omega_j - c_j)]_+$, agent i envies j's sacrifice, contradicting sacrifice-no-envy. $\square$

Proposition 3. *The claims domain $\mathscr{C}^*$ is the maximal domain on which an assignment mechanism and an individually rational exchange rule combined together can generate sacrifice-envy-free allocations. Moreover, on this maximal claims domain, individual rationality, sacrifice-no-envy, and efficiency are compatible.*

Proof. Let $(N,\Omega,c) \in \mathscr{C} \setminus \mathscr{C}^*$. Let φ be an assignment mechanism and F an individually rational exchange rule, such that, when combined together, can generate sacrifice-envy-free allocations. Then, by Lemma 2, $\varphi(N,\Omega,c) = \omega^{es}(N,\Omega,c)$ but as $(N,\Omega,c) \notin \mathscr{C}^*$, this contradicts the non-negativity condition in the definition of assignment mechanisms.

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[31]More precisely, $p \cdot \omega_j^{es} < p \cdot \omega_j$ implies that $p \cdot (\omega_j - c_j) > p \cdot (\Omega - \sum_{i \in N} c_i)/n$ and, therefore, $p \cdot [c_i + (\omega_j - c_j)]_+ \geqq p \cdot (c_i + (\omega_j - c_j)) > p \cdot \omega_i^{es} > p \cdot \omega_i$.

On the other hand, using the constrained equal losses mechanism and the Walrasian exchange rule, the second statement follows from the First Fundamental Theorem of Welfare Economics. □

Proposition 4. *Let F be an exchange rule satisfying individual rationality. If an assignment mechanism φ satisfies resource monotonicity and, when combined with F, leads to a social choice rule satisfying sacrifice-no-envy on $\mathscr{E}^*$, then, $\varphi(N,\Omega,c) = \varphi^{CEL}(N,\Omega,c)$, for each $(N,\Omega,c) \in \mathscr{C}$, with $|N| = 2$.*

Proof. Let F be an exchange rule satisfying individual rationality, and φ an assignment mechanism satisfying resource monotonicity such that, when combined with F, leads to a social choice rule satisfying sacrifice-no-envy on $\mathscr{E}^*$. Furthermore, let $(N,\Omega,c) \in \mathscr{C}$, with $|N| = 2$. Without loss of generality, assume that $N \equiv \{1,2\}$.

For each $l = 0,\ldots,\ell$, let $\mathscr{C}^*(N,l) \equiv \{(N,\Omega,c) \in \mathscr{C}(N) : \forall k \geq l+1, \forall i \in N, (\sum_{j \in N} c_{jk} - \Omega_k)/2 \leq c_{ik}\}$. Then, $\mathscr{C}^*(N,0) \equiv \mathscr{C}^*(N)$, and $\mathscr{C}^*(N,\ell) = \mathscr{C}(N)$. We show that φ coincides with φ^{CEL} on $\mathscr{C}(N,k)$ for each $k = 0,1,\ldots,\ell$, using mathematical induction.

By Lemma 2, φ coincides with φ^{CEL} on $\mathscr{C}^*(N,0)$. Let $l \in \{1,\ldots,\ell\}$, Suppose, by induction, that φ coincides with φ^{CEL} on $\mathscr{C}^*(N,k)$ for each $k \leq l-1$. We now prove that φ coincides with φ^{CEL} on $\mathscr{C}^*(N,l)$.

Let $(N,\Omega,c) \in \mathscr{C}^*(N,l) \backslash \mathscr{C}^*(N,l-1)$ be such that $c_{1l} \leq c_{2l}$. Then, $\varphi_l^{CEL}(N,\Omega,c) = (0,\Omega_l) \leq (0,c_{2l} - c_{1l})$. Let $\Omega'_l \equiv c_{2l} - c_{1l}$ and, for each $k \neq l$, $\Omega'_k = \Omega_k$. Then $(N,\Omega',c) \in \mathscr{C}^*(N,l-1)$ and, by the induction hypothesis,

$$\varphi(N,\Omega',c) = \varphi^{CEL}(N,\Omega',c). \tag{3.6}$$

In particular, $\varphi_l^{CEL}(N,c,\Omega') = (0,c_{2l} - c_{1l}) = \varphi_l(N,c,\Omega')$. By resource monotonicity and non-negativity, $\varphi_{1l}(N,c,\Omega) = 0$ and so $\varphi_l(N,c,\Omega) = (0,\Omega_l) = \varphi_l^{CEL}(N,c,\Omega)$.

As $\Omega'_k = \Omega_k$ for each $k \neq l$, then applying resource monotonicity to both φ and φ^{CEL}, we have $\varphi_k(N,c,\Omega) = \varphi_k(N,c,\Omega')$ and $\varphi_k^{CEL}(N,c,\Omega') = \varphi_k^{CEL}(N,c,\Omega)$. Hence, using (3.6), for each $k \neq l$, we conclude the proof. □

This result for two-agent economies can be extended to all economies with more than two agents, when a consistency axiom is added as stated next in the main result of this section.

Theorem 2. *An assignment mechanism satisfies resource monotonicity and zero-award-out-consistency, and, when combined with an individually rational exchange rule, leads to a social choice rule satisfying sacrifice-no-envy on $\mathscr{E}^*$ if and only if it is the constrained equal losses mechanism.*

Proof. It is straightforward to see that the constrained equal losses mechanism satisfies resource monotonicity and zero-award-out-consistency and, when combined with an individually rational exchange rule (e.g., the Walrasian rule), leads to a social choice rule satisfying no-envy on $\mathscr{E}^*$. We focus on the converse implication, which we prove for the case $\ell = 1$.[32] Let φ be an assignment mechanism satisfying resource monotonicity and zero-award-out-consistency that, when combined with an individually rational exchange rule, leads to a social choice rule satisfying sacrifice-no-envy on $\mathscr{E}^*$. By Lemma 2, we only have to show that, for each $(N, c, \Omega) \in \mathscr{C} \backslash \mathscr{C}^*$, $\varphi(N, c, \Omega) = \varphi^{CEL}(N, c, \Omega)$. The proof is by induction. More precisely, let $(N, c, \Omega) \in \mathscr{C} \backslash \mathscr{C}^*$. For ease of exposition, and without loss of generality, we assume that $N = \{1, \ldots, n\}$ and that $c_1 \leq c_2 \leq \cdots \leq c_n$. Now, for each $m \in \{1, 2, \ldots, n-1\}$, let

$$\mathscr{C}_m(N) = \{(N, c, \Omega) \in \mathscr{C} : \sum_{i=m+1}^{n} c_i - (n-m)c_{m+1} \leq \Omega < \sum_{i=m}^{n} c_i - (n-m+1)c_m\}.$$

It is straightforward to see that $\mathscr{C}(N) \backslash \mathscr{C}^*(N) = \bigcup_{m=1}^{n-1} \mathscr{C}_m(N)$. We now show, by induction, that, for each $m \in \{1, 2, \ldots, n-1\}$, $\varphi(N, c, \Omega) = \varphi^{CEL}(N, c, \Omega)$, for each $(N, c, \Omega) \in \mathscr{C}_m(N)$.

Case $m = 1$. Let $(N, c, \Omega) \in \mathscr{C}_1(N)$. Then, $\varphi^{CEL}(N, c, \Omega) = (0, c_1 - \lambda, \ldots, c_n - \lambda)$, where $\lambda = (\sum_{i=2}^{n} c_i - \Omega)/(n-1)$. Let $\Omega' \equiv \sum_{i \in N} c_i - nc_1$. Note that $(N, \Omega', c) \in \mathscr{C}^*$ and

$$\varphi^{CEL}(N, \Omega', c) = \left(c_i - \frac{\sum_{j=1}^{n} c_j - \Omega'}{n} \right)_{i \in N} = (0, c_2 - c_1, c_3 - c_1, \ldots, c_n - c_1).$$

Then, by Lemma 2, $\varphi(N, \Omega', c) = \varphi^{CEL}(N, \Omega', c)$ and, in particular, $\varphi_1(N, \Omega', c) = 0$. By resource monotonicity, and non-negativity, $\varphi_1(N, \Omega, c) = 0$. By zero-award-out-consistency,

$$\varphi(N \backslash \{1\}, \sum_{j \in N \backslash \{1\}} \varphi_j(N, \Omega, c), c_{-1}) = \varphi(N \backslash \{1\}, \Omega, c_{-1})$$

$$= \varphi_{N \backslash \{1\}}(N, \Omega, c).$$

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[32] The general proof is provided in the appendix.

As $(N\backslash\{1\},\Omega,c_{-1}) \in \mathscr{C}^*$, then, by Lemma 2,

$$\varphi(N\backslash\{1\},\Omega,c_{-1}) = \varphi^{CEL}(N\backslash\{1\},\Omega,c_{-1}) = \left(c_i - \frac{\sum_{j=2}^{n} c_j - \Omega}{n-1}\right)_{i \in N\backslash\{1\}}.$$

Therefore, $\varphi(N,\Omega,c) = \varphi^{CEL}(N,\Omega,c)$.

Case $m \to m+1$. Suppose then that, for each $m \in \{1,2,\ldots,n-2\}$, $\varphi(N,c,\Omega) = \varphi^{CEL}(N,c,\Omega)$, for each $(N,c,\Omega) \in \mathscr{C}_m(N)$. Let $(N,c,\Omega) \in \mathscr{C}_{m+1}(N)$. We aim to show that $\varphi(N,c,\Omega) = \varphi^{CEL}(N,c,\Omega) = (0,\ldots,0,c_{m+2}-\lambda,\ldots,c_n-\lambda)$, where $\lambda = \frac{\sum_{i=m+2}^{n} c_i - \Omega}{n-m-1}$. Let $t > 0$ be such that $c_{m+1} = \frac{\sum_{i=m+1}^{n} c_i - (\Omega+t)}{n-m}$. Let $\Omega' \equiv \Omega + t$ and $\lambda' \equiv \frac{\sum_{i=m+1}^{n} c_i - (\Omega+t)}{n-m}$. Then, by the induction hypothesis, $\varphi(N,\Omega',c) = \varphi^{CEL}(N,\Omega',c) = (0,\ldots,0,c_{m+1}-\lambda',\ldots,c_n-\lambda')$. In particular, $\varphi_i(N,\Omega',c) = 0$, for each $i \le m+1$. By resource monotonicity, and non-negativity, $\varphi_i(N,\Omega,c) = 0$, for each $i \le m+1$. Let $N' \equiv \{m+2,\ldots,n\}$. By zero-award-out-consistency,

$$\varphi(N',\Omega,c_{N'}) = \varphi_{N'}(N,\Omega,c).$$

As $(N',\Omega,c_{N'}) \in \mathscr{C}^*$, then, by Lemma 2,

$$\varphi(N',\Omega,c_{N'}) = \varphi^{CEL}(N',\Omega,c_{N'}) = \varphi^{CEL}_{N'}(N,\Omega,c).$$

Therefore, $\varphi(N,\Omega,c) = \varphi^{CEL}(N,\Omega,c)$. $\qquad\square$

Remark 2. Zero-award-out-consistency can be strengthened in the statement of the theorem to consistency. In this case, the "only if" part can be proven using Proposition 4 and the so-called Elevator Lemma (e.g., Thomson, 2007).

3.3 Relative-no-envy and the proportional mechanism

We explore in this section the implications of relative-no-envy. In contrast with the previous two sections, one can find an assignment mechanism that, when combined with an individually rational exchange rule, leads to a social choice rule satisfying relative-no-envy on the "whole" domain. A simple example, among others, is the combination of proportional mechanism and the no-trade exchange rule.

Thus, the maximal claims domain on which an assignment mechanism and an individually rational exchange rule combined together can generate relative-envy-free allocations is the universal claims domain. This is in contrast with the

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counterpart results for no-envy (Proposition 1) and sacrifice-no-envy (Proposition 3).

The next result shows that if any other assignment mechanism, different from the proportional one is used, relative-envy is inevitable.

Lemma 3. *Let F be an exchange rule satisfying individual rationality. Let φ be an assignment mechanism that, when combined with F, generates a social choice rule satisfying relative-no-envy. Then, for each $(N, \Omega, c) \in \mathscr{C}$, $\varphi(N, \Omega, c) = \varphi^{pro}(N, \Omega, c)$.*

Let F be an individually rational exchange rule and φ be an assignment mechanism that, when combined with F, generates a social choice rule satisfying relative-no-envy. Suppose, by contradiction, that there exists $(N, \Omega, c) \in \mathscr{C}$ such that $\omega \equiv \varphi(N, \Omega, c) \neq \varphi^{pro}(N, \Omega, c)$. Thus, there exist $i, j \in N$, and $\hat{l} \in \{1, \ldots, \ell\}$, such that $\omega_{i\hat{l}} > \Omega_{\hat{l}} \cdot \frac{c_{i\hat{l}}}{\sum_{h \in N} c_{h\hat{l}}}$ and $\omega_{j\hat{l}} < \Omega_{\hat{l}} \cdot \frac{c_{j\hat{l}}}{\sum_{h \in N} c_{h\hat{l}}}$. Then $c_{i\hat{l}} > 0$, as otherwise the first inequality contradicts claims boundedness. Likewise, $c_{j\hat{l}} > 0$, as otherwise the second inequality contradicts non-negativity. Hence $\frac{\omega_{i\hat{l}}}{c_{i\hat{l}}} > \frac{\omega_{j\hat{l}}}{c_{j\hat{l}}}$. Then, there exists a vector of prices $p \in \mathbb{R}^{\ell}_{++}$ such that $p \cdot (c_j \times \frac{\omega_i}{c_i}) > p \cdot \omega_j$. Let $e = (N, \Omega, c, R) \in \mathscr{E}$, where R is such that R_i is represented by $U_i(x) \equiv p \cdot x$, R_h is strictly convex for some $h \in N$, and ω is Pareto efficient at R. Then, ω is the only feasible allocation that satisfies individual rationality. Hence, $\omega = F(N, \omega, R)$. Now, as $p \cdot c_j \times \frac{\omega_i}{c_i} > p \cdot \omega_j$, agent j envies i's relative sacrifice, which proves the statement of the lemma.

As explained earlier, on the whole claims domain, individual rationality and relative-no-envy can be met when the proportional mechanism is used. This lemma shows that it is the only such mechanism. However, as stated in the next proposition, relative-no-envy and efficiency are incompatible, which is also in contrast with the corresponding results for no-envy (Proposition 1) and sacrifice-no-envy (Proposition 3) presented above.

Proposition 5. *On the universal claims domain, there exist an assignment mechanism and an individually rational exchange rule such that, when combined together, can generate relative-envy-free allocations. However, relative-no-envy and efficiency are incompatible.*[33]

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[33]It is worth mentioning that the counter example we use for the proof, is set in an exchange economy. Thus, the incompatibility remains valid in the standard model of exchange economies.

Proof. The first statement follows from the discussion preceding the previous proposition. To show the second statement, consider an exchange economy with two agents and two goods. Let $c_1 \equiv (200/3, 100/3)$ and $c_2 \equiv (100/3, 200/3)$. Let $\Omega \equiv (100, 100)$. Preferences of agents 1 and 2 are represented, respectively, by $u_1(x_1, x_2) \equiv \alpha x_1 + x_2$ and $u_2(x_1, x_2) \equiv \beta x_1 + x_2$, with $1/2 < \alpha < \beta < 2$. Then, the set of *efficient* allocations is $\{((0, x_2), (100, 100 - x_2)) : x_2 \in [0, 100]\} \cup \{((x_1, 100), (100 - x_1, 0)) : x_1 \in [0, 100]\}$. We show that no *efficient* allocation in this economy satisfies *relative-no-envy*. First, consider the *efficient* allocations $((0, x_2), (100, 100 - x_2))$ with $x_2 \in [0, 100]$. for each $x_2 \in [0, 100]$, all these efficient allocations fail relative-no-envy due to agent 1's envy. If agent 1 makes the relative sacrifice of agent 2, her consumption becomes $(2 \times 100, (100 - x_2)/2)$. As $\alpha > 1/2$, then $(400\alpha + 100)/3 > 100$. This implies $x_2 < 200\alpha + 50 - x_2/2$, which means $u_1(0, x_2) < u_1(2 \times 100, (100 - x_2)/2)$, i.e., agent 1 prefers making the relative sacrifice of agent 2 to making his own. In the case of efficient allocations $((x_1, 100), (100 - x_1, 0))$ with $x_1 \in [0, 100]$, using $\beta < 2$ and the same argument as above for agent 2, we can show that they fail relative-no-envy due to agent 2's envy. □

Now we are ready to state the third characterization result.

Theorem 3. *An assignment mechanism, when combined with an individually rational exchange rule, leads to a social choice rule satisfying relative-no-envy if and only if it is the proportional mechanism.*

Proof. It is straightforward to see that the proportional mechanism, when combined with the no-trade exchange rule (which is, trivially, *individually rational*) guarantees *relative-no-envy*.[34] As for the converse implication, let φ be an assignment mechanism such that, when combined with an individually rational exchange rule, leads to a social choice rule satisfying relative-no-envy. By Lemma 3, φ must be the proportional mechanism. □

In contrast with the results from the previous sections involving other notions of no-envy, the Walrasian rule cannot be used here as equilibrium allocations may violate *relative-no-envy*. In the case of no-envy, the Walrasian budget set from the equal division provides equal opportunities of consumption for each agent, which guarantees no-envy at equilibrium allocations. Likewise, the Walrasian

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[34]Apart from the no-trade rule, there are many other *individually rational* exchange rules that always select allocations satisfying *relative-no-envy* (as well as *efficiency*, whenever *relative-no-envy* and *efficiency* are compatible).

budget set provides equal opportunities of trades for each agent. Thus, when the initial endowment is chosen at the allocation with equal sacrifice, all equilibrium allocations necessarily satisfy sacrifice-no-envy. However, the Walrasian budget set does not provide equal opportunities of "relative trades", i.e., the ratio of the final consumption and the initial endowment across consumers.

Ratio-no-envy and proportional Walrasian equilibrium

An alternative method of measuring the "rates" of sacrifices, or rewards, is to use a price vector that reflects "subjective" marginal rate of substitutions between any two goods. We next introduce the corresponding axiom of no-envy. In contrast with relative-no-envy, it is compatible with efficiency, as we explain later.

Let z be a feasible allocation for an economy $e \equiv (N, \Omega, c, R)$. For each $i \in N$, denote a *supporting normal vector of i's indifference set at z_i* by p^i, that is, for all x with $x R_i z_i$, $p^i \cdot x \geq p^i \cdot z_i$. Then p^i gives i's marginal rate of substitution between any two goods for agent i and may be called as i's subjective valuation of goods. Note that $\frac{z_i}{p^i \cdot c_i}$ is the vector consisting of the amounts of each good l person i gets for each unit value of her claim. Likewise, from i's point of view, $\frac{z_j}{p^i \cdot c_j}$ is the vector consisting of the amounts of each good l person j gets for each unit value (measured by i's price vector) of j's claim. The comparative axiom of fairness with regard to these vectors can be defined as follows.

Allocation z satisfies *ratio-no-envy at $e \equiv (N, \Omega, c, R)$* if, for each $i \in N$, there is a supporting normal vector p^i of i's indifference curve at z_i such that, for each $j \in N$,

$$p^i \cdot c_i \times \frac{z_i}{p^i \cdot c_i} = z_i R_i p^i \cdot c_i \times \frac{z_j}{p^i \cdot c_j} = \frac{p^i \cdot c_i}{p^i \cdot c_j} \times z_j.$$

To show the existence of allocations satisfying ratio-no-envy and efficiency, an extended notion of Walrasian equilibrium introduced by Peleg (1996) and Korthues (2000) can be used.[35] An allocation z is a *proportional Walrasian equilibrium* if there is a price p such that for each i, i's income $w_i(p, c, \Omega)$ is given by

$$w_i(p, c, \Omega) \equiv \frac{p \cdot c_i}{p \cdot \sum c_j} p \cdot \Omega,$$

and z is a Walrasian equilibrium with equilibrium price p and the profile of individual incomes $(w_i(p, c, \Omega))_i$, that is, for each i, $p \cdot z_i \leq w_i(p, c, \Omega)$ and for each x with $p \cdot x \leq w_i(p, c, \Omega)$, $z_i R_i x$, and $\sum_i z_i = \Omega$.

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[35]They consider the framework where the total claims of each good may be more or less than the total endowment; thus, all economies in our model are examples of theirs.

Note that

$$\frac{w_i(p,c,\Omega)}{w_j(p,c,\Omega)} = \frac{p \cdot c_i}{p \cdot c_j}.$$

When z and p constitute a proportional Walrasian equilibrium, as the ratio of $w_i(p,c,\Omega)$ and $w_j(p,c,\Omega)$ is the same as the ratio of $p \cdot c_i$ and $p \cdot c_j$, we get

$$p \cdot \left(p^i \cdot c_i \times \frac{z_j}{p^i \cdot c_j} \right) = \frac{p^i \cdot c_i}{p^i \cdot c_j} \times p \cdot z_j \leq \frac{p^i \cdot c_i}{p^i \cdot c_j} \times w_j(p,c,\Omega) = w_i(p,c,\Omega).$$

This means that with i's wealth $w_i(p,c,\Omega)$, i can afford to obtain the consumption bundle $\frac{p^i \cdot c_i}{p^i \cdot c_j} \times z_j$ where, out of each unit of the value of her claim, she gets the same amount of each good as j does at z_j. As z_i maximizes i's preference satisfaction over her budget set,

$$z_i R_i p^i \cdot c_i \times \frac{z_j}{p^i \cdot c_j}.$$

Thus, ratio-no-envy is satisfied. Existence of ratio-Walrasian equilibrium is shown by Peleg (1996). Therefore, there exists an allocation satisfying both efficiency and ratio-no-envy.

4   Concluding remarks

We have presented in this paper a general model of exchange economies to study the allocation of disputed properties, while accommodating the three levels in which fairness can be scrutinized in this context; namely, fairness in the initial allocation of rights on disputed properties, fairness in the transaction of allocated rights, and fairness of the end-state allocation. We have focused, in such context, on the combination of assignment mechanisms (to assign each profile of conflicting demands and supply an initial endowment) and Walrasian or other individually rational market exchange. We have characterized two assignment mechanisms that, when combined with Walrasian exchange, give rise to *efficiency,* as well as two related forms of *no envy.* More precisely, we have shown that the composition of the constrained equal awards (losses) assignment mechanism with the Walrasian exchange rule is essentially the only way to obtain (sacrifice-)envy-free (and efficient) allocations via such a two-step process.[36]

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[36]The composition of the proportional assignment mechanism with Walrasian exchange has not been characterized in this paper, although we have seen how a suitable notion of envy free-

The composition of the constrained equal awards mechanism with Walrasian exchange, an example of the rules we characterize in Theorem 1, is akin to the so-called *Walrasian rule from equal division* (or *competitive equilibrium correspondence from equal division*) that has been singled out as a focal social choice rule in the literature of fair allocation and distributive justice (e.g., Varian, 1974; Dworkin, 1981; Yaari and Bar-Hillel, 1984; Roemer, 1996; Fleurbaey and Maniquet, 2011). When the equal division is within the claims-bound of all agents, they coincide. In the model of classical exchange economies, the Walrasian equilibrium from equal division has been axiomatized in several ways. In particular, Yaari (1982) shows that it is the unique exchange rule that guarantees, not only no-envy among individual consumers, but also "coalitional" no-envy (no-envy among equal-sized coalitions),[37] provided the number of agents is sufficiently large.[38]

The composition of the constrained equal losses assignment mechanism with Walrasian exchange, an example of the rules we characterize in Theorem 2, does not have a counterpart in the literature on fair allocation, as claims, which are collectively not feasible, are the standard for making interpersonal fairness comparisons and for determining the initial endowment from which Walrasian exchange takes place. Nevertheless, this rule and the end-state fairness concept (sacrifice-no-envy) used for the characterization, resemble the Walrasian rule and the axiom of fair net trade, or trade-no-envy, introduced by Schmeidler and Vind (1972) in exchange economies. The Walrasian rule is shown to be the unique rule satisfying "coalitional" trade-no-envy if the number of agents is sufficiently large (Gabszewicz, 1975).

Using no-envy as both procedural and end-state principles of fairness, Kolm (1972), Feldman and Kirman (1974), Goldman and Sussangkarn (1980), and Thomson (1982), among others, investigate whether procedural fairness induces the end-state fairness. The results are negative. The combination of envy-free initial allocation (equal division) and a sequence of envy-free trades may lead to a core allocation with envy. Our three main characterization results impose three versions of no-envy as the principle of end-state fairness and obtain no-envy, "with some constraints", of the initial allocation as an implication. We do

ness points towards it. It remains as an open question to find appealing axioms, from a fairness viewpoint, that would allow to provide a full-fledged characterization of such mechanism in this context.

[37]That is, any coalition does not find that it could use the resources allotted to the other equal-sized coalition in such a way that would make all its members better off than they are with what has actually been allotted to them.

[38]With a small number of agents, other mechanisms also exist.

not impose no-envy as a procedural requirement. Instead, other standard axioms (such as claims-boundedness, resource monotonicity, and consistency for rights-assignment, and individual rationality and efficiency for exchange) are used as procedural requirements.[39]

In standard exchange economies, Thomson (1983) is also concerned with the three levels of justice, fair initial position (endowment), fair trade, and end-state fairness. In his approach, the principle of fair trade plays a central role and the principle of fair initial position is formulated through the possibility of changing the initial positions of agents (as in the definition of no-envy) and their objections based on the principle of fair trade from any reshuffled position. Thus, the key idea of no-envy is behind his notion of fair initial position. He shows that *no-envy* is the unique end-state fairness concept that is obtained from this procedural approach using individual rationality as the principle of fair trade (Proposition 1). His main result is to show that Walrasian trade and the principle of fair initial position defined via Walrasian trade give rise to the same outcomes as the Walrasian rule from equal division (Proposition 2). In a sense, this result says that if one accepts Walrasian trade to be a fair rule of trade and the possibility of changing initial positions among agents, then the only fair initial position is equal division. Our Theorem 1 reinforces this conclusion in the extended framework and using a different procedural approach. While Thomson's procedural approach is hypothetically oriented (reminiscent of the veil of ignorance or the original position) à la Harsanyi and Rawls, our procedural approach is practically oriented. We deal with an environment where we face the issue of conflicting claims as well as allocating resources. Our procedural approach is practical in the sense that it is informationally simple and decentralizable and is also representative of actual institutions. The first procedure of rights assignment allows us to utilize the findings in the vast literature of claims problems (bankruptcy or taxation problems, surplus sharing, cost sharing, etc). The second procedure of exchange is assumed to meet the very basic condition for a decentralized system, individual rationality. Unlike Thomson (1983), we use no-envy as the end-state fairness axiom and characterize egalitarian rights assignment mechanisms for the first procedure.

Our model is also related with a model introduced by Thomson (2007).[40] The aim in that model is to formulate appropriate notions of consistency dealing with the exchange economy augmented by some social deficit to be shared among

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[39]Our model and the procedural approach follow the lesson on procedural fairness delineated by Thomson (2011, pp.419~422).

[40]See also Peleg (1996) and Korthues (2000).

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agents. Mathematically, the model of "deficit-sharing exchange economies" (on p.184, Thomson 2007) is identical to ours, if the deficit T in his model is considered as the difference between the sum of the claims and the aggregate endowment in our model. However, we do not impose consistency on the whole procedure of social choice, but impose consistency only on the first procedure of assignment mechanisms. Thus, due to the claims-boundedness condition for assignment mechanisms, defining consistency is not an issue in our context.

In the definition of assignment mechanisms, we have imposed the condition of claims-boundedness. It is a standard condition in the literature of rationing and can capture, in the most convincingly minimal sense, the thesis of self-ownership when claims can represent self-ownership rights.[41] Nevertheless, one may wonder about the consequence of dropping the condition. A result similar to Theorem 1, but without resource monotonicity and full-award-out-consistency, prevails after replacing the constrained equal awards mechanism with the equal awards (equal division) mechanism.[42] In a sense, Theorem 1 is simplified after dropping claims-boundedness. However, the other theorems will hold in the same shape as we currently have them.

To conclude, it is worth mentioning that the validity of our three end-state fairness axioms should be judged context by context. In some applications, claims may be perceived as indicating "credits", and in other applications, as indicating discredits (or responsibility). In bankruptcy problems, claims are investment amounts; the greater they are, the more credits are attributed to the claimants. On the other hand, in allocating pollution permits, claims represent past emissions and greater claims mean greater discredits or more responsibility for the environmental damage.

No-envy fully ignores claims information in judging for fairness. On the other hand, both sacrifice-no-envy and relative-no-envy make agents' claims be the benchmark outcome for fairness judgments. They require that the final outcome should move away from this benchmark without breaking the particular "fair" balance across agents, as specified in the two axioms.

When claims are credits, claims deserve special attention in judging for fairness and so no-envy does not qualify as a valid criterion for fairness. In such scenario, our results pin down proportional and constrained equal losses mechanisms as the only fair assignment mechanisms.

When claims are discredits, using these claims as the benchmark in judging

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[41] See Footnote 12.

[42] This will follow directly from Lemma 1.

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for fairness, as in sacrifice-no-envy and relative-no-envy, does not go well with our moral intuition. In particular, when claims represent discredits of past generations, upon which the advantage of the current generation is based, these two axioms dismiss "historical accountability", and our results show that they yield 'grandfathering' rules, namely, constrained equal losses and proportional rules.

For example, in allocating GHG emission rights, it is abnormal to ask people in continuous poverty (and, therefore, with little responsibility for the current GHG problem) to set their benchmark for fairness to be at their past record of poor consumptions (with almost zero GHG emission) while setting the same benchmark at the luxurious life style (with high GHG emission) for very rich people. Thus, in this case, the two axioms can be invalidated right away and no-envy is a valid axiom for fairness. In such scenario, our results, then, point toward constrained equal awards as the unique fair assignment mechanism. This is our justification for the scheme of equal per capita allocation of GHG emission rights. We take as its moral foundation the axiom of no-envy that is a focal axiom of fairness in Economics and Political Philosophy. The axiom of no-envy does not directly involve the allocation of rights; it emphasizes the welfare consequences of this allocation upon individual agents. Appendix C gives a formal presentation of this case as an example of our model.

A   Proof of Theorem 1

Proof. As mentioned above, it is straightforward to see that the constrained equal awards mechanism satisfies resource monotonicity and full-award-out-consistency and, when combined with an individually rational exchange rule, leads to a social choice rule satisfying no-envy on $\mathscr{E}^0$. We focus on the converse implication. Let φ be an assignment mechanism satisfying resource monotonicity and full-award-out-consistency that, when combined with an individually rational exchange rule, leads to a social choice rule satisfying no-envy on $\mathscr{E}^0$. For each $l \in \{1, \ldots, \ell - 1\}$, let $\mathscr{C}(l) \equiv \{(N, c, \Omega) \in \mathscr{C} : \forall k \geq l + 1, \forall i \in N, \ \Omega_k/|N| \leq c_{ik}\}$. Let $\mathscr{C}(\ell) \equiv \mathscr{C}$ and $\mathscr{C}(0) \equiv \mathscr{C}^0$.

We show that φ coincides with φ^{CEA} on $\mathscr{C}(k)$ for each $k = 0, 1, \ldots, \ell$, using mathematical induction.

By Lemma 1, φ coincides with φ^{CEA} on $\mathscr{C}(0)$. Let $l \in \{1, \ldots, \ell - 2\}$. Suppose, by induction, that φ coincides with φ^{CEA} on $\mathscr{C}(k)$ for each $k \leq l$. In what follows, we prove that φ coincides with φ^{CEA} on $\mathscr{C}(l)$, which completes the induction argument. Let $(N, c, \Omega) \in \mathscr{C}(l)$. For ease of exposition, assume, without

loss of generality, that $N = \{1,\ldots,n\}$ and that claims on the l-th commodity are increasingly ordered, i.e., $c_{1l} \leq \cdots \leq c_{nl}$. For each $m \in \{1,2,\ldots,n\}$, let[43]

$$\mathscr{C}(N)(l)^m = \{(N,c,\Omega) \in \mathscr{C}(N)(l) : \sum_{i=1}^{m-1} c_{il} + (n-m+1)c_{ml} < \Omega_l \leq \sum_{i=1}^{m} c_{il} + (n-m)c_{(m+1)l}\}.$$

Furthermore, let

$$\mathscr{C}(N)(l)^0 = \{(N,c,\Omega) \in \mathscr{C}(l) : \Omega_l \leq nc_{1l}\}.$$

Then, it is straightforward to see that $\mathscr{C}(N)(l) = \bigcup_{m=0}^{n} \mathscr{C}(N)(l)^m$. Note that, as $\mathscr{C}(N)(l)^0 \subset \mathscr{C}(N)(l-1)$, φ coincides with φ^{CEA} on $\mathscr{C}(N)(l)^0$. We now show, by induction, that it also happens for the remaining subsets in the partition, i.e., for each $m \in \{1,\ldots,n\}$, $\varphi(N,c,\Omega) = \varphi^{CEA}(N,c,\Omega)$, for each $(N,c,\Omega) \in \mathscr{C}(N)(l)^m$.

Case $m = 1$. Let $(N,c,\Omega) \in \mathscr{C}(N)(l)^1$. Then, $\varphi_l^{CEA}(N,c,\Omega) = (c_{1l},\lambda,\ldots,\lambda)$, where $\lambda = (\Omega_l - c_{1l})/(n-1)$. Let $\Omega' \in \mathbb{R}_{++}^{\ell}$ be such that $\Omega_l' \equiv nc_{1l}$ and, for each $l' \neq l$, $\Omega_{l'}' \equiv \Omega_{l'}$. Then, $(N,c,\Omega') \in \mathscr{C}(N)(l-1)$ and, therefore, $\varphi(N,c,\Omega') = \varphi^{CEA}(N,c,\Omega')$. In particular, $\varphi_l(N,c,\Omega') = (c_{1l},c_{1l},\ldots,c_{1l})$ and, for each $l' \geq l$, $\varphi_{l'}(N,c,\Omega') = (\Omega_{l'}/n,\cdots,\Omega_{l'}/n)$. By resource monotonicity, and claims boundedness, $\varphi_{1l}(N,c,\Omega) = c_{1l}$.

Now, consider the problem $(N\backslash\{1\},c_{-1},\bar{\Omega})$, where $\bar{\Omega} \equiv \Omega - \varphi_1(N,c,\Omega)$. Note that, for each $l' \geq l+1$, $\varphi_{1l'}(N,c,\Omega) = \Omega_{l'}/n$ and, for each $i \in N$, $\Omega_{l'}/n \leq c_{il'}$. Then, $\bar{\Omega}_{l'}/(n-1) = \Omega_{l'}/n \leq c_{il'}$, for each $i \in N$. Moreover, as $\bar{\Omega}_l/(n-1) = (\Omega_l - c_{1l})/(n-1) \leq c_{2l} \leq \cdots \leq c_{nl}$, it follows that $(N\backslash\{1\},c_{-1},\bar{\Omega}) \in \mathscr{C}(l-1)$. Thus, $\varphi(N\backslash\{1\},c_{-1},\bar{\Omega}) = \varphi^{CEA}(N\backslash\{1\},c_{-1},\bar{\Omega})$.

By full-award-out-consistency, $\varphi_{il}(N,c,\Omega) = \varphi_{il}(N\backslash\{1\},c_{-1},\bar{\Omega})$, for each $i \in N\backslash\{1\}$. Therefore, $\varphi_{il}(N,c,\Omega) = \varphi_{il}^{CEA}(N,c,\Omega)$, for each $i \in N\backslash\{1\}$, completing the proof of this case.

Case $m \rightarrow m+1$. Suppose then that, for each $m \in \{1,2,\ldots,n-1\}$, $\varphi(N,c,\Omega) = \varphi^{CEA}(N,c,\Omega)$, for each $(N,c,\Omega) \in \mathscr{C}(N)(l)^m$. Let $(N,c,\Omega) \in \mathscr{C}(N)(l)^{m+1}$. We aim to show that $\varphi(N,c,\Omega) = \varphi^{CEA}(N,c,\Omega)$. Note that, $\varphi_l^{CEA}(N,c,\Omega) = (c_{1l},\ldots,c_{m-1,l},\lambda,\ldots,\lambda)$, where $\lambda = [\Omega - \sum_{k=1}^{m-1} c_{kl}]/(n-m+1)$.

Let $\Omega' \equiv \sum_{k=1}^{m-1} c_{kl} + (n-m+1)c_{(m-1)l}$. Then, $(N,c,\Omega') \in \mathscr{C}(N)(l)^m$, and, therefore, $\varphi(N,c,\Omega') = \varphi^{CEA}(N,c,\Omega')$. In particular, for each $k \leq m-1$, $\varphi_{kl}(N,c,\Omega') = c_{kl}$. As $\Omega_{l'}' = \Omega_{l'}$, for each $l' \neq l$, it follows, by resource monotonicity, that $\varphi_{il'}(N,c,\Omega) = \varphi_{il'}(N,c,\Omega')$ and $\varphi_{il'}^{CEA}(N,c,\Omega) = \varphi_{il'}^{CEA}(N,c,\Omega')$, for each $i \in N$. Therefore, for each $l' \neq l$ and $i \in N$, $\varphi_{il'}(N,c,\Omega) = \varphi_{il'}^{CEA}(N,c,\Omega)$. As,

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[43]We use the notational convention that $\sum_{i=1}^{0} c_i = 0$.

for each $i \leq m-1$, $\varphi_{il}(N,c,\Omega') = c_{kl}$, then, by resource monotonicity and claims boundedness, $\varphi_{il}(N,c,\Omega) = c_{il}$, for each $i \leq m-1$.

Finally, let $\bar{\Omega} \equiv \Omega - \sum_{i=1}^{m-1} \varphi_i(N,c,\Omega)$. Then $\bar{\Omega}_l = \Omega_l - \sum_{i=1}^{m-1} c_{il}$. By full-award-out-consistency, $\varphi_{il}(N,c,\Omega) = \varphi_{il}(\{m,\ldots,n\},(c_m,\ldots,c_n),\bar{\Omega})$, for each $i \in N\backslash\{1,\ldots,m-1\}$. As $(\{m,\ldots,n\},(c_m,\ldots,c_n),\bar{\Omega}) \in \mathscr{C}(l-1)$, it follows that $\varphi_{il}(N,c,\Omega) = \varphi_{il}^{CEA}(N,c,\Omega)$, for each $i \leq m-1$, which completes the proof. $\square$

B Proof of Theorem 2

Proof. As mentioned above, it is straightforward to see that the constrained equal losses mechanism satisfies resource monotonicity and zero-award-out-consistency and, when combined with an individually rational exchange rule, leads to a social choice rule satisfying sacrifice-no-envy on $\mathscr{E}^*$. We focus on the converse implication. Let φ be an assignment mechanism satisfying resource monotonicity and zero-award-out-consistency that, when combined with an individually rational exchange rule, leads to a social choice rule satisfying sacrifice-no-envy on $\mathscr{E}^*$.

Let $\mathscr{C}^*(\ell) \equiv \mathscr{C}$ and $\mathscr{C}^*(0) \equiv \mathscr{C}^*$. For each $l \in \{1,\ldots,\ell-1\}$, let $\mathscr{C}^*(l) \equiv \{(N,c,\Omega) \in \mathscr{C} : \forall k \geq l+1, \forall i \in N, (\sum_{j\in N} c_{jk} - \Omega_k)/|N| \leq c_{ik}\}$.

We show that φ coincides with φ^{CEL} on $\mathscr{C}^*(l)$ for each $l = 0,1,\ldots,\ell$, using mathematical induction.

By Lemma 2, φ coincides with φ^{CEL} on $\mathscr{C}^*(0)$. Let $l \in \{1,\ldots,\ell-2\}$. Suppose, by induction, that φ coincides with φ^{CEL} on $\mathscr{C}^*(k)$. for each $k \leq l$. In what follows, we prove that φ coincides with φ^{CEL} on $\mathscr{C}^*(l)$, which completes the induction argument. Let $(N,c,\Omega) \in \mathscr{C}^*(l)$. For ease of exposition, assume, without loss of generality, that $N = \{1,\ldots,n\}$ and that claims on the l-th commodity are increasingly ordered, i.e., $c_{1l} \leq \cdots \leq c_{nl}$. For each $m \in \{1,2,\ldots,n-1\}$ let

$$\mathscr{C}(l)_m = \{(N,c,\Omega) \in \mathscr{C}(l) : \sum_{i=m+1}^{n} c_{il} - (n-m)c_{(m+1)l} \leq \Omega < \sum_{i=m}^{n} c_{il} - (n-m+1)c_{ml}\}.$$

Furthermore, let

$$\mathscr{C}(l)_0 = \{(N,c,\Omega) \in \mathscr{C}(l) : \Omega_l \geq \sum_{i=1}^{n} c_{il} - nc_{1l}\}.$$

Then, it is straightforward to see that $\mathscr{C}^*(l) = \bigcup_{m=0}^{n-1} \mathscr{C}(l)_m$. Note that, as $\mathscr{C}(l)_0 \subset \mathscr{C}^*(l-1)$, φ coincides with φ^{CEL} on $\mathscr{C}(l)_0$. We now show, by induction, that it

also happens for the remaining subsets of the partition, i.e., for each $m \in \{1, \dots, n\}$, $\varphi(N, c, \Omega) = \varphi^{CEL}(N, c, \Omega)$, for each $(N, c, \Omega) \in \mathscr{C}(l)_m$.

Case $m = 1$.

Let $(N, c, \Omega) \in \mathscr{C}(l)_1$. Then, $\varphi_l^{CEL}(N, \Omega, c) = (c_{1l}, \lambda, \dots, \lambda)$, where $\lambda = (\sum_{i=2}^{n} c_{il} - \Omega_l)/(n-1)$.

Let $\Omega' \in \mathbb{R}_{++}^{\ell}$ be such that $\Omega_l' \equiv \sum_{i \in N} c_{il} - nc_{1l}$ and, for each $l' \neq l$, $\Omega_{l'}' \equiv \Omega_{l'}$. Note that $(N, \Omega', c) \in \mathscr{C}^*(l-1)$ and

$$\varphi_{il}^{CEL}(N, \Omega', c) = (0, c_{2l} - c_{1l}, c_{3l} - c_{1l}, \dots, c_{nl} - c_{1l})$$

Thus, $\varphi(N, \Omega', c) = \varphi^{CEL}(N, \Omega', c)$. In particular, $\varphi_{1l}(N, \Omega', c) = 0$, whereas $\varphi_{il'}(N, \Omega', c) = c_{il'} - [\sum_{j \in N} c_{jl'} - \Omega_{l'}]/n$. As $\Omega_{l'}' = \Omega_{l'}$ for each $l' \neq l$, by resource monotonicity, $\varphi_{il'}(N, c, \Omega) = \varphi_{il'}(N, c, \Omega') = \varphi_{il'}^{CEL}(N, c, \Omega') = \varphi_{il'}^{CEL}(N, c, \Omega)$, for each $i \in N$. In particular, $\varphi_{il'}(N, c, \Omega) = c_{il'} - [\sum_{j \in N} c_{jl'} - \Omega_{l'}]/n$, for each $l' \geq l+1$ and $i \in N$. By resource monotonicity, and non-negativity, $\varphi_{1l}(N, \Omega, c) = 0$.

Now, consider the problem $(N \setminus \{1\}, c_{-i}, \bar{\Omega})$, where $\bar{\Omega} \equiv \Omega - \varphi_1(N, c, \Omega)$. It is not difficult to show that $(N \setminus \{1\}, c_{-i}, \bar{\Omega}) \in \mathscr{C}^*(l-1)$. Therefore, $\varphi(N \setminus \{1\}, c_{-1}, \bar{\Omega}) = \varphi^{CEL}(N \setminus \{1\}, c_{-1}, \bar{\Omega})$. By zero-award-out-consistency, $\varphi(N \setminus \{1\}, \bar{\Omega}, c_{-1}) = \varphi_{N \setminus \{1\}}(N, \Omega, c)$, and $\varphi_{N \setminus \{1\}}(N, \Omega, c) = \varphi^{CEL}(N \setminus \{1\}, c_{-1}, \bar{\Omega})$. Note that $\varphi_1(N, c, \Omega) = \varphi_1^{CEL}(N, c, \Omega)$. By consistency, $\varphi^{CEL}(N \setminus \{1\}, c_{-1}, \bar{\Omega}) = \varphi_{N \setminus \{1\}}^{CEL}(N, c, \Omega)$. Therefore, $\varphi_{N \setminus \{1\}}(N, \Omega, c) = \varphi_{N \setminus \{1\}}^{CEL}(N, \Omega, c)$, completing the proof for the case $m = 1$.

Case $m \to m+1$. Suppose then that, for each $m \in \{1, 2, \dots, n-1\}$, $\varphi(N, c, \Omega) = \varphi^{CEL}(N, c, \Omega)$, for each $(N, c, \Omega) \in \mathscr{C}(l)_m$. Let $(N, c, \Omega) \in \mathscr{C}(l)_{m+1}$. We aim to show that $\varphi(N, c, \Omega) = \varphi^{CEL}(N, c, \Omega)$. Note that, $\varphi_l^{CEL}(N, c, \Omega) = (0, \dots, 0, c_{(m+2)l} - \lambda, \dots, c_{nl} - \lambda)$, where $\lambda = \frac{\sum_{i=m+2}^{n} c_{il} - \Omega_l}{n-m-1}$. Let $t > 0$ be such that $c_{m+1,l} = \frac{\sum_{i=m+1}^{n} c_{il} - (\Omega_l + t)}{n-m}$. Let $\Omega' \in \mathbb{R}_{++}^{\ell}$ be such that $\Omega_l' \equiv \Omega_l + t$ and $\Omega_{l'}' = \Omega_{l'}$, for each $l' \neq l$. Then, $\varphi_l^{CEL}(N, c, \Omega) = (0, \dots, 0, c_{(m+2)l} - \lambda', \dots, c_{nl} - \lambda')$, where $\lambda' \equiv \frac{\sum_{i=m+1}^{n} c_{il} - (\Omega_l + t)}{n-m}$. Furthermore, $(N, c, \Omega') \in \mathscr{C}(l)_m$, and, therefore, $\varphi(N, c, \Omega') = \varphi^{CEL}(N, c, \Omega')$. In particular, for each $i \leq m+1$, $\varphi_{il}(N, \Omega', c) = 0$. Then, by resource monotonicity, $\varphi_{il}(N, \Omega, c) = 0 = \varphi_{il}^{CEL}(N, \Omega, c)$, for each $i \leq m+1$. Also, as $\Omega_{l'}' = \Omega_{l'}$ for each $l' \neq l$, then, by resource monotonicity, $\varphi_{il'}(N, \Omega', c) = \varphi_{il'}(N, \Omega, c)$ and $\varphi^{CEL}(N, \Omega', c) = \varphi^{CEL}(N, \Omega, c)$. From here, it follows that $\varphi_{il'}(N, \Omega, c) = \varphi_{il'}^{CEL}(N, \Omega, c)$, for each $i \in N$ and $l' \neq l$.

Finally, let $N' \equiv \{m+2, \dots, n\}$ and consider the problem $(N', \bar{\Omega}, c_{N'})$ where $\bar{\Omega} \equiv \Omega - \sum_{i=1}^{m+1} \varphi_i(N, c, \Omega) = \Omega - \sum_{i=1}^{m+1} \varphi_i^{CEL}(N, c, \Omega)$.

By zero-award-out-consistency,

$$\varphi_{jl}(N', \bar{\Omega}, c_{N'}) = \varphi_{jl}(N, \Omega, c),$$

for each $j \in N'$. Now, as $(N', \bar{\Omega}, c_{N'}) \in \mathscr{C}^*(l)$,

$$\varphi(N', \bar{\Omega}, c_{N'}) = \varphi^{CEL}(N', \bar{\Omega}, c_{N'}).$$

As $\varphi^{CEL}(N', \bar{\Omega}, c_{N'}) = \varphi^{CEL}_{N'}(N, \Omega, c)$,

$$\varphi_{jl}(N, \Omega, c) = \varphi^{CEL}_{jl}(N, \Omega, c),$$

for each $j \in N'$, which completes the proof. $\qquad\square$

C Allocating rights for greenhouse gas emission: A simple general equilibrium model

Consider a simple general equilibrium model of allocating GHG (green house gas) emission rights. There are two goods, energy x and money m for buying all other goods. Assume that energy can be produced using money under a constant returns to scale technology with a fixed marginal cost of $\kappa \geq 0$. Each consumer i is endowed with money M_i used to consume energy $\bar{x}_i$. The total energy consumption should be reduced to E where $E \leq \sum_{i \in N} \bar{x}_i$.

Consider a scheme of cap-and-trade with the total energy consumption capped at E. In order for each $i \in N$ to consume x_i units of energy, i needs to purchase the same units of permit at price r. An allocation $(x_i, m_i)_{i \in N}$ is *feasible* if $\sum_i x_i \leq E$ and $\sum_{i \in N} m_i + \kappa \sum_{i \in N} x_i \leq \sum_{i \in N} M_i$, where the first inequality means that the aggregate energy consumption (also quantity of permit demand) does not exceed the aggregate energy supply (also permit supply) and the second inequality means that the total money demand (by consumers and the energy producer who uses money to produce energy) does not exceed the total money supply. When $i \in N$ has initial *permit endowment* e_i, i's budget constraint can be written as $(\kappa + r)x_i + m_i \leq re_i + M_i$. A *cap-and-trade equilibrium* is composed of permit price r^* and allocation $(x_i^*, m_i^*)_{i \in N}$ such that $(x_i^*, m_i^*)_{i \in N}$ is feasible and, for each $i \in N$, (x_i^*, m_i^*) maximizes i's preferences over the set of bundles satisfying i's budget constraint.

This model of cap-and-trade can be transformed into our model as follows. Let $p \equiv \kappa + r$. Then i's budget constraint with i's permit endowment e_i is written

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as $px_i + m_i \leq pe_i + M_i - \kappa e_i$. Let $c_i \equiv (\bar{x}_i, M_i)$ be i's claims profile. Let $\Omega \equiv (E, \sum_{i \in N} M_i - \kappa E)$ be the social endowment of the two goods, permit (or energy) and money (net of inputs used to produce E units of energy). Now, with this social endowment, associating individual endowments (property rights over permit and money) $\omega_i \equiv (\omega_{ie}, \omega_{im})$ ($\sum_{i \in N} \omega_i = \Omega$), any Walrasian equilibrium under the initial endowments with price p^* and allocation (x_i^*, m_i^*) coincides with the cap-and-trade equilibrium under the permit allocation $(\omega_{ie})_{i \in N}$, with permit price $r^* \equiv p^* - \kappa$.

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