Equity in Production Economies with Dedicated Factors

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Abstract

In this paper I consider the problem of determining an equitable allocation of resources in production economies in which factors must be dedicated to production. That is, although they might be used to produce various consumption goods, they cannot be consumed directly. I first establish that in such economies, envy-free and efficient allocations exist under standard assumptions, whereas that is not the case for general production economies [Pazner and Schmeidler (1974)]. However, I argue that no-envy is not a suitable notion of fairness in this context. I then introduce two new concepts of fairness and I explore the compatibility of each with Pareto efficiency. A novel aspect of these is that they include the influence of each agent on the factor allocation decision as a component of fairness. Finally, I provide an example to demonstrate the new concepts.

1 Introduction

In this paper I consider the problem of determining an equitable allocation of resources in production economies in which factors must be dedicated to production and cannot be consumed directly.[1] Given an endowment of such factors, the basic questions are: How should they be employed in producing consumption goods? And how should the output be allocated among the consumers?[2] Equitable solutions to the latter have been proposed in the literature. A novel aspect of the present work is that it includes the influence of each agent on the factor allocation decision as a component of fairness. Here, I introduce two new concepts of fairness, namely, *opportunity-envy-freedom*[3] and *compromise value*. I also explore the compatibility of these with Pareto efficiency. Whereas the former is necessarily consistent with efficiency only in the case of two consumers, the latter is consistent more generally.

Historically, the problem of formulating an appropriate ordinal[4] equity criterion for use in production economies has proven to be difficult. The most common notion is *no-envy* (Foley, 1967, and Tinbergen, 1946), where an allocation is said to be envy-free if no agent prefers the consumption bundle of another agent to its own. Under very weak restrictions this notion of fairness is compatible with Pareto efficiency in exchange economies; that is, generally there are allocations that satisfy both criteria.[5] However, as demonstrated

[1] The focus here is on extractive resources and other raw materials, and, in particular, it excludes labor. However, it might be extended to include intermediate or manufactured inputs. Also, while the results and concepts discussed here are for economies with dedicated inputs, it is an open question whether they can be extended to economies with dual-use inputs.

[2] An appropriate interpretation is that the aggregate endowment of factors is commonly owned.

[3] This differs from the notion of no-envy of opportunities introduced by Thomson (1994). (See also Thomson and Varian, 1985, and Thomson, 2016.)

[4] That is, one that does not require interpersonal utility comparisons.

[5] For example, under standard assumptions, a competitive equilibrium from equal division is envy-free and Pareto efficient. (See Varian, 1974.)

by Pazner and Schmeidler (1974), that need not be the case for economies involving production. Faced with this, one recourse is to identify subclasses of production economies for which no-envy and efficiency are compatible, and thus to determine the scope of the problem.[6] Another is to consider other notions of fairness.[7] In this paper I do both. I first show that in this class of economies envy-free and efficient allocations exist under standard assumptions. But I argue that this notion of fairness is wholly unsuitable for the present context.

Instead, I propose two new notions of fairness, and I explore their compatibility with efficiency. To motivate the first, suppose one consumer, with a taste bias for good 1, could dictate how factors were to be allocated and, subsequently, it would receive most of the resulting product. Then another consumer with a taste bias for good 2 might not be envious of the outcome since s/he does not like good 1 as much as good 2, but nevertheless devoting a disproportionate amount of resources to producing good 1 would be unfair. Thus, I would argue that it is not what consumer 1 *would do* with the factors that is the source of envy but what it *could do* with them. This leads to a notion I call *opportunity-envy-freeness*, where no agent prefers an outcome that another agent might feasibly achieve.

To motivate the second notion, one might argue that as a principle of fairness agents should have equal voice in determining how factors are allocated or, equivalently, what outputs are produced. However, it is difficult for a consumer to advocate that more factors should be devoted to, say, the production of good 1 and fewer to good 2 if they do not know how such changes will affect them personally. For example, suppose more of good 1 is

[6] Cf., Piketty (1994) who has shown that if the technology is linear, then a sufficient condition for the existence of envy-free and efficient allocations is that preferences satisfy a Spence-Mirrlees-like single-crossing condition.

[7] Such notions include egalitarian equivalence (Pazner and Schmeidler, 1978a), income fairness (Pazner and Schmeidler, 1978b, and Varian, 1974), and wealth fairness (Varian, 1974), each of which is problematic. See Thomson and Varian (1985), Arnsperger (1994) and Thomson (2016).

produced but they get less of it. Therefore, it is necessary that agents know how output will be divided in order to express such a preference. Suppose, therefore, that consumers were to agree ex ante on a "fair" method for distributing the output. Then, conditional on such an agreement, I consider each consumer's most preferred production plan. A *weighted compromise value* reconciles such disparate "demands" by considering their weighted average. This appropriately takes into consideration each agent's opinion in determining the allocation of factors and hence the production of outputs. Together with the agreed upon "fair distribution of outputs," this constitutes a new notion of fair allocation.

The main results of the paper are the following. First, focusing on the case in which the equal ownership competitive mechanism is used to resolve both the output division problem as well as the overall allocation problem, I show that such an outcome is necessarily opportunity-envy-free and efficient only when there are two consumers; with three or more consumers, that need not be the case. In particular, an equal ownership competitive equilibrium need not be opportunity-envy-free. Turning to the second notion, I show more generally that when preferences and technologies are sufficiently well-behaved[8], then there exists an efficient weighted compromise value. Thus, as described above, suppose agents were to agree to the use of a particular output division rule that resolves the exchange problem equitably and efficiently. Then, conditional on the subsequent use of such a procedure, if each consumer was to announce its most preferred production vector, then there is a compromise among such demands that is both equitable and efficient for the problem as a whole.

The paper is organized as follows. Section 2 contains the basic model. The next two sections, 3 and 4, study the notions of opportunity-envy-freeness and compromise value, respectively. In Section 5, I provide a numerical

[8]Specifically, this includes satisfying the Inada conditions and that the MRS varies monotonically.

example to demonstrate the previous concepts. Finally, Section 6 contains a brief conclusion and questions for future research.

2 Model

Let $N = \{1, ..., n\}$ be the set of consumers, $L = \{1, ..., \ell\}$ the set of factors of production, and $M = \{1, ..., m\}$ the set of consumption or manufactured goods. Consumer i has preferences over consumption bundles $x^i = (x^i_1, ..., x^i_m)$ which are represented by the utility function $u^i(x^i)$. Let $\mathbf{x} = (x^1, ..., x^n)$.

Assumption $A.1$: For all $i \in N$, $u^i : \mathbb{R}^m_+ \to \mathbb{R}$ is $\mathcal{C}^2$, strictly increasing, strictly quasiconcave, and satisfies the Inada conditions.[9]

For $j, j' \in M$, let $MRS^i_{jj'}(x^i) = \frac{\frac{\partial u^i}{\partial x^i_j}(x^i)}{\frac{\partial u^i}{\partial x^i_{j'}}(x^i)}$.

The economy is endowed with a positive quantity of each of the factors, $\overline{y} \in \mathbb{R}^\ell_{++}$, and these are commonly owned. Consumption good k is manufactured according to the technology $f^j(y^j)$, where $y^j = (y^j_1, ..., y^j_\ell)$. Let $\mathbf{y} = (y^1, ..., y^m)$, $f = (f^1, ..., f^m)$ and $f(\mathbf{y}) = (f^1(y^1), ..., f^m(y^m))$.

Assumption $A.2$: For all $j \in M$, $f^j : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ is $\mathcal{C}^2$, strictly increasing, strictly concave and $f^j(0) = 0$.

For $k, k' \in L$, let $MRTS^j_{kk'}(y^j) = \frac{\frac{\partial f^j}{\partial y^j_k}(y^j)}{\frac{\partial f^j}{\partial y^j_{k'}}(y^j)}$.

An *allocation* is a list $\mathbf{z} = (\mathbf{y}, \mathbf{x}) = ((y^1, ..., y^m), (x^1, ..., x^n)) \in \mathbb{R}^{m\ell}_+ \times \mathbb{R}^{nm}_+$ consisting of an "input allocation" $\mathbf{y}$ and an "output or consumption allocation" $\mathbf{x}$. $\mathbf{z}$ is feasible if $\Sigma_{k \in M} y^k \leq \overline{y}$ ("$\mathbf{y}$ is feasible") and $\Sigma_{i \in N} x^i \leq f(\mathbf{y})$

[9] Formally, $\lim\limits_{x^i_k \to \infty} \frac{\partial u^i}{\partial x^i_k} = 0$ and $\lim\limits_{x^i_k \to 0} \frac{\partial u^i}{\partial x^i_k} = \infty$. Thus, for each consumer some small positive amount of each consumption good is indispensable.

("$\mathbf{x}$ is feasible at $\mathbf{y}$").[10] Let $\mathbf{Z}$ denote the set of feasible allocations. (Thus, $\mathbf{Z}$ can be decomposed into $\mathbf{Z}^y$ and $\mathbf{Z}^x(\mathbf{y})$, where $\mathbf{Z}^y$ denotes the feasible $\mathbf{y}$ and $\mathbf{Z}^x(\mathbf{y})$ denotes the feasible $\mathbf{x}$ at $\mathbf{y}$.) Also, let $\mathbf{F}$ denote the production possibility set, that is $\mathbf{F} = \{x \in \mathbb{R}^m_+ \mid x \leq f(\mathbf{y})$ for some $\mathbf{y} \in \mathbf{Z}^y\}$, and let $\overline{\mathbf{F}}$ denote the (undominated) boundary of $\mathbf{F}$, i.e., the production possibility frontier.[11] Note that since f is strictly concave, the inverse mapping $f^{-1}: \overline{\mathbf{F}} \rightarrow \mathbf{Z}^y$ is well-defined.

For $j, j' \in M$, let $MRT_{jj'}(x) = \frac{\frac{\partial \overline{\mathbf{F}}}{\partial x_j}(x)}{\frac{\partial \overline{\mathbf{F}}}{\partial x_{j'}}(x)}$.

An allocation $\mathbf{z} \in \mathbf{Z}$ is *Pareto efficient*, or simply *efficient*, if there is no $\mathbf{z}' = (\mathbf{y}', \mathbf{x}') \in \mathbf{Z}$ such that $u^i(x'^i) \geq u^i(x^i)$ for all i and $u^i(x'^i) > u^i(x^i)$ for some i. Let $\mathbf{P}$ denote the efficient set. An allocation $\mathbf{z} \in \mathbf{Z}$ is *envy free* if $u^i(x^i) \geq u^i(x^j)$ for i, j. Let $\mathbf{V}$ denote the envy-free set.

In the case of exchange economies, Foley (1967), Schmeidler and Vind (1972) and Kolm (1972) noted that envy-free and efficient allocations exist under standard assumptions (continuity, convexity and nonsatiation) since a competitive equilibrium from equal division satisfies both criteria.[12] A similar result holds here. Formally,

Definition 1 *An equal ownership competitive equilibrium (EOCE) consists of a price vector* $(\mathbf{p}^*, \mathbf{w}^*) \in \mathbb{R}^m_+ \times \mathbb{R}^\ell_+$, $(\mathbf{p}^*, \mathbf{w}^*) \neq \mathbf{0}$, *and an allocation* $\mathbf{z}^* = (\mathbf{y}^*, \mathbf{x}^*) \in \mathbb{R}^{nm}_+ \times \mathbb{R}^{m\ell}_+$ *such that:*

(i) $\Sigma_{k \in M} y^{*k} \leq \overline{y}$ *and* $\Sigma_{i \in N} x^{*i} \leq f(\mathbf{y}^*)$,

(ii) y^{*k} *maximizes* $p^*_k f^k(y^k) - \mathbf{w}^* y^k$ *over* $y^k \in \mathbb{R}^\ell_+$, *for all* $k \in M$,

(iii) x^{*i} *maximizes* $u^i(x^i)$ *over* $B^i(\mathbf{p}^*, \mathbf{w}^*) \equiv \{x^i \in \mathbb{R}^m_+ \mid \mathbf{p}^* x^i \leq \mathbf{w}^* \frac{\overline{y}}{n} + \frac{1}{n} \sum_{k \in M} \left(p^*_k f^k(y^{*k}) - \mathbf{w}^* y^{*k}\right)$, *for all* $i \in N$.

[10] Vector inequalities are denoted $\geq$, $>$, and $\gg$, where $x \geq x'$ if $x_i \geq x'_i$ for all i, $x > x'$ if $x \geq x'$ and $x \neq x'$, and $x \gg x'$ if $x_i > x'_i$ for all i.

[11] Although a slight abuse of notation, I use $\overline{\mathbf{F}}$ to denote both the production possibility frontier (a set) as well as the (implicit) transformation function $\overline{\mathbf{F}}(x) = 0$ describing the points on the frontier.

[12] Varian (1974) showed that it is possible to drop the convexity assumption under the very weak condition that there are no two weakly efficient allocations which all agents regard as indifferent.

Clearly, an EOCE is envy-free since all agents have the same budget set. Moreover, it is efficient by the First Welfare Theorem. However, this notion of fairness is wholly unsuitable for the class of economies under consideration. Consider, the following example. Let $n = m = 2$ and suppose $u^1(x_1^1, x_2^1) = x_1^1$ and $u^2(x_1^2, x_2^2) = x_2^2$. Then $((\overline{y}, 0), ((f^1(\overline{y}), 0), (0, 0)))$ is envy-free and efficient, that is, the entire endowment is devoted to producing good 1 and person 1 consumes the entire output.[13] Clearly, the problem in this case is that $(\overline{y}, 0)$ is not a fair use of the factors of production.

What is a fair use of resources when the agents have different tastes, for example, if 1 has a taste bias in favor of good 1 (but perhaps not as extreme as in the above example) and 2 has a taste bias in favor of good 2? Seemingly, person 1 would prefer that more of the resources be devoted to good 1 production than good 2, and person 2 would prefer the opposite. However, without knowing how the output changes would affect each agent, such a statement is untenable. It might be the case that person 1 would be better off with a larger portion of a smaller pie, or it might receive a better bundle even if production were skewed in favor of good 2. Consequently, to evaluate the effect of output changes, it is necessary to know how outputs would be distributed. For example, if the product is always divided equally between the agents, then it is easy for person 1 to compare alternative output bundles.

To operationalize this, I decompose the issue of fairness into two components: first, determining an equitable distribution of factors, and, second, determining an equitable division of outputs. I begin with the former, treating the output division problem as given, and then return to consider the latter.

Definition 2 *An output division rule is a mapping $\phi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^{nm}$ which associates with each vector of consumption goods (outputs) x some feasible*

[13] Note that the problem is not exclusive to such extreme cases since, by continuity, small perturbations of the preferences would yield similar but interior results.

division $\phi(x) = (\phi^1(x), ..., \phi^n(x)) \in \mathbb{R}^{nm}_+$ *such that* $\Sigma_{i\in N}\phi^i(x) = x$.[14]

Let Φ denote the class of output division rules, and take $\phi \in \Phi$ as given (for now). Let $\mathbf{Z}^\phi \subset \mathbf{Z}$ denote the set of allocations $\mathbf{z} = (\mathbf{y}, \mathbf{x})$ that are feasible under ϕ, i.e., for which $\mathbf{x} = \phi(f(\mathbf{y}))$. Also, define $\Lambda^{i\phi} := \{\phi^i(x) \in \mathbb{R}^m_+ \mid x \in \mathbf{F}\}$. $\Lambda^{i\phi}$ is the set of consumption bundles that i can achieve under ϕ at some feasible output vector x, or simply i's *achievable set under* ϕ.[15] $\overline{\Lambda}^{i\phi}$ denotes the boundary of $\Lambda^{i\phi}$. Figure 1 depicts the production possiblity set and achievable sets for the case in which $n = m = 2$ (for typical ϕ). Note that for each point on the frontier of $\mathbf{F}$ (e.g., c), there is a feasible division into points on the respective frontiers of $\Lambda^{A\phi}$ and $\Lambda^{B\phi}$ (e.g., b and c).

(insert Figure 1)

Thus, $\phi(x)$ describes how x is allocated among the consumers. Moreover, knowing ϕ, the agents can anticipate the effects of any change in production. Two questions remain: (1) How is the allocation of factors is to be determined? and (2) What is a suitable division rule? I will discuss two possibilities.

3 No Envy

Once the aggregate production vector $x = f(\mathbf{y})$ has been determined, the second half of the problem, namely, allocating these quantities between the

[14] To avoid ambiguity, I make the strong assumption that the output division rule is single-valued. Otherwise, if the same allocation procedure might result in multiple outcomes, then agents would have to assess the expected effect of a change in output on their consumption. Similarly, if the output division rule were unknown, then agents would have to assess the likelihood of different divisions.

[15] For later reference, note that under the Walrasian mechanism, agent i's achievable, or opportunity, set *does not* correspond to its budget set but rather to the set of consumption bundles it might receive at some Walrasian equilibrium, that is, to the projection of the graph of the Walrasian correspondence onto i's consumption set.

consumers, is simply an exchange problem. In order to solve the overall production problem equitably and efficiently, it is necessary (but not sufficient) that ϕ solve the second half of the problem equitably and efficiently as well. To distinguish between efficiency in exchange and overall efficiency, I will refer to the former as "x-efficiency."

Let Φ^e denote the class of equitable and x-efficient exchange solutions. One example of an output division rule in Φ^e would be a selection from the envy-free and x-efficient correspondence. Other equity notions that are compatible with x-efficiency include *egalitarian equivalence* (Pazner and Schmeidler (1978a)) and *equal (average) shadow wealth* (Kranich (2015)). (See Thomson (2016) for a survey.) In this section, I consider no envy.

Returning to the previous extreme example, suppose agent 1 was able to dictate how $\overline{y}$ is used. Then 1 would indeed use all of the factors to produce good 1, and efficiency would require that s/he receive the entire output. As mentioned, this is envy-free. Thus, it is not what 1 *would do* with $\overline{y}$ that is enviable, but rather what 1 *could do*. Formally, I will say *i envies j's opportunities (under ϕ) at $x \in$* $\mathbf{F}$ if there exists $x' \in \mathbf{F}$ such that $u^i(\phi^j(x')) > u^i(\phi^i(x))$, that is, there is an achievable bundle for j that i prefers to it's own, or i would be better off with j's achievable set than with $\phi^i(x)$.

Definition 3 *Given $\phi \in \Phi^e$, an allocation $\mathbf{z} = (\mathbf{y}, \mathbf{x}) \in \mathbf{Z}^\phi$ is ϕ-opp-envy-free if neither agent envies the other's opportunities at $f(\mathbf{y})$.*[16,17] $\mathbf{z}$ *is opp-envy-free if there exists $\phi \in \Phi^e$ for which it is ϕ-opp-envy-free.*

[16] Since $x^j = \phi^j(x) \in \Lambda^{j\phi}$, a ϕ-opp-envy-free allocation is necessarily envy-free.

[17] Regarding the previous example, under the division rule which allocates the entire output to A, B is envious of A's opportunities, even though it is not envious of A's choice or bundle, $(f(\bar{L}, \bar{K}), 0)$. The key issue is that some of the factors *could be* allocated to y production, and if A would then receive the entire output, B would be envious.

3.0.1 Equal Ownership Competitive Equilibrium

In this subsection, I focus on the competitive (Walrasian) mechanism from equal division as a canonical fair allocation procedure both for the output allocation problem (division rule) as well as the overall problem. For completeness, when restricted to the output division (exchange) problem, the EOCE mechanism reduces to the following.

Definition 4 *Given the aggregate output* $x \in \mathbb{R}^m_{++}$*, an equal income competitive x-equilibrium (EICE$_x$) consists of prices and quantities* $\mathbf{p}^* \in \mathbb{R}^m_+$*,* $\mathbf{p}^* \neq 0$*, and* $\mathbf{x}^* \in \mathbb{R}^{nm}_+$ *such that:*

(i) $\Sigma_{i \in N} x^{*i} \leq x$*, and*

(ii) x^{*i} *maximizes* $u^i(x^i)$ *over* $B^i(\mathbf{p}^*) \equiv \{x^i \in \mathbb{R}^m_+ \mid \mathbf{p}^* x^i \leq \mathbf{p}^* \frac{x}{n}\}$*, for all* $i \in N$*.*

Let ϕ^{EI} denote the mapping which associates with each exchange problem its EICE$_x$ allocation.[18] As mentioned above, $\phi^{EI} \in \Phi^e$. Next, I establish that for the two-agent case, an EOCE is not only envy-free and efficient, but it is opp-envy-free.

Theorem 5 *For* $n = 2$*, if* $((\mathbf{p}^*, \mathbf{w}^*), \mathbf{z}^*)$ *is an EOCE, then* $\mathbf{z}^*$ *is* ϕ^{EI}*-opp-envy-free and efficient.*

Proof. Let $n = 2$ and let $((\mathbf{p}^*, \mathbf{w}^*), \mathbf{z}^*)$ be an EOCE. By the First Welfare Theorem, $\mathbf{z}^*$ is efficient. To show that $\mathbf{z}^*$ is ϕ^{EI}-opp-envy-free, suppose to the contrary, that agent 1 envies 2's opportunities at $x^* = f(\mathbf{y}^*)$. Then there exits $x' \in \mathbf{F}$ such that $u^1(x'^2) > u^1(x^{*1})$, where $x'^2 = \phi^{EI2}(x')$. Therefore,

$$\mathbf{p}^* x'^2 > \mathbf{p}^* x^{*1} = \mathbf{p}^* x^{*2}. \tag{1}$$

[18] Again to avoid the issue of assessing the likelihood of different competitive equilibria, I will assume that such an equilibrium is unique.

Let $\mathbf{p}'$ denote the equilibrium prices associated with $\phi^{EI}(x')$. Since x'^1 is maximal over $B^1(\mathbf{p}') = B^2(\mathbf{p}')$, it must be the case that $u^1(x'^1) \geq u^1(x'^2)$. Hence, $u^1(x'^1) > u^1(x^{*1})$. Therefore,

$$\mathbf{p}^* x'^1 > \mathbf{p}^* x^{*1}. \tag{2}$$

Adding the left- and right-hand sides of (1) and (2), respectively,

$$\mathbf{p}^* x'^1 + \mathbf{p}^* x'^2 > \mathbf{p}^* x^{*1} + \mathbf{p}^* x^{*2}. \tag{3}$$

Since $x' \in \mathbf{F}$, there is a feasible $\mathbf{y}'$ such that $x' \leq f(\mathbf{y}')$. Hence, aggregating the budget constraints, $\mathbf{p}^* x'^1 + \mathbf{p}^* x'^2 \leq \mathbf{p}^* f(\mathbf{y}')$ and $\mathbf{p}^* x^{*1} + \mathbf{p}^* x^{*2} = \mathbf{p}^* f(\mathbf{y}^*)$. But then $\mathbf{p}^* f(\mathbf{y}') - \mathbf{w}^* \overline{y} > \mathbf{p}^* f(\mathbf{y}^*) - \mathbf{w}^* \overline{y}$, which violates the profit maximization condition (ii) aggregated across k. ■

This is easily illustrated for the 2-good case.[19] (See Figure 2.) Following the above argument, consider the common budget line that both agents face under EOCE at output prices p^* and denote the consumption allocation $\mathbf{x}^*$. If agent 1 were to be envious of 2's opportunities, then there is another EOCE with output prices p' and consumption allocation $\mathbf{x}'$ such that 1 would be better off with 2's bundle x'^2. But since x'^2 is in 1's budget set at p' and x'^1 is utility maximizing, $u^1(x'^1) \geq u^1(x'^2) > u^1(x^{*1})$. Again, this implies $\mathbf{p}^* x'^2 > \mathbf{p}^* x^{*1} = \mathbf{p}^* x^{*2}$, $\mathbf{p}^* x'^1 > \mathbf{p}^* x^{*1}$ and thus $\mathbf{p}^* x'^1 + \mathbf{p}^* x'^2 > \mathbf{p}^* x^{*1} + \mathbf{p}^* x^{*2}$.

(insert Figure 2)

Unfortunately, the previous theorem does not extend beyond the two-agent case, as the following graphical argument demonstrates.[20] (See Figure 3.) Figures 2 and 3 differ only in that the latter includes a third agent. However, this is sufficient to undermine the previous argument. Because

[19] This will also prove useful for the 3-agent case discussed below.

[20] Thus, with different EICEs associated with different output vectors each is envy-free and x-efficient and yet an individual might strictly prefer one equilibrium to another.

$\mathbf{p}^* x'^3 < \mathbf{p}^* x^{*1} = \mathbf{p}^* x^{*3}$, it is no longer the case that opportunity envy necessarily would violate the profit maximization condition. Indeed, there is nothing aberrational about the example in which 1 is envious of 2's opportunities.

(insert Figure 3)

Closer observation of the previous proof reveals that, here, it is not the fact that 1 envies 2's opportunities that is the source of the difficulty but rather that agent 1 would prefer the equilibrium allocation $\mathbf{z}' = (\mathbf{y}', \mathbf{x}')$ (at equilibrium prices $(\mathbf{p}', \mathbf{w}')$) to the allocation $\mathbf{z}^* = (\mathbf{y}^*, \mathbf{x}^*)$ at $(\mathbf{p}^*, \mathbf{w}^*)$. That is, under ϕ^{EI}, if 1 envies 2's opportunities at $x^* = f(\mathbf{y}^*)$, then 1 would prefer it's own bundle at $\mathbf{z}'$ even more, independent of 2's treatment. This suggests the following alternative approach, which considers each agent's best bundle in $\Lambda^{i\phi^{EI}}$.

4 Compromise Value

In this section, I again begin by treating the output division rule ϕ as given. Moreover, I assume that ϕ^i is invertible and denote the inverse by ϕ^{i-1}. Then ϕ^{i-1} associates with each consumption bundle $x^i \in \Lambda^{i\phi}$ the aggregate production vector $x \in \mathbf{F}$ that generates it under ϕ.[21] In this way, i's preferences over x^i induce preferences over $\mathbf{F}$ conditional on ϕ. Denoting the latter by $\succsim_{i\phi}$, $x \succsim_{i\phi} x'$ if and only if $u^i(\phi^i(x)) \geq u^i(\phi^i(x'))$. Let $x^{i\phi} = \underset{x^i \in \Lambda^{i\phi}}{\arg\max} u^i(x^i)$. Then $\phi^{i-1}(x^{i\phi})$ is i's most preferred output bundle in $\mathbf{F}$ under ϕ. The question is then how to reconcile the agents' different preferred output vectors $\phi^{1-1}(x^{1\phi}), ..., \phi^{n-1}(x^{n\phi})$, or "demands."

First, let Δ^{m-1} denote the $(m-1)$-dimensional unit simplex $\{x \in \mathbb{R}^m_+ \mid \sum_{j \in M} x_j = 1 \text{ and } x_j \geq 0 \text{ for all } j \in M\}$, and note that under assumption $A.2$, $\overline{\mathbf{F}}$ is isomorphic to Δ^{m-1} under the projection $\pi(x) = (\frac{x_1}{\Sigma_j x_j}, ..., \frac{x_m}{\Sigma_j x_j})$.

[21] Here, too, it is assumed that $\phi^{i-1}(x^i)$ is single-valued or that ϕ^i is a bijection.

One way to reconcile the demands is to consider a weighted average. However, given the strict convexity of the production set, a convex combination of $\phi^{1-1}(x^{1\phi}), ..., \phi^{n-1}(x^{n\phi})$ is generally in the interior of $\mathbf{F}$. Therefore, although a slight abuse of terminology, I will say $x_\alpha^{*\phi}$ is a *convex combination of* $\phi^{1-1}(x^{1\phi}), ..., \phi^{n-1}(x^{n\phi})$, for $\alpha \in \Delta^{n-1}$, if $x_\alpha^{*\phi} = \pi^{-1}(\Sigma_i \alpha_i \pi(\phi^{i-1}(x^{i\phi})))$. That is, $x_\alpha^{*\phi}$ is obtained by projecting $\phi^{1-1}(x^{1\phi}), ..., \phi^{n-1}(x^{n\phi})$ onto Δ^{m-1}, constructing the convex combination of $\pi(\phi^{1-1}(x^{1\phi})), ..., \pi(\phi^{n-1}(x^{n\phi}))$ according to α, and projecting $\Sigma_i \alpha_i \pi(\phi^{i-1}(x^{i\phi}))$ back onto $\overline{\mathbf{F}}$. Let $\mathbf{H}^\phi$ denote the convex hull of $\phi^{1-1}(x^{1\phi}), ..., \phi^{n-1}(x^{n\phi})$ in $\overline{\mathbf{F}}$, i.e., $\mathbf{H}^\phi = \{x_\alpha^{*\phi} \in \overline{\mathbf{F}} \mid \alpha \in \Delta^{n-1}\}$.

Definition 6 *The α-weighted ϕ-compromise value is given by $(f^{-1}(x_\alpha^{*\phi}), \phi(x_\alpha^{*\phi}))$. That is, it consists of the input allocation $f^{-1}(x_\alpha^{*\phi})$ that produces outputs $x_\alpha^{*\phi}$ together with the output allocation $\phi(x_\alpha^{*\phi})$ that results from using ϕ to allocate $x_\alpha^{*\phi}$.*

Clearly, such an allocation exists and is unique for each α. I now turn to the question of the consistency between this notion of fairness and Pareto efficiency.

4.1 Compromise Value and Efficiency

In this subsection I establish sufficient conditions under which there exists an efficient weighted compromise value. I focus on environments in which two natural conditions are satisfied. First, as mentioned previously, it is possible that an agent with a taste bias for one good might receive a better bundle if production were skewed in a different direction. The first condition rules out such perverse cases.

Definition 7 *$\phi \in \Phi$ is regular if for all $x, x' \in \overline{\mathbf{F}}$ and for all $k \in M$, $x'_k > x_k$ if and only if $\phi_k^i(x') > \phi_k^i(x)$, for all $i \in N$.*

Under regularity, the direction of improvement for agent i in $\Lambda^{i\phi}$ according to u^i must coincide with the direction of improvement in $\mathbf{F}$ according to

$\succsim_{i\phi}$. Hence, an agent with a taste bias for one good would prefer that more resources be devoted to the production of that good.

The second condition (on u^i) is that $MRS^i_{jj'}(x^i)$ is monotonic. That is, if $MRS^i_{jj'}(\cdot)$ decreases from x^i to x'^i, then it decreases throughout the interval (x^i, x'^i). This is implied by homotheticity but is significantly weaker. It suggests that the trade-offs between goods behave consistently and do not oscillate. Formally,

Definition 8 *u^i has monotone curvature if for all $x^i, x'^i \in \mathbb{R}^m_+$ and for all $j, j' \in M$, if $MRS^i_{jj'}(x^i) > MRS^i_{jj'}(x'^i)$, then $MRS^i_{jj'}(x^i) > MRS^i_{jj'}(x''^i) > MRS^i_{jj'}(x'^i)$, for all $x''^i \in (x^i, x'^i)$.*

Theorem 9 *If $\phi \in \Phi^e$ is regular and continuous and if u^i has monotone curvature for all $i \in N$, then there exist weights $\alpha \in \Delta^{n-1}$ such that $(f^{-1}(x^{*\phi}_\alpha), \phi(x^{*\phi}_\alpha))$ is Pareto efficient.*

In other words, there is some weighted average of the agents' most preferred output bundles which is efficient to produce and to distribute according to ϕ.

Proof Suppose ϕ and u^i are as stated, for all $i \in N$. Given the convexity of preferences and concavity of the technologies, the necessary and sufficient conditions for an allocation $\mathbf{z} = (\mathbf{y}, \mathbf{x})$ to be Pareto efficient are (1) $MRTS^j_{kk'}(y^j) = MRTS^{j'}_{kk'}(y^{j'})$ for all $k, k' \in L$ and for all $j, j' \in M$; (2) $MRS^i_{jj'}(x^i) = MRS^{i'}_{jj'}(x^{i'})$ for all $j, j' \in M$ and for all $i, i' \in N$; and (3) $MRS^i_{jj'}(x^i) = MRT_{jj'}(x)$ for all $j, j' \in M$, where $x = f(\mathbf{y})$.

For any $\alpha \in [0,1]$, $x^{*\phi}_\alpha \in \overline{\mathbf{F}}$, which ensures that (1) is satisfied at the input allocation $f^{-1}(x^{*\phi}_\alpha)$. Moreover, since ϕ is x-efficient, (2) is satisfied at $\phi(x^{*\phi}_\alpha)$ for all α. It remains to be shown that there is some $\alpha \in [0,1]$ such that (3) holds at $x^{*\phi}_\alpha$ and $\phi(x^{*\phi}_\alpha)$. The proof proceeds as follows. First, I search for a bundle $x^i \in \overline{\Lambda}^{i\phi}$ such that $MRS^i_{jj'}(x^i) = MRT_{jj'}(\phi^{i-1}(x^i))$ for all $j, j' \in M$. This is sufficient to ensure that (3) holds since, by construction, each such x^i is paired with other bundles $x^{i'} \in \overline{\Lambda}^{i'\phi}$ for $i' \neq i$ such that the output allocation is $\mathbf{x} = \phi(\phi^{i-1}(x^i))$. I then show that such an output vector must lie between the preferred vectors $x^{i\phi}$ and hence correspond to $x^{*\phi}_{\alpha}$ for some α.

First, the j^{th} face of Δ^{m-1} is given by $\Delta^{m-1}_j = \{v \in \Delta^{m-1} \mid v_j = 0\}$. Δ^{m-1} is taken as the set of normal vectors in $\mathbb{R}^m_+$. Also, under the stated assumptions, $\overline{\Lambda}^{i\phi}$ and $\overline{\mathbf{F}}$ are both isomorphic to Δ^{m-1}. Correspondingly, I refer to the j^{th} *edge* of $\overline{\Lambda}^{i\phi}$ as the image of the j^{th} face of Δ^{m-1} under the isomorphism (projection).

Since preferences satisfy the Inada conditions, approaching the j^{th} edge of $\overline{\Lambda}^{i\phi}$, v_j approaches zero for any normal vector v to an indifference surface for i. Moreover, since ϕ is x-efficient, v is normal to the indifference surfaces of the other agents as well at their corresponding consumption bundles. The set of such normal vectors as x^i spans $\overline{\Lambda}^{i\phi}$, which I refer to as the *normal space of u^i over* $\overline{\Lambda}^{i\phi}$, is the entire set Δ^{m-1}.[22] It remains to be established that there is some $v^* \in \Delta^{m-1}$ that is normal to both the indifference surface at $x^i \in \overline{\Lambda}^{i\phi}$ and to $\overline{\mathbf{F}}$ at $\phi^{i-1}(x^i)$. The proof involves the following fixed point argument.

Consider a vector $v^\circ \in \Delta^{m-1}$. Under Assumption A.1 there exists $x^{i\circ} \in \overline{\Lambda}^{i\phi}$ such that v° is normal to the indifference surface at $x^{i\circ}$.[23] Moreover, since u^i has monotone curvature, $x^{i\circ}$ is unique. Denote the mapping from v° to $x^{i\circ}$ by ∇^{-1}, i.e., $x^{i\circ} = \nabla^{-1}(v^\circ)$. Therefore, $MRS^i_{j1}(x^{i\circ}) = \frac{v^\circ_j}{v^\circ_1}$. Next, consider $\phi^{i-1}(x^{i\circ})$ and evaluate $MRT_{j1}(\phi^{i-1}(x^{i\circ}))$. The normal vector is then modified as follows. Let $g_1(v^\circ) = \max\{v^\circ_1, \varepsilon\}$, for some $\varepsilon > 0$, and for $j \neq 1$ let

[22] To clarify, I refer to the projections of $\overline{\Lambda}^{i\phi}$ and $\overline{\mathbf{F}}$ onto Δ^{m-1} for the pupose of constructing the convex combination of most preferred output vectors. However, I also separately associate Δ^{m-1} with the normal vectors to u^i.

[23] That is, $\nabla u^i(x^{i\circ}) = v^\circ$.

$$g_j(v^\circ) = \begin{cases} \max\left\{v_j^\circ - a\left(MRS_{j1}^i(x^{i\circ}) - MRT_{j1}(\phi^{i-1}(x^{i\circ}))\right), 0\right\} & \text{if} \quad -\lambda \le MRS_{j1}^i(x^{i\circ}) - MRT_{j1}(\phi^{i-1}(x^{i\circ})) \le \lambda \\ \max\left\{v_j^\circ - a\lambda, 0\right\} & \text{if} \quad MRS_{j1}^i(x^{i\circ}) - MRT_{j1}(\phi^{i-1}(x^{i\circ})) > \lambda \\ v_j^\circ + a\lambda & \text{if} \quad MRS_{j1}^i(x^{i\circ}) - MRT_{j1}(\phi^{i-1}(x^{i\circ})) < -\lambda, \end{cases}$$

where a and λ are positive scalars.

Since $g(v^\circ)$ is not necessarily in Δ^{m-1}, it is renormalized according to $h_j(v^\circ) = \frac{g_j(v^\circ)}{\Sigma_j g_j(v^\circ)}$.

Then Δ^{m-1} is nonempty, compact and convex, and $h : \Delta^{m-1} \to \Delta^{m-1}$ is continuous. Therefore, by the Brouwer Fixed Point Theorem, h has a fixed point, or there is some $v^* \in \Delta^{m-1}$ such that $h(v^*) = v^*$. But this requires that $MRS^i_{j1}(x^{i*}) = MRT_{j1}(\phi^{i-1}(x^{i*}))$, where $x^{i*} = \nabla^{-1}(v^*)$. Hence, condition (3) holds.

Finally, by way of contradiction, let x^* denote $\phi^{i-1}(x^{i*})$ and suppose $x^* \notin \mathbf{H}^\phi$. Then by regularity, it must be the case that for some $j, j' \in M$, $MRS^i_{jj'}(\phi^i(x^*)) > MRT_{jj'}(x^*)$ uniformly for all $i \in N$. That is, agents would unanimously prefer that more of good j and less of good j' were produced. But this contradicts the fact that $(f^{-1}(x^*), \phi(x^*))$ is efficient. ■

Corollary 10 *For $n = 2$, the α-weighted ϕ^{EI}-compromise value is opp-envy-free.*

5 An Example

To demonstrate the notion of a compromise value, consider the following example involving two consumers, two consumption goods, and a single factor of production. For simplicity, I modify the notation of the previous sections and refer to the as consumers A and B, the consumption goods as x and y, and the factor as K. The consumers' preferences are represented by $u^A(x, y) = x^2 y$ and $u^B(x, y) = xy^2$, respectively, and the technologies are given by $x = K$ and $y = \sqrt{K}$.[24] The aggregate endowment of K is 16. Thus, an allocation is a list $((K^x, K^y), ((x^A, y^A), (x^B, y^B)))$ specifying a division of K between x and y production and quantities of x and y allocated to A and B. Here, the production possibility frontier is given by the expression $x + y^2 = 16$. (See Figure 4.) Moreover, since there is a single factor, the

[24] Although the x technology is concave rather than strictly concave; the example suffices since the production possibility set is strictly convex.

requirements for efficiency (conditions (2) and (3) of the previous section) are $MRS^A(x^A, y^A) = MRS^B(x^B, y^B) = MRT(\overline{x}, \overline{y})$, where $\overline{x} = x^A + x^B = K^x$, $\overline{y} = y^A + y^B = \sqrt{K^y}$ and $K^x + K^y = 16$.

Next, suppose output is divided according to the equal income competitive x-equilibrium rule ϕ^{EI} described earlier. For any production vector $(\overline{x}, \overline{y})$, the x-equilibrium allocation will be $((\frac{2}{3}\overline{x}, \frac{1}{3}\overline{y}), (\frac{1}{3}\overline{x}, \frac{2}{3}\overline{y}))$. Thus, the equation for agent A's achievable set $\Lambda^{A\phi^{EI}}$ is $3x + 18y^2 = 32$, and the equation for $\Lambda^{B\phi^{EI}}$ is $12x + 9y^2 = 64$. A's most preferred bundle in $\Lambda^{A\phi^{EI}}$ is $(8.53, 0.60)$ (indicated by a in Figure 4.) and its utility is 43.66. Under ϕ^{EI}, this would result from the aggregate production vector $(12.80, 1.80)$ (indicated by b), where $(K^x, K^y) = (12.80, 3.24)$. That is, under ϕ^{EI}, A's most preferred division of the factor would be to allocate 12.80 units to x production and 3.24 units to y.

Similarly, B's most preferred bundle in $\Lambda^{B\phi^{EI}}$ is $(2.67, 1.89)$ (indicated by c) and its utility is 9.54. This would result from the aggregate production vector $(8.01, 2.84)$ (indicated by d) and the factor division $(8.01, 8.07)$.[25] Note that $MRS^A(8.53, 0.60) = 0.14 < MRT(12.80, 1.80) = 0.28$ and $MRS^B(2.67, 1.89) = 0.35 > MRT(8.01, 2.84) = 0.18$. Thus, neither A's most preferred outcome or B's is efficient. Here, there is a unique efficient α-weighted ϕ^{EI}-compromise value $\mathbf{z}^* = ((10.67, 5.34), ((7.11, 0.77), (3.56, 1.54)))$ (indicated by e on $\overline{\mathbf{F}}$ with consumption components f and g on $\Lambda^{A\phi^{EI}}$ and $\Lambda^{B\phi^{EI}}$) corresponding to the weights $(\alpha^A, \alpha^B) = (0.78, 0.22)$.

Summarizing, conditional on using the equal income competitive equilibrium division rule, agent A would prefer that outputs $(12.80, 1.80)$ are produced, and B would prefer outputs $(8.01, 2.84)$. The α-weighted compromise would entail producing quantities $(10.67, 2, 31)$ and distributing them according to $((7.11, 0.77), (3.56, 1.54))$.

Finally, since there are only two consumers, an EOCE allocation is ϕ^{EI}-

[25] In light of the symmetry of their utility functions, one might conclude that A benefits by virtue of the fact that it has a taste bias for the good that is easier to produce.

opp-envy-free and efficient by Theorem 5. And since the preceding discussion of the compromise value focused on the division rule ϕ^{EI}, $\mathbf{z}^*$ is indeed an EOCE allocation. To verify that it is ϕ^{EI}-opp-envy-free, note that $u^A(7.11, 0.77) = 38.93$ and $u^B(3.56, 1.54) = 8.44$ whereas the highest utilities A and B can achieve over $\Lambda^{B\phi^{EI}}$ and $\Lambda^{A\phi^{EI}}$, respectively, are 21.49 and 4.68. Thus, neither consumer would prefer the other's achievable set.

(insert Figure 4)

6 Conclusion

First, this paper demonstrates that the source of the incompatibility between no-envy and efficiency in production economies, as demonstrated by Pazner and Schmeidler (1974), is the existence of dual-use factors which can be either consumed or used in production. However, even if this problem could be resolved, I have argued that no-envy is not a suitable notion of fairness in this context for it fails to take into consideration the influence of each agent on the production decision. Instead, I have proposed two new notions of fairness, opportunity-envy-freedom and the weighted compromise value. Both are conditioned on the output division rule. That is, it is presumed that a suitable (equitable and efficient) procedure can be found for dividing any given quantities of goods. Of the two notions, the weighted compromise value is the more appealing due to its greater compatibility with Pareto efficiency and its greater practicability. This would entail determining each agent's most preferred output vector (according to $\succsim_{i\phi}$) conditional on the given output rule, and then considering a weighted average of such vectors. Nevertheless, there are several interesting open questions concerning both notions. Are there reasonable restrictions under which opportunity-envy-freedom is compatible with efficiency for more than two agents? Can the conditions described in Theorem 9 be (significantly) relaxed? Possibly even to allow dual-use factors? What, if any, significance might be attached to the appropriate α weights? How are they related to the fundamentals of the economy and how are they to be determined? In the example of Section 5, the unique, efficient compromise value attaches 3.5 times the weight to agent A's preferred production vector than to agent B's. Is this related to the fact that A has a taste bias for the good that is easier to produce? And, if so, how? This also suggests that a more general inquiry into the issue of fairness and factor intensity and/or responsibility for hard-to-please tastes may be warranted. These and other questions are left for future research.

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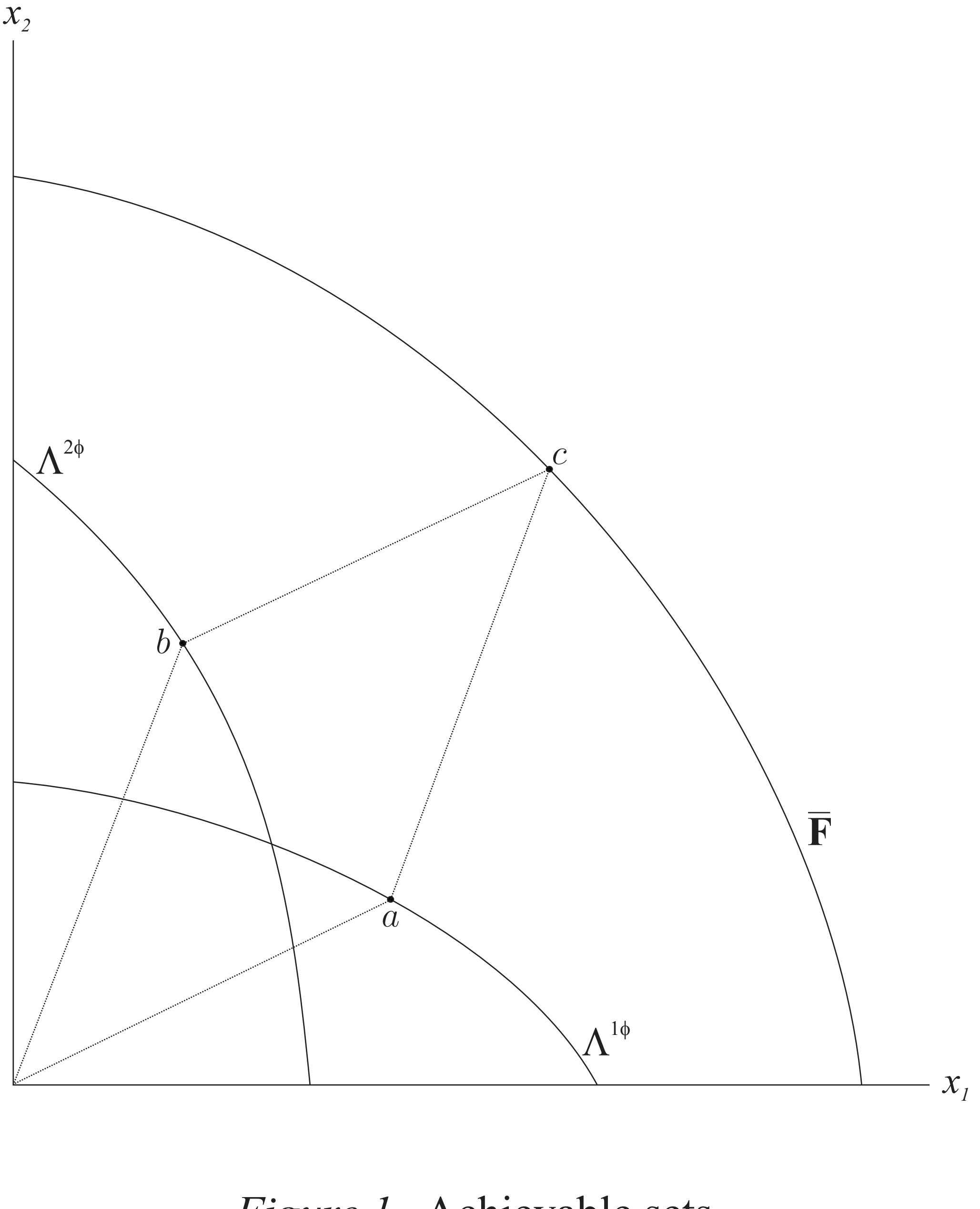

Figure 1. Achievable sets.

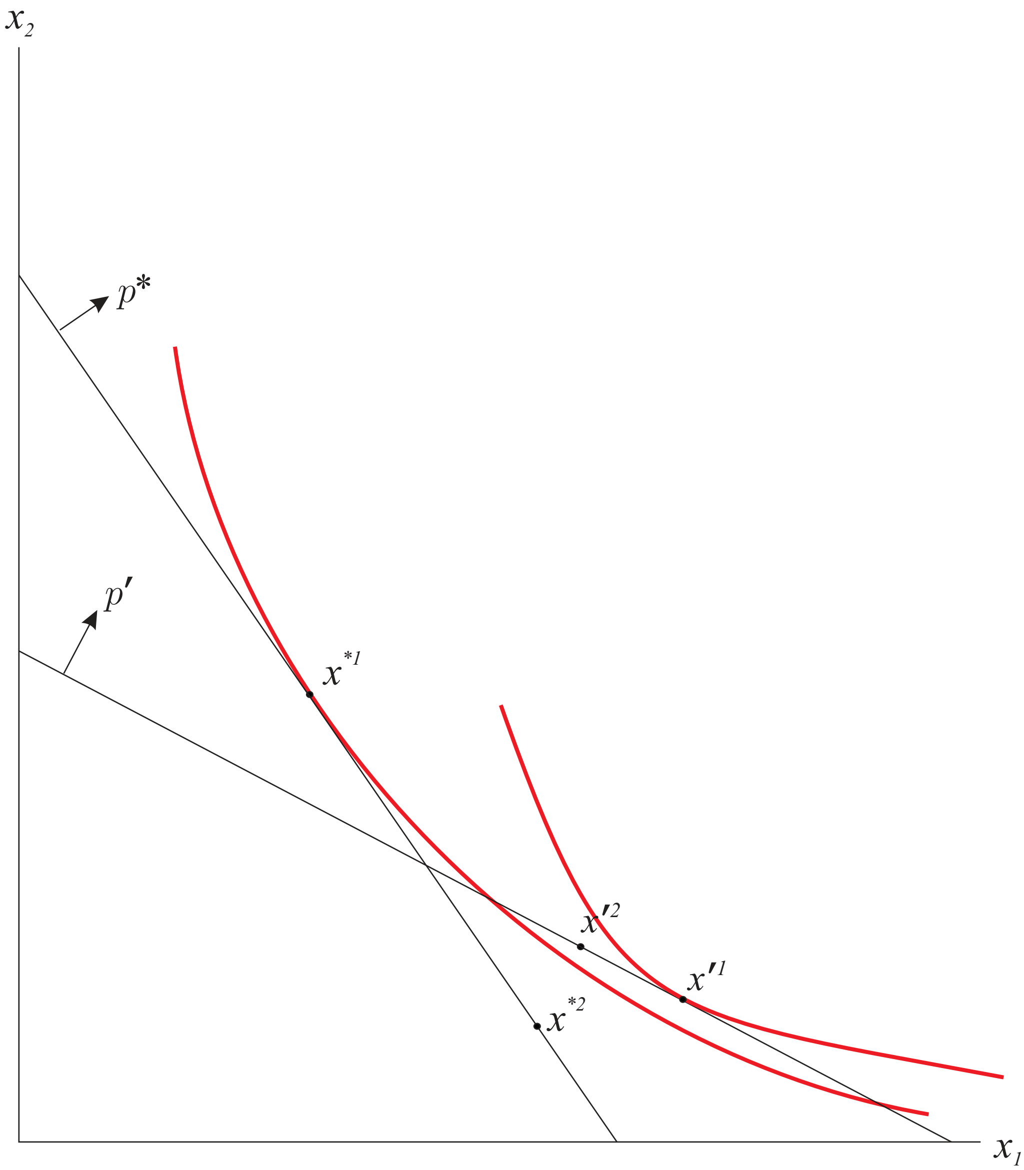

Figure 2. For $n=2$, an EOCE is opp-envy-free.

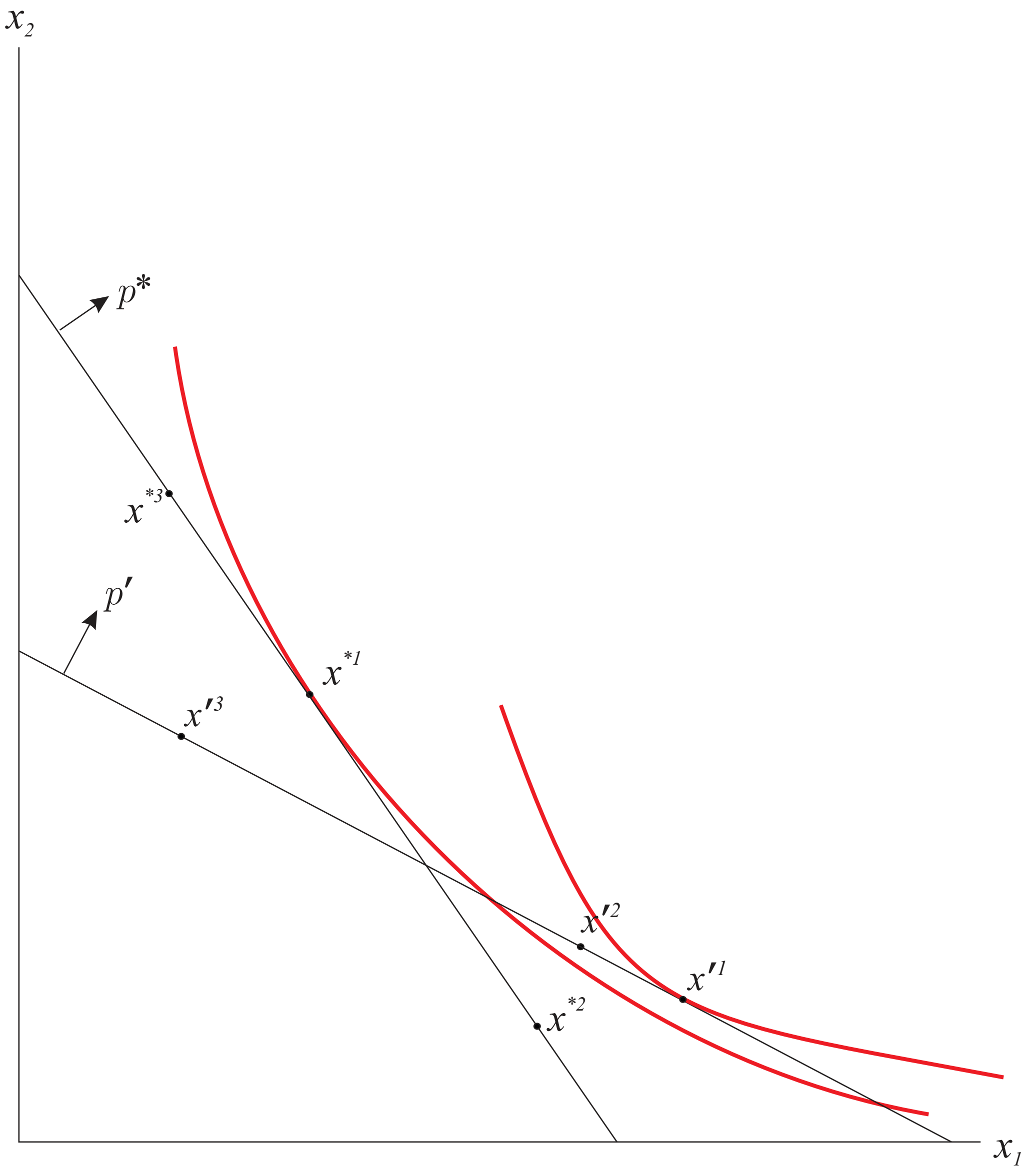

Figure 3. An EOCE with opportunity envy.

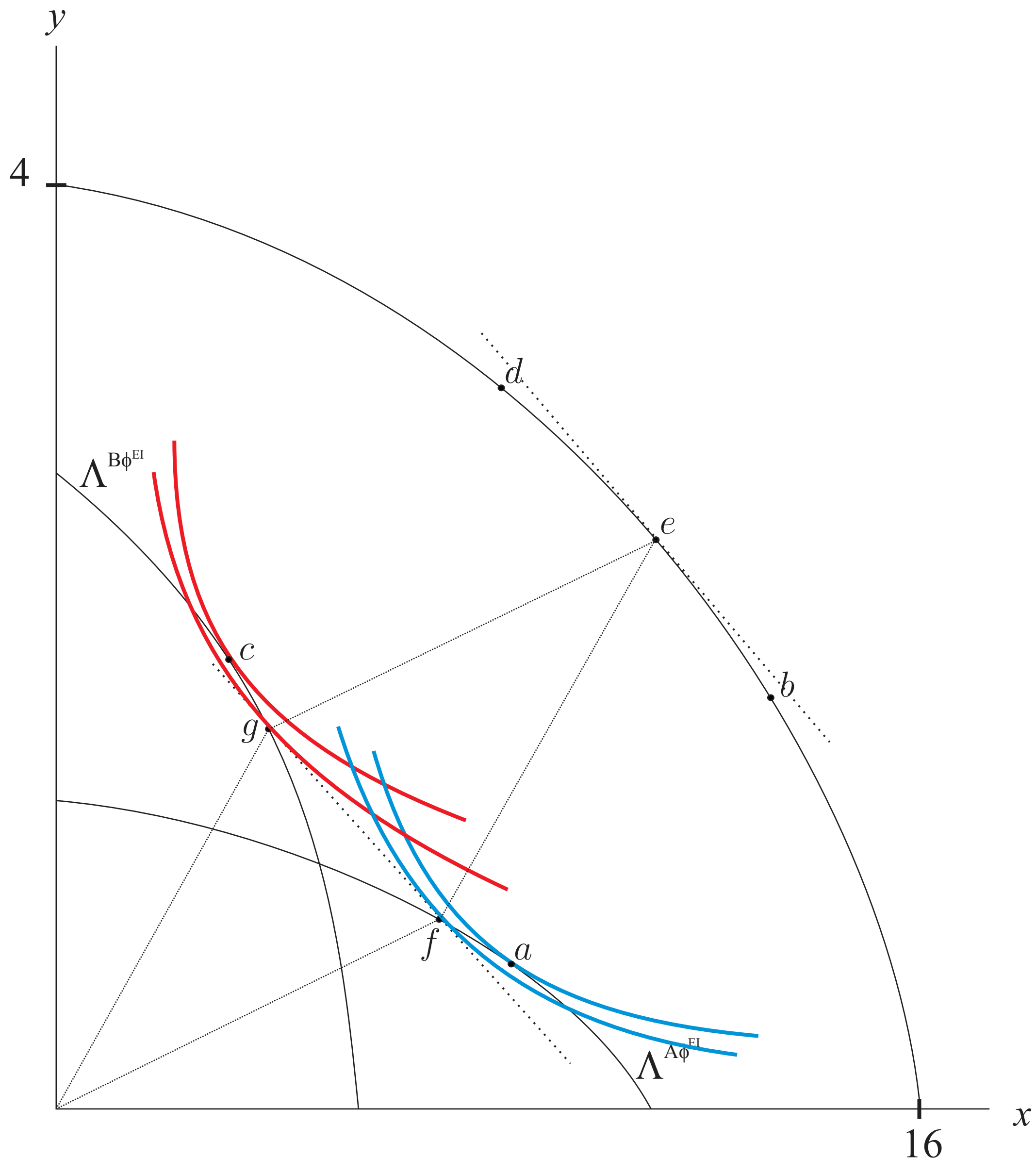

Figure 4. Weighted Compromise Value.