Expanding "Choice" in School Choice

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Abstract: Gale-Shapley's deferred acceptance (henceforth DA) mechanism has emerged as a prominent candidate for placing students to public schools. While the DA has desirable fairness and incentive properties, it limits the applicants' abilities to communicate their preference intensities, which entails ex-ante inefficiency when ties at school preferences are broken randomly. We propose a variant of deferred acceptance mechanism that allows students to influence how they are treated in ties. It inherits much of the desirable properties of the DA but performs better in ex ante efficiency.

Keywords: Gale-Shapley's deferred acceptance algorithm, choice-augmented deferred acceptance, tie breaking, ex ante Pareto efficiency.

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1 Introduction

The promise of school choice programs is to expand students' access to public schools beyond their residential boundaries.[1] In reality, this promise is tampered by the fact that public schools face capacity constraints. When too many students demand limited seats at a school, a decision must be made on who are admitted and who are turned away. Such a decision is to a degree moderated by the school's priorities over the applicants, but schools' priorities are typically very coarse. For instance, Boston Public Schools (BPS) prioritize applicants based only on sibling attendance and "walk zone," leaving many in the same priority class. So the fundamental issue remains: how to assign scarce seats at schools.

Allocating scarce resources is after all the fundamental role of markets. Competitive markets allocate goods efficiently based on individuals' preference intensities since prices of goods adjust to select those individuals willing to pay the most for them, namely those with the highest preference intensities. However, selling seats at public schools is not a viable option, nor is it desirable given the principle of free public education. But the "cardinal"efficiency of school assignment—that seats at a popular school must go to those who would lose relatively more by being assigned the next best school—is an important issue.[2] Can the market efficiency be achieved without monetary exchange? We suggest that this is indeed possible, and can be done by applying the lesson from competitive markets. We show that school choice can be designed to harness the pricing function of the market and generate a more efficient outcome in the cardinal sense, and this can be done within the framework of the most popular choice mechanism, namely Gale-Shapley's (Gale and Shapley, 1962) student-proposing deferred acceptance (henceforth, DA) algorithm, without sacrificing much of its beneficial properties.

Since proposed by Abdulkadiroğlu and Sönmez (2003), DA has emerged as one of the most prominent candidates for school admissions design. In 2003, New York City Department of Education adopted the DA.[3] In 2005, the BPS also adopted DA in place of the existing priority

[1]Government policies promoting school choice take various forms, including interdistrict and intradistrict public school choice as well as open enrollment, tax credits and deductions, education savings accounts, publicly funded vouchers and scholarships, private voucher programs, contracting with private schools, home schooling, magnet schools, charter schools and dual enrollment. See an interactive map at http://www.heritage.org/research/Education/SchoolChoice/SchoolChoice.cfm for a comprehensive list of choice plans throughout the US.

[2]We use the term "cardinal" in order to distinguish the efficiency we focus on from the ordinal efficiency. This distinction is made clearer by the example introduced shortly and by subsequent sections.

[3]The developments in NYC were initiated independently when, being aware of his pioneering work on market design in the entry level labor markets (Roth, 1984; Roth and E. Peranson, 1999), the New York City Department of Education contacted Alvin Roth to inquire about the appropriateness of a system like the National Residency Matching Program for the NYC high school match.

rule known as the "Boston" mechanism. Beyond school choice, DA has a celebrated history, having been successfully applied to the matching of doctors to hospital for their internships and residencies (see Roth, 2008). The algorithm works as follows. Once students submit their ordinal rankings of schools and school priorities are determined, it iterates the following procedure in successive rounds: Each student applies to her most preferred school that has not rejected her yet; each school *tentatively* admits up to its capacity according to its priority order from new applicants and those it has tentatively admitted in the previous round, rejecting those not admitted. The tentative assignment becomes final when no student is rejected. The Boston mechanism operates similarly, however, in contrast to DA, admissions at each round of the Boston mechanism are *permanent*.

When school priorities are strict, the DA has several desirable properties. It ensures fairness by eliminating justified envy; that is, no student ever loses a seat at a desired school to somebody with a lower priority at that school (Gale and Shapley, 1962; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003).[4] If schools' priorities are strict, then the assignment also Pareto dominates all other fair assignments (Gale and Shapley, 1962). Furthermore, DA is strategy-proof, meaning that the students have a dominant strategy of revealing their preference rankings of schools truthfully (Dubins and Freedman, 1981; Roth, 1982).[5]

These qualities notwithstanding, the DA does not respond to applicants' cardinal preferences. When two applicants tie in priority, the DA randomly determines who will be admitted and who will be rejected. That is, when priorities do not determine the allocation, DA completely ignores the underlying preference intensities of students. Luck of "draw" instead determines an applicant's fate. This is in contrast with competitive markets where agents can express their preference intensities via their willingness to pay, and also with the Boston mechanism where students' rankings have priority over random lottery numbers. This apparent "irony" was not lost in the run up to the BPS' adoption of the DA when parents observed:

> ... if I understand the impact of Gale Shapley, and I've tried to study it and I've met with BPS staff... I understood that in fact the random number ... [has] preference over your choices... (Recording from the BPS Public Hearing, 6-8-05).
>
> I'm troubled that you're considering a system that takes away the little power that parents have to prioritize... what you call this strategizing as if strategizing is a dirty word... (Recording from the BPS Public Hearing, 5-11-04).

[4] Justified envy is mathematically equivalent to the standard notion of blocking in two-sided matching.

[5] Aside from the ease with which parents make their choice, strategy-proofness "levels the playing field," by putting those strategically unsophisticated at no disadvantage relative to those who are more sophisticated (Abdulkadiroğlu, Pathak, Roth and Sönmez, 2006; Pathak and Sönmez, 2008).

This lack of responsiveness to cardinal preferences, or the inability by the parents to influence how they are treated at a tie, entails real welfare loss. The following example illustrates this drawback of DA.

Example 1. *Suppose there are three students,* $\{1, 2, 3\}$*, and three schools,* $\{a, b, c\}$*, each with one seat. Schools have no intrinsic priorities over students, and students' preferences are represented by the following von-Neumann Morgenstern (henceforth, vNM) utility values, where* v_j^i *is student* i*'s vNM utility value for school* j*:*

	v_j^1	v_j^2	v_j^3
$j = a$	4	4	3
$j = b$	1	1	2
$j = c$	0	0	0

Since schools have no priorities, ties must be broken before applying DA. Consider a DA algorithm that generates the student rankings at schools by a uniform lottery; that is, ties among students are broken randomly. By strategy-proofness, all three students submit truthful rankings of the schools. Consequently, the students are assigned each school with probability 1/3. In other words, the DA mechanism reduces to a pure lottery assignment. The students obtain expected utility of $EU_1^{DA} = EU_2^{DA} = EU_3^{DA} = \frac{5}{3}$. This assignment makes no distinction between students 1, 2 and student 3, despite the fact that the first two would suffer more by being assigned the next best alternative b than student 3.

Not surprisingly, reallocating the assignment probabilities can make all agents strictly better off. Suppose instead student 3 is assigned school b for sure, and students 1 and 2 are assigned a and c with equal probability 1/2. Each student obtains the expected utility of $EU_1^B = EU_2^B = EU_3^B = 2$ strictly higher than $\frac{5}{3}$ they enjoyed in the DA.[6] Moreover, this new assignment can be implemented in an incentive compatible way. Suppose for instance that the students are offered the two different lotteries; sure assignment at school b versus a uniform lottery between a and c. Intuitively, this mechanism offers a 1/2 chance of getting assigned the best school a, for the "price" of 1/2 chance of getting assigned the worst school c. The first two students (who will suffer a lot by being placed to the second best school instead of the best) will pay that price, but the third student (who will not suffer as much) will not. Hence, this mechanism implements the desirable assignment.

[6] In fact, the Boston mechanism can implement this better assignment since in equilibrium student 3 ranks B at the top while the other two rank truthfully. As we mention in Conclusion, our companion paper (Abdulkadiroğlu, Che and Yasuda (2011)) generalizes this observation to show that the Boston mechanism weakly Pareto dominates DA in every symmetric equilibrium when ordinal preferences over schools are aligned, schools have no priorities and break ties symmetrically.

We propose a way to modify the DA algorithm to harness the "pricing" feature of this mechanism. The idea is to allow the students to signal their cardinal preferences by sending an additional message, and this message is used to break ties at schools. In what we call "Choice-Augmented Deferred Acceptance" (CADA) algorithm, each student submits an ordinal list of schools just as before, but she also submits the name of a "target" school. A school then elevates the priority standing of those who targeted that school and favor them in breaking ties among those with the same priority at the school. The iterative procedure of DA is then implemented with the rankings generated in this way. The CADA inherits the desirable properties of the standard DA: It is fair in the sense of eliminating justified envy, and strategy-proof with respect to the ordinal preference lists. Clearly, targeting involves strategic behavior, but its importance is limited by the priorities. If schools' priorities are strict, then there are no ties, so CADA coincides with the standard DA.

If the priorities are coarse, then targeting allows individuals to signal their preference intensities, and in the process serves to "price" the schools based on their demands. Intuitively, if a school is targeted by more students, one finds it more difficult to raise the odds of assignment at that school via targeting that school; effectively the price of that school has risen. As will be seen, this feature of CADA allows competitive markets to operate for a set of popular schools, attaining ex ante efficiency within these schools. For this reason, for a large economy (both in the size of student body and school capacities), the CADA performs better in ex ante welfare than the DA with standard random tie-breaking rules. For instance, in the above example, the unique Nash equilibrium of the CADA has students 1 and 2 targeting a, and student 3 targeting b, so the desirable outcome is implemented.[7]

The issue of cardinal welfare, or ex ante efficiency that captures cardinal welfare, has not received much attention in the debate of school choice design. The existing debate has largely focused on ex post efficiency or ordinal efficiency as a welfare concept. Cardinal welfare would not matter much if either students' preferences are diverse or if the schools' priorities are strict. In the former case, the preferences do not conflict, so they can be easily accommodated. In the latter case, even if the preferences conflict, school priorities pin down the assignment, so there is no scope for assignment to respond to cardinal welfare.

[7] All students will submit their rankings truthfully, so $a - b - c$ in this order. Given the targeting behavior, school a will then rank students 1 and 2 ahead of 3 (but randomly between the first two), and school b will rank student 3 ahead of 1 and 2 (again randomly between these two). In the first round of CADA, all students apply to a and it will choose between 1 and 2, and the two rejected students, including student 3, will then apply to school b in the second round, and school b will admit student 3. Hence, student 3 is assigned b for sure, and 1 and 2 are assigned between a and c with equal probability, thus implementing the superior assignment. It is routine to check that there is no unilateral profitable deviation from the assumed targeting behavior.

If neither is true, however, the cardinal welfare issue becomes important. Suppose for an extreme case that the students have the same ordinal preferences and schools have no priorities. In that case, as seen in Example 1, it matters how the students are assigned based on their relative preference intensities of the schools. By contrast, ex post efficiency loses its bite; as can be seen in the example, *all* assignments are ex post efficient, thus indistinguishable on this ground. In particular, DA reduces to a pure lottery assignment and CADA may do strictly better in such a case. Although this extreme scenario is special, it captures salient features of reality. In reality, priorities are coarse and parents tend to value the similar qualities about schools (e.g., safety, academic excellence, etc.), leading them to have similar (ordinal) preferences. Indeed, the BPS data exhibits strong correlation across students' ordinal preferences over schools. In 2007-2008, only 8 out of 26 schools (at grade level 9) were overdemanded whereas an average of 22.21 (std 0.62) schools should have been overdemanded if students' preferences had been uncorrelated.[8] As we point out in this paper, in such an environment, CADA performs particularly well relative to the DA with standard random tie-breaking rules.

Besides CADA, we also offer two methodological contributions. Analysis of CADA requires an understanding of students' strategic behavior with regard to their targeting. Such an analysis is not tractable in the general finite economy model. Instead we study a matching model with a continuum of students and finitely many schools with mass capacities as an approximation to large finite matching models.[9] This model produces a clear insight and captures salient features of large strategic environments without sacrificing tractability, and is well founded as the approximation of large finite economy models, as discussed in Section 6.6. Second, we offer a novel and convenient way of measuring efficiency. Realistic mechanisms often trade off efficiency for practicality, and thus may not attain full Pareto efficiency. To compare such mechanisms in efficiency, we introduce the notion "scope of efficiency" which refers to the set of schools that are allocated efficiently from an ex ante point of view. The size of this set can measure efficiency, enabling us to rank different mechanisms.

The rest of the paper is organized as follows. Section 2 defines CADA more precisely and shows that it inherits the desired properties of standard DA. Section 3 introduces the formal

[8]This comparison is based on submitted preferences under the DA introduced in 2005. Since the DA is strategy-proof and BPS paid significant attention in communicating that feature of the DA to the public, we assume that those submitted preferences are a good approximation of the underlying true preferences. For the counter-factual, we generated 100 different preference profiles by drawing a school as first choice for each student uniformly randomly from the set of schools and computed the number of overdemanded schools given school capacities.

[9]Following the initial draft of the current paper, Che and Kojima (2010), Azevedo and Leshno (2012) and Azevedo (2011) have also adopted continuum models of matching, and the first two study the asymptotic properties of a model that converges to the same model in the limit.

model and welfare criterion. Section 4 provides welfare comparison across the three alternative procedures. Section 5 presents simulation to quantify the welfare benefits of CADA. Section 6 then considers the implication of enriching the message used in the CADA and the robustness of our results to some students not behaving in a strategically sophisticated way. Section 7 concludes. Appendix contains most of the proofs. Those not contained in the Appendix are available in the Supplementary Appendix ("not for publication").

2 Choice-Augmented DA Algorithm: Definition and Finite Economy Properties

This section defines CADA formally in a finite economy setting. To begin, consider an economy with a finite set I of students and a finite set S of schools such that a school $s \in S$ has a finite number q_s of seats. Let $\emptyset$ denote a null set, meaning the outcome of not being assigned any school. An **assignment** x is a mapping $x : I \to S \cup \emptyset$ such that $|x^{-1}(s)| \leq q_s$ for each $s \in S$; that is, x is a many-to-one matching with the property that the number of students assigned a school does not exceed its capacity. We assume that each student in I has strict ordinal preferences over schools S, and each school in S has priority ordering of students, which may contain thick indifference classes.

A mechanism specifies a message space for each student and an assignment as a function of messages and other parameters of the model, such as school priorities and capacities. We will consider mechanisms that have ordinal preferences and potentially some other auxiliary messages as their massage space. Within this class, a mechanism is **ordinally strategy-proof** if for every student, for every auxiliary message behavior for her, and for all strategies of other students, it is optimal for her to report her ordinal preferences truthfully. This notion coincides with the standard strategy-proofness notion when a mechanism consists of only ordinal preferences as message space.

Furthermore, a mechanism is **individually rational** if its assignment is weakly preferred to null assignment for each student and every assigned student is eligible for the school she is assigned. A mechanism **eliminates justified envy** if, for every preference and priority profile, there exists no pair of a student and a school such that the student prefers the school over her assignment and the school either has a vacant seat or admits a student with a lower priority.

Since standard DA requires the rankings to be strict on both sides, using DA requires a procedure to break ties if schools' priorities are non-strict. There are two common procedures for breaking ties. **Single tie-breaking (STB)** assigns every student a single lottery number uniform-randomly to break ties at every school, whereas **multiple tie-breaking (MTB)** assigns a distinct lottery number to each student at every school. Clearly, a DA algorithm is well defined with respect to the strict priority list generated by either method. We refer to the DA algorithms using single and multiple tie-breaking by **DA-STB** and **DA-MTB**, respectively.

CADA involves an alternative way to break a tie, one that allows students to influence its outcome based on their messages. It proceeds in the following three steps:

- **Step 1:** All students submit ordinal preferences, plus an "auxiliary message," naming a "target" school. If a student names a school for a target, she is said to have "targeted" the school.

- **Step 2:** The schools' strict priorities over students are generated based on their *intrinsic priorities* and the students' auxiliary messages as follows. First, each student is uniform-randomly assigned two lottery numbers. Call one *target lottery number* and the other *regular lottery number*. Each school's *strict priority list* is then generated as follows: (i) First consider the students in the school's highest priority group. Within that group, rank at the top those who name the school as their target. List them in the order of their target lottery numbers, and list below them the rest (who didn't target that school) according to their regular lottery numbers. (ii) Move to the next highest priority group, list them below in the same fashion, and repeat this process until all students are ranked in a strict order.

- **Step 3:** The students are then assigned schools via the DA algorithm, using *each student's ordinal preferences* from Step 1 and *each school's strict priority list* compiled in Step 2.

To illustrate Step 2, suppose there are five students $I = \{1, 2, 3, 4, 5\}$ and two schools $S = \{a, b\}$, neither of which has intrinsic priority ordering over the students. Suppose students $1, 3$ and 4 targeted a and 2 and 5 targeted b, and that students are ordered according to their target and regular lottery orders the students as follows:

$$\mathbf{T}(I) : 3 - 5 - 2 - 1 - 4; \quad \mathbf{R}(I) : 3 - 4 - 1 - 2 - 5.$$

Then the priority list for school a first reorders students $\{1, 3, 4\}$, who targeted that school, based on $\mathbf{T}(I)$, to $3 - 1 - 4$, and reorders the rest, $\{2, 5\}$, based on $\mathbf{R}(I)$, to $2 - 5$, which produces a complete list for a:

$$\mathbf{P}_a(I) = 3 - 1 - 4 - 2 - 5.$$

Similarly, the priority list for b is:

$$\mathbf{P}_b(I) = 5 - 2 - 3 - 4 - 1.$$

The process of compiling the priority lists resembles the STB in that the same lottery is used by different schools, but only within each group. Unlike STB, though, different lotteries are used across different groups. This ensures that students are treated identically at their non-target schools regardless of the schools they target, which plays a role in our welfare characterization (as will be explained in footnote 22). Besides, this feature has an added fairness benefit of giving a student with a bad draw at her target school a “new lease on life” with another independent draw at other schools.

The desirable properties of the DA are all preserved.

Theorem 1. *(i) CADA eliminates justified envy; it is ordinally strategy-proof. (ii) Given an arbitrary targeting behavior and truthful reporting of ordinal preferences by students and any realization of the corresponding CADA assignment, there is no individually rational assignment that every student strictly prefers over the CADA assignment.*

Part (i) follows from the fact that CADA is a form of DA and that the ordinal rankings submitted by the students under CADA are used in the same way as DA. Part (ii) follows from Theorem 6 of Roth (1982).

3 A Large Economy Model

The main focus of this paper is to understand the cardinal welfare properties of CADA and the standard DA mechanisms. This requires analysis of students' strategic behavior with regard to the targeting of schools under CADA and the resulting ex ante welfare consequences (as well as ex ante welfare properties of the standard DA mechanisms). Such an analysis is difficult for a general finite economy model. To obtain a clear insight, we instead study a model in which there are a continuum of students and finitely many schools with mass capacities. For analytical tractability we also assume that schools have no priorities.

The large economy assumption, we believe, is fairly descriptive of the typical school choice environment. For instance, NYC serves about a million students at K-12 grade levels and about 100,000 pupils go through a centralized admissions process to be assigned to one of about 700 high school programs. About 60,000 pupils are served at K-12 grade level in Boston. However, as will be shown in Section 5, the welfare comparison from the large market model holds even when the size of the market is considerably smaller. The assumption that schools have no priorities captures the coarseness of the priority structure in general. But it applies exactly to several school choice environments. For instance, the school choice programs in Korea and the second round of NYC programs involve no priorities on the school side. In addition, a majority of high schools in NYC, including Educational Option schools for half of the seats and unscreened schools, do not prioritize students.

3.1 Primitives

There are $n \geq 2$ schools, $S = \{1, 2, ..., n\}$, each with a unit mass of seats to fill. There are mass n of students who are indexed by vNM values $\mathbf{v} = (v_1, ..., v_n) \in \mathcal{V} := [0, 1]^n$ they attach to the n schools, where the outside option for students is zero.[10] The set of student types, $\mathcal{V}$, is equipped with a measure μ. A tuple (S, μ) constitutes a **(school choice) problem**.

We assume that μ involves no atom and admits strictly positive density in the interior of $\mathcal{V}$. This implies that almost every student has strict preferences, that is, $\mu(\{\mathbf{v} : v_i = v_j \text{ for some } i \neq j\}) = 0$. The assumptions that the aggregate measure of students (interested in public schooling) equal aggregate capacities of schools and that all students find every school acceptable are made for convenience and will not affect our main results (see Subsection 6.5).

The students' vNM values induce their ordinal preferences. For any $\mathbf{v} \in \mathcal{V}$, let $\pi_k(\mathbf{v}) \in S$ denote the k-th preferred school for the type-$\mathbf{v}$ student. Formally, $\pi := (\pi_1, ..., \pi_n) : \mathcal{V} \to S^n$ be such that $\pi_i(\mathbf{v}) \neq \pi_j(\mathbf{v})$ if $i \neq j$ and that $v_{\pi_i(\mathbf{v})} > v_{\pi_j(\mathbf{v})}$ implies $i < j$. That is, $\pi(\mathbf{v})$ lists the schools in the descending order of the preferences for a type-$\mathbf{v}$ student. Let Π denote the set of all ordered lists of S. Then, for each $\tau \in \Pi$,

$$m_\tau := \mu(\{\mathbf{v}|\pi(\mathbf{v}) = \tau\})$$

represents the measure of students whose ordinal preferences are τ. By the full support assumption, $m_\tau > 0$ for each $\tau \in \Pi$. Finally, let $\mathfrak{m} := \{m_\tau\}_{\tau\in\Pi}$ be a profile of measures of all ordinal types. Let $\mathfrak{M} := \{\{m_\tau\}_{\tau\in\Pi} | \sum_{\tau\in\Pi} m_\tau = n\}$ be the set of all possible measure profiles. We say a property holds *generically* if it holds for a subset of $\mathfrak{m}$'s that has the same Lebesgue measure as $\mathfrak{M}$.

Each school has no priorities on students and is willing to admit any student if it has a vacant seat. This latter assumption is consistent with the policy that every student is entitled to a public school seat. Given this assumption, we focus throughout on assignment in which all students are assigned to public schools. Formally, an **assignment**, denoted by $\mathbf{x}$, is a probability distribution over S, and this is an element of a simplex, $\Delta := \{(x_1, ..., x_n) \in \mathbb{R}^n_+ | \sum_{a\in S} x_a = 1\}$. We are primarily interested in how a procedure determines the assignment for each student *ex ante*, prior to conducting the lottery. To this end, we define an **allocation** to be a measurable function $\phi := (\phi_1, ..., \phi_n) : \mathcal{V} \mapsto \Delta$ such that $\int \phi_a(\mathbf{v}) d\mu(\mathbf{v}) = 1$ for each $a \in S$, with the interpretation that student $\mathbf{v}$ is assigned by mapping $\phi = (\phi_1, ..., \phi_n)$ to school a with probability $\phi_a(\mathbf{v})$. Let $\mathcal{X}$ denote the set of all allocations.[11]

[10] As in Introduction, examples below consider vNM values outside $[0, 1]^n$, for convenience.

[11] The definition of allocation may raise the following question: Can an allocation in a continuum economy be implemented via a lottery over deterministic assignments? This issue does not arise for the three alternative

3.2 Welfare Standards

To begin, we say allocation $\tilde{\phi} \in \mathcal{X}$ **weakly Pareto dominates** allocation $\phi \in \mathcal{X}$ if it provides almost every student with a higher expected utility. That is, for almost every $\mathbf{v}$,

$$\sum_{a \in S} v_a \tilde{\phi}_a(\mathbf{v}) \geq \sum_{a \in S} v_a \phi_a(\mathbf{v}), \tag{1}$$

$\tilde{\phi}$ **Pareto dominates** ϕ if the former weakly dominates the latter and if the inequality of (1) is strict for a positive measure of $\mathbf{v}$'s. We also say $\tilde{\phi} \in \mathcal{X}$ **ordinally dominates** $\phi \in \mathcal{X}$ if the former has higher chance of assigning each student to her more preferred school than the latter in the sense of first-order stochastic dominance: for a.e. $\mathbf{v}$,

$$\sum_{i=1}^{k} \tilde{\phi}_{\pi_i(\mathbf{v})}(\mathbf{v}) \geq \sum_{i=1}^{k} \phi_{\pi_i(\mathbf{v})}(\mathbf{v}), \ \forall k = 1, ..., n-1, \tag{2}$$

with the inequality being strict for some k, for a positive measure of $\mathbf{v}$'s. The left and right hand sides of the inequality are the probability that type-$\mathbf{v}$ student is assigned one of her top k choices by $\tilde{\phi}$ and ϕ, respectively.

Our welfare notion concerns the scope of efficiency, measured by the subset of schools that are efficiently allocated. To this end, fix any allocation ϕ. A *within-K reallocation of* $\phi \in \mathcal{X}$ is an element of a set

$$\mathcal{X}^K_\phi := \{\tilde{\phi} \in \mathcal{X} | \tilde{\phi}_a(\mathbf{v}) = \phi_a(v), \forall a \in S \setminus K \text{ and for a.e. } \mathbf{v} \in \mathcal{V}\}.$$

We then look for a within-K reallocation of allocation ϕ in which students have exhausted the (mutually beneficial) opportunities for trading shares of schools within K.

Definition 1. *(i) For any $K \subset S$, an allocation $\phi \in \mathcal{X}$ is **Pareto efficient (PE) within** K if there is no within-K reallocation of ϕ that Pareto dominates ϕ. (ii) For any $K \subset S$, an allocation $\phi \in \mathcal{X}$ is **ordinally efficient (OE) within** K if there is no within-K reallocation of ϕ that ordinally dominates ϕ. (iii) An allocation is **PE** (resp. **OE**) if an allocation is PE (resp. OE) within S. (iv) An allocation is **pairwise PE** (resp. **pairwise OE**) if it is PE (resp. OE) within every $K \subset S$ with $|K| = 2$.*

These welfare criteria are quite intuitive. Suppose the students are initially endowed with ex ante shares ϕ of schools, and they can trade these shares among them. Can they trade mutually beneficially if the trading is restricted to the shares of K? The answer is no if allocation ϕ is

mechanisms we consider since they are variants of DA with explicit rules for tie-breaking. See Uhlig (1996) and Budish et al (2011) for a broader discussion.

PE within K. In other words, the size of the latter set represents the restriction on the trading technologies and thus determines the scope of markets within which efficiency is realized. The bigger this set is, the less restricted the agents are in realizing the gains from trade, so the more efficient the allocation is. Clearly, if an allocation is Pareto efficient within the set of all schools, then it is fully Pareto efficient. In this sense, we can view the size of such a set as a **scope of efficiency**.

A similar intuition holds with respect to ordinal efficiency. In particular, ordinal efficiency can be characterized by the inability to form a cycle of traders who can beneficially swap their probability shares of schools. Formally, let $\rhd^{\phi}$ be the binary relation on S defined by

$$a \rhd^{\phi} b \Longleftrightarrow \exists V \subset \mathcal{V}, \mu(V) > 0, \text{ s.t. } v_a > v_b \text{ and } \phi_b(\mathbf{v}) > 0, \forall \mathbf{v} \in V;$$

that is, if a positive measure students prefer a to b but are assigned b with positive probabilities. We say that ϕ *admits a trading cycle within* K if there exist $a_1, a_2, ...a_l \in K$ such that $a_1 \rhd^{\phi} a_2, ..., a_{l-1} \rhd^{\phi} a_l$, and $a_l \rhd^{\phi} a_1$. The next lemma is adapted from Bogomolnaia and Moulin (2001).

Lemma 1. *An allocation ϕ is OE within $K \subset S$ if and only if ϕ does not admit a trading cycle within K.*

Before proceeding further, we observe how different notions relate to one another.

Lemma 2. *(i) If an allocation is PE (resp. OE) within K, then it is PE (resp. OE) within $K' \subset K$; (ii) if an allocation is PE within $K \subset S$, then it is OE within K; (iii) for any K with $|K| = 2$, if an allocation is OE within K, then it is PE within K; (iv) if an allocation is OE, then it is pairwise PE.*

Part (i) follows since a Pareto improving within-K' reallocation constitutes a Pareto improving within-K reallocation for any $K \supset K'$. Likewise, a trading cycle within any set forms a trading cycle within its superset. Part (ii) follows since if an allocation is not ordinally efficient within K, then it must admit a trading cycle within K, which produces a Pareto improving reallocation. Part (iii) follows since, whenever there exists an allocation that is not Pareto efficient within a pair of schools, one can construct a trading cycle involving a (positive-measure) set of agents who would benefit from swapping their probability shares of these schools. Part (iv) then follows from Part (iii).[12]

[12] These characterizations are tight. The converse of Part (iii) does not hold for any K with $|K| > 2$. In Example 1 from the introduction, the DA allocation is OE but not PE. Likewise, an allocation that is PE within K need not be OE within any $K' \supsetneqq K$, since an allocation could be Pareto improved upon only via a trading cycle that includes a school in $K' \backslash K$. In that case, the allocation may be PE within K, yet it will not be OE within K'.

3.3 Alternative School Choice Procedures

We consider three alternative procedures for assigning students to the schools: (1) Deferred Acceptance with Single Tie-breaking (DA-STB), (2) Deferred Acceptance with Multiple Tie-Breaking (DA-MTB), and (3) Choice-Augmented Deferred Acceptance (CADA).

The alternative procedures differ only in the way the schools break ties. The tie-breaking rule is well-defined for DA-STB and DA-MTB, and it follows from Step 2 of Section 2 in the case of CADA. These rules can be extended to the continuum of students in a natural way. The formal descriptions are provided in the Supplementary Appendix; here we offer the following heuristic descriptions:

• **DA-STB:** Each student draws a number $\omega \in [0,1]$ at random according to the uniform distribution. A student with a lower number has a higher priority at *every* school than does a student with a higher number.

• **DA-MTB:** Each student draws n independent random numbers $(\omega_1, ..., \omega_n)$ from $[0,1]^n$ according to the uniform distribution. The a-th component, ω_a, of student's random draw then determines her priority at school a, with a lower number having a higher priority than does a higher number.

• **CADA:** Each student draws two random numbers $(\omega_T, \omega_R) \in [0,1]^2$ according to the uniform distribution. Each school then ranks those students who targeted that school, based on their values of ω_T, and then ranks the others based on the values of $1 + \omega_R$ (with a lower number having a higher priority in both cases). In other words, those who didn't target the school receive a penalty score of 1.

For each procedure, the DA algorithm is readily defined using the appropriate tie-breaker and the students' ordinal preferences as inputs. The Supplementary Notes provide a precise algorithm, which is sketched here. At the first step, each student applies to her most preferred school. Every school tentatively admits up to unit mass from its applicants according to its priority order, and rejects the rest if there are any. In general, each student who was rejected in the previous step applies to her next preferred school. Each school considers the set of students it has tentatively admitted and the new applicants. It tentatively admits up to unit mass from these students in the order of its priority, and rejects the rest. The process converges when the set of students that are rejected has zero measure. Although this process might not complete in finite time, it converges in the limit and the allocation in the limit is well defined.[13] We focus on that limiting allocation.

Strategy-proofness of DA-STB and DA-MTB with respect to students' ordinal preferences

[13]See Azevedo and Leshno (2012) for an example of failure of convergence in finite steps. See Supplementary Appendix for the limit results.

is well-known in the finite case (Dubins and Freedman 1982, Roth 1982) and it extends to CADA in the finite case (Theorem 1). More importantly, all three procedures continue to be ordinally strategy-proof in the large economy:

Theorem 2. (Ordinal strategy-proofness) *Each of the three mechanisms is ordinally strategy-proof.*

Proof: The proof is in the Supplementary Appendix.

3.4 Characterization of Equilibria

□ *DA-STB* and *DA-MTB*

In either form of DA algorithm, the resulting allocation is conveniently characterized by a "cutoff" for each school — namely the highest lottery number a student can have to get into that school. Specifically, the DA-STB process induces a cutoff $c_a \in [0, 1]$ for each school a such that a student who ever applies to school a gets admitted by that school if and only if her (single) draw ω is less than or equal to c_a. Intuitively, the existence of such a cutoff follows from the deferred acceptance feature; namely, regardless of how a student ranks a school, she is treated by each school only according to the lottery draw she has. Formally, our proof in the Appendix explicitly constructs the cut-offs for both DA-STB and DA-MTB and establishes uniqueness of these cutoffs.[14]

Lemma 3. *DA-STB admits a unique set of cutoffs $\{c_a\}_{a\in S}$ for the schools in every problem. Each cutoff is strictly positive and one of them equals 1. For a generic $\mathfrak{m}$, the cutoffs are all distinct.*

Importantly, these cutoffs pin down the allocation of all students. To see this, consider any student with $\mathbf{v}$ and a school a with cutoff c_a. If a is her most preferred school, then she will be assigned a if and only if $w < c_a$. If not, let b be the school that has the largest cutoff among those schools that are preferred to a by that student. If the cutoff of school b has $c_b > c_a$, then the student will never get assigned school a since whenever she has a draw $\omega < c_a$ (good enough for a), she will get into school b or better. If $c_b < c_a$, however, then she will get into school a if and only if she receives a draw $\omega \in [c_b, c_a]$. The probability of this event is precisely the distance between the two cutoffs, $c_a - c_b$. Formally, let $S(a, \mathbf{v}) := \{b \in S|v_b > v_a\}$ denote the

[14]That the DA-STB admits a unique set of cutoffs can be also deduced from the fact that DA algorithm is well-defined (Theorem 0 in the Supplementary Appendix), along with the law of large numbers. We thank Eduardo Azevedo for pointing this out.

set of schools more preferred to a by type-$\mathbf{v}$ students. Then, the allocation ϕ^{STB} arising from DA-STB is given by

$$\phi_a^{STB}(\mathbf{v}) := \max\{c_a - \max_{b\in S(a,\mathbf{v})} c_b, 0\}, \forall \mathbf{v}, \forall a \in S,$$

where $c_\emptyset := 0$.

DA-MTB is similar to DA-STB, except that each student has independent draws $(\omega_1, ..., \omega_n)$, one for each school. The DA process again induces a cutoff $\tilde{c}_a \in [0,1]$ for each school a such that a student who ever applies to school a gets assigned to it if and only if her draw for school a, ω_a, is less than $\tilde{c}_a$. These cutoffs are well defined.[15]

Lemma 4. *DA-MTB admits a unique set of cutoffs $\{\tilde{c}_a\}_{a\in S}$ in every problem. Each cutoff is strictly positive and one of them equals 1. For a generic $\mathfrak{m}$, the cutoffs are all distinct.*

Given the cutoffs $\{\tilde{c}_a\}_{a\in S}$, a type $\mathbf{v}$-student receives school a whenever she has a rejectable draw $\omega_b > \tilde{c}_b$ for each $b \in S(a, \mathbf{v})$ she prefers to school a and when she has an acceptable draw $\omega_a < \tilde{c}_a$ for school a. Formally, the allocation ϕ^{MTB} from DA-MTB is determined by:

$$\phi_a^{MTB}(\mathbf{v}) := \tilde{c}_a \prod_{b\in S(a,\mathbf{v})} (1 - \tilde{c}_b), \forall \mathbf{v}, \forall a \in S,$$

with the convention $\tilde{c}_\emptyset := 0$.

□ *CADA*

As with the two other procedures, given the students' strategies on their messages, the DA process induces cutoffs for the schools, one for each school in $[0, 2]$. Of particular interest is the equilibrium in the students' choices of messages. Given Theorem 2, the only nontrivial part of the students' strategy concerns her "auxiliary message." Let $\sigma = (\sigma_1, ..., \sigma_n) : \mathcal{V} \mapsto \Delta$ denote the students' mixed strategy, whereby a student with $\mathbf{v}$ targets a with probability $\sigma_a(\mathbf{v})$. Non-atomic distribution of types ensures existence of a pure-strategy Nash equilibrium (Mas-Colell 1984).

Theorem 3. (Existence) *There exists an equilibrium σ^* in pure strategies.*

We say that a student **applies to** school a if she is rejected by all schools she lists ahead of a in her (truthful) ordinal list. We say that a student **subscribes to** school $a \in S$ if she targets school a *and* applies to that school during the DA process. Targeting school a does not necessarily imply applying to a. The latter event depends on where the student lists

[15]Extending our analysis of cut-offs, Azevedo and Leshno (2012) characterize stable matchings in a large economy school choice model.

school a in her ordinal list and the other students' strategies, as well as the outcome of tie breaking. Let $\bar{\sigma}_a^*(\mathbf{v})$ be the probability that a student $\mathbf{v}$ subscribes, i.e. targets and applies, to school a in equilibrium, and $\bar{\sigma}^*(\mathbf{v}) = (\bar{\sigma}_a^*(\mathbf{v}))_{a\in S}$. We say a school $a \in S$ is *oversubscribed* if $\int \bar{\sigma}_a^*(\mathbf{v})d\mu(\mathbf{v}) \geq 1$ and *undersubscribed* if $\int \bar{\sigma}_a^*(\mathbf{v})d\mu(\mathbf{v}) < 1$. In equilibrium, there will be at least (generically, exactly) one undersubscribed school which anybody can get admitted to (that is, even when she fails to get into any other schools she listed ahead of that school). Formally, a school $w \in S$ is said to be "worst" if its cutoff on $[0, 2]$ equals precisely 2. A worst school always exists, otherwise a positive measure of students would remain unassigned, constituting a contradiction. Then, we have the following lemma.

Lemma 5. *(i) Any student who prefers the worst school the most is assigned that school with probability 1 in equilibrium. (ii) If her most preferred school is undersubscribed and it is not the worst school, then she targets that school in equilibrium. (iii) In every equilibrium, almost every student whose top choice is not the worst school subscribes to a school with the probability that she targets that school, that is,* $\sigma^*(\mathbf{v}) = \bar{\sigma}^*(\mathbf{v})$.

In light of Lemma 5-(iii), we shall refer to "targets school a" simply as "subscribes to school a."

4 Welfare Properties of Alternative Procedures in Large Economy

It is useful to begin with an example. Let us consider the following simplified situation.

Example 2. *Suppose there are three schools,* $S = \{a, b, c\}$*, and three types of students* $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$*, each with* $\mu(\mathbf{v}^i) = 1$*, and their vNM values are described as follows.*

	v_j^1	v_j^2	v_j^3
$j = a$	5	4	1
$j = b$	1	2	5
$j = c$	0	0	0

Consider first DA-MTB. Each student draws three lottery numbers, $(\omega_a, \omega_b, \omega_c)$, one for each school. Given truthful reporting of preferences, the cutoffs of the schools a, b and c are determined at $\tilde{c}_a \approx 0.39, \tilde{c}_b \approx 0.45$, and $\tilde{c}_c = 1$, respectively.[16] The resulting allocation is

[16] The cutoffs for the schools are determined by the requirements that the measure of students assigned to each school equals one, its quota. The mass of students assigned a are those of type 1 and 2 students with

$\phi^{MTB}(\mathbf{v}^1) = \phi^{MTB}(\mathbf{v}^2) \approx (0.39, 0.27, 0.33)$ and $\phi^{MTB}(\mathbf{v}^3) \approx (0.22, 0.45, 0.33)$. This allocation is PE within $\{a, c\}$ and within $\{b, c\}$, but not OE (or PE) within $\{a, b\}$. The ordinal inefficiency within $\{a, b\}$ can be seen by the fact that type-$\{\mathbf{v}^1, \mathbf{v}^2\}$ students have positive shares of school b, and type-$\mathbf{v}^3$ students have positive share of school a, which they can swap with each other to do better. This feature stems from the independent drawings of priority lists for the schools. For instance, as in Figure 1, type-$\{\mathbf{v}^1, \mathbf{v}^2\}$ students may draw (ω_a, ω_b) and type-$\mathbf{v}^3$ students may draw (ω'_a, ω'_b). Hence, we have $a \rhd^{\phi^{MTB}} b \rhd^{\phi^{MTB}} a$. (Note that the cutoff for school c is 1, which explains why the allocation is PE within $\{a, c\}$ and within $\{b, c\}$.)

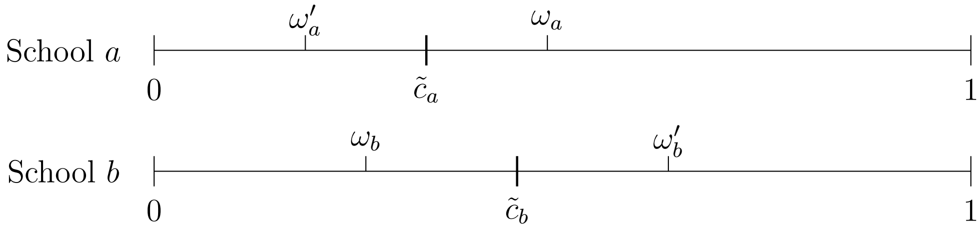

Figure 1: Ordinal inefficiency within $\{a, b\}$ under DA-MTB.

DA-STB avoids this problem, since each student draws only one lottery number for all schools. In this example, the cutoffs of schools a, b and c are $c_a = 1/2$, $c_b = 2/3$, and $c_c = 1$, respectively.[17] The resulting allocation is $\phi^{STB}(\mathbf{v}^1) = \phi^{STB}(\mathbf{v}^2) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ and $\phi^{STB}(\mathbf{v}^3) = (0, \frac{2}{3}, \frac{1}{3})$. This allocation is OE, and thus pairwise PE (by Lemma 2).

Figure 2: Ordinal efficiency of DA-STB

lottery draw less than $\tilde{c}_a$ plus type 1 students who have bad draws at b but good draws at a (i.e., less than $\tilde{c}_a$). Hence, the requirement reduces to

$$1 = 2 \times \tilde{c}_a + 1 \times (1 - \tilde{c}_b)\tilde{c}_a.$$

In this way, one obtains three equations, each corresponding to a "demand = supply" condition for each school, thus determining the cutoffs. See the proof of Lemma 4 in the Appendix.

[17] The cutoffs are again determined by the requirement that the measure of those students assigned to each school equals its supply. Since a is more popular than b, we have $c_a < c_b$. Given this, those assigned a are the type $\mathbf{v}^1$ and $\mathbf{v}^2$ students with $w < c_a$. Next, those assigned b are type $\mathbf{v}^3$ students with $w < c_b$ and type $\mathbf{v}^1$ and $\mathbf{v}^2$ students with $w \in [c_a, c_b]$. Therefore, c_a and c_b must solve

$$1 = 2 \times c_a \Leftrightarrow c_a = 1/2.$$

and

$$1 = 1 \times c_b + 2 \times (c_b - c_a).$$

See the proof of Lemma 3 for more detail.

To see this, consider any students who strictly prefer school b to school a. In our example, type-$\mathbf{v}^3$ students have such preference. These students can never be assigned school a since, whenever they have draws acceptable for school a (for instance $\omega < c_a$ in Figure 2), they will choose school b and admitted by it. Hence, we cannot have $b \rhd^{\phi^{STB}} a$. A similar logic implies that we cannot have $c \rhd^{\phi^{STB}} b$.[18] Hence, ϕ^{STB} admits no trading cycle. Despite the superiority over DA-MTB, the DA-STB allocation is not fully PE; type-$\mathbf{v}^1$ students can profitably trade with type-$\mathbf{v}^2$ students, selling probability shares of schools a and c in exchange for probability share of school b.

Consider lastly CADA. As with the two DA mechanisms, all students rank the schools truthfully; and type-$\{\mathbf{v}^1, \mathbf{v}^2\}$ students target school a and type-$\mathbf{v}^3$ target school b. The resulting equilibrium allocation is $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^2) = (\frac{1}{2}, 0, \frac{1}{2})$ and $\phi^*(\mathbf{v}^3) = (0, 1, 0)$. Notice that no type-$\{\mathbf{v}^1, \mathbf{v}^2\}$ students are ever assigned school b, which means in this case the allocation is fully PE.[19]

These observations are generalized as follows:

Theorem 4. (DA-MTB) *(i) The allocation ϕ^{MTB} from DA-MTB is PE within $\{a, w\}$ for each $a \in S \setminus \{w\}$. (ii) Generically, for any $K \subset S$ with $|K| \geq 2$, ϕ^{MTB} is not PE within K if K contains more than 2 schools or K does not contain the worst school w.*

Theorem 5. (DA-STB) *(i) The allocation ϕ^{STB} from DA-STB is OE and is thus pairwise PE. (ii) For a generic $\mathfrak{m}$, there exists no $K \subset S$ with $|K| > 2$ such that ϕ^{STB} is PE within K.*

Theorem 6. (CADA) *(i) Every equilibrium allocation ϕ^* of CADA is OE and is thus pairwise PE. (ii) Every equilibrium allocation of CADA is PE within the set of oversubscribed schools. (iii) If all but one school is oversubscribed in an equilibrium, then the equilibrium allocation of CADA is PE.*

In sum, DA-STB yields an ordinally efficient allocation in the large economy, but this is the most that can be expected from DA-STB, in the sense that the scope of efficiency is generically limited to (sets of) two schools. CADA is also ordinally efficient. However, the set of oversubscribed schools, which is determined endogenously in equilibrium, may contain more than two schools. Therefore CADA achieves a broader scope of efficiency.

[18] In this example, no student prefers school c to b, but the logic applies even if there were students with such a preference.

[19] While the equilibrium allocation of CADA is PE and that of DA is not, it does not imply that the former (ex ante) Pareto dominates the latter. Actually, $\mathbf{v}^3$ students become better off while $\mathbf{v}^1$ and $\mathbf{v}^2$ students become *worse off* in CADA compared to DA-STB. This example illustrates that allocations of CADA cannot be Pareto ranked with those of DA in general. However, under certain condition, we also show that CADA indeed Pareto dominates DA (Theorem 7).

Theorem 6-(ii) and (iii) offers the main characterization of CADA, which showcases the ex ante efficiency benefit associated with CADA. The benefit parallels that of a competitive market. Essentially, CADA activates "competitive markets" for oversubscribed schools.

This insight is borne out by our proof (in Appendix). Intuitively, suppose that students can trade their equilibrium probability shares of oversubscribed schools in an exchange market. PE within oversubscribed schools implies that no trade occurs in this exchange market. More formally, consider any equilibrium ϕ^* of CADA, which yields an assignment of $\phi^*(\mathbf{v})$ for student of type $\mathbf{v}$, which constitutes the student's endowment in the exchange market. Then each individual student can be seen effectively to choose shares $(x_a)_{a\in K}$ of oversubscribed schools $K \subset S$ to

$$\text{maximize} \sum_{a\in K} v_a x_a$$

subject to

$$\sum_{a\in K} p_a x_a \le \sum_{a\in K} p_a \phi^*_a(\mathbf{v}),$$

where "price" p_a is given by the mass of students targeting school a. More precisely, a student can attain any shares satisfying the "budget" constraint. Since $\phi^*(\mathbf{v})$ solves her maximization problem, she will attain the maximized value in equilibrium.[20]

Intuitively, each student is given a "budget" of unit probability she can allocate across alternative schools for targeting. A given unit probability can "buy" different numbers of shares for different schools, depending on how many others target those schools. If a mass $p_a \ge 1$ of students targets school a, allocating a unit budget (i.e., probability of targeting a) can only buy a share $1/p_a$.[21] In other words, the relative congestion at alternative schools, or their relative popularity, serves as relative "prices" for these schools.[22] In a large economy, individual students take these prices as given, so the prices play the usual role of allocating resources efficiently. Hence, the welfare benefit obtains much in the same fashion as the First Welfare Theorem.

[20] Note that the right hand side of the budget constraint becomes 1 if a student subscribes to an oversubscribed school and 0 if he or she subscribes to an undersubscribed school.

[21] To obtain a share x_a of an oversubscribed school $a \in K$, a student can target school a with probability $p_a x_a$, in which case she will be selected by school a with probability $\frac{p_a x_a}{p_a} = x_a$ since once she targets a she competes with mass p_a of students for a unit mass of seats at a.

[22] Given the DA format, a student may be assigned an undersubscribed school after targeting (and failing to get into) an oversubscribed school. This may cause a potential spill-over from consumption of an over-subscribed school toward undersubscribed schools. This spill-over does not undermine the efficient allocation, however. Under our CADA procedure, targeting alternative oversubscribed schools have no impact on the conditional probability of assignment with undersubscribed schools, since the tie breaking at non-target schools are determined by a *separate* random priority lists.

Why are competitive markets limited only to oversubscribed schools? Why not undersubscribed schools? Recall that one can be assigned an undersubscribed school in two different ways: she can target it, in which case she gets assigned it for sure if she applies to it. Alternatively, she can target an oversubscribed school but the school may reject her, in which case she may still get assigned that undersubscribed school via the usual DA channel. Clearly, assignment via this latter channel does not respond to, or reflect, the "prices" set by the targeting behavior. Consequently, competitive markets do not extend to the undersubscribed schools.

Finally, Part (i) asserts ordinal efficiency for CADA. At first glance, this feature may be a little surprising in light of the fact that different schools use different priority lists. As is clear from DA-MTB, this feature is susceptible to ordinal inefficiency. The CADA equilibrium is OE, however. To see this, observe first that any student who is assigned an oversubscribed school with positive probability must strictly prefer it to any undersubscribed school (or else she should have secured assignment to the latter school by targeting it). Thus, we cannot have $b \rhd^{\phi^*} a$ if school b is undersubscribed and school a is oversubscribed. This means that if the allocation admits any trading cycle, it must be within oversubscribed schools or within undersubscribed schools. The former is ruled out by Part (ii) and the latter by the same argument as Theorem 5-(i).

The characterization of Theorem 6-(ii) is tight in the sense that there is generally no bigger set that includes all oversubscribed schools *and some undersubscribed school* that supports Pareto efficiency.[23]

Theorem 6 refers to an endogenous property of an equilibrium, namely the set of over/undersubscribed schools. We provide a sufficient condition for this property. For each school $a \in S$,

[23] To see this, suppose there are four schools, $S = \{1, 2, 3, 4\}$, and four types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4\}$, with $\mu(\mathbf{v}^1) = \frac{3-\varepsilon}{2}$, $\mu(\mathbf{v}^2) = \frac{1+\varepsilon}{2}$, $\mu(\mathbf{v}^3) = \frac{3-\varepsilon}{2}$, and $\mu(\mathbf{v}^4) = \frac{1+\varepsilon}{2}$ where ε is a small number.

	v_j^1	v_j^2	v_j^3	v_j^4
$j = 1$	10	10	20	20
$j = 2$	3	5	9	8
$j = 3$	1	4	8	1
$j = 4$	0	0	0	0

In this case, type 1 and 3 students subscribe to school 1, and type 2 and 4 students subscribe to school 2. More specifically, the allocation ϕ^* has $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^3) = (\frac{1}{3-\varepsilon}, 0, \frac{2-\varepsilon}{2(3-\varepsilon)}, \frac{2-\varepsilon}{2(3-\varepsilon)})$ and $\phi^*(\mathbf{v}^2) = \phi^*(\mathbf{v}^4) = (0, \frac{1}{1+\varepsilon}, \frac{\varepsilon}{2(1+\varepsilon)}, \frac{\varepsilon}{2(1+\varepsilon)})$. Although schools 1 and 2 are oversubscribed, this allocation is not PE within $\{1, 2, 3\}$ since type 1 students can trade probability shares of school 1 and 3 in exchange for probability share at 2, with type 2 students. The allocation is not PE within $\{1, 2, 4\}$ either, since type 3 students can trade probability shares of school 1 and 4 in exchange for probability share at 2, with type 4 students. Therefore $\{1, 2\}$ is the largest set of schools that support Pareto efficiency.

let $m_a^* := \mu(\{\mathbf{v} \in \mathcal{V} | \pi_1(\mathbf{v}) = a\})$ be the measure of students who prefer a the most. We then say a school a is **popular** if $m_a^* \geq 1$, namely, the size of the students whose most preferred school is a is as large as its capacity.

It is easy to see that every popular school must be oversubscribed in equilibrium. Suppose to the contrary that a popular school a is undersubscribed. Then, by Lemma 5-(ii), every student with $\mathbf{v}$ with $\pi_1(\mathbf{v}) = a$ must subscribe to a, a contradiction. Since every popular school is oversubscribed, the next result follows from Theorem 6.

Corollary 1. *Any equilibrium allocation of CADA is PE within the set of popular schools.*

It is worth emphasizing that the popularity of a school is sufficient, but not necessary, for that school to be oversubscribed. In realistic settings, many non-popular schools will be oversubscribed. In this sense, Corollary 1 understates the benefit of CADA. This point is confirmed by our simulation in Section 5. Even though full PE is not ensured by any of the three mechanisms, our results imply the following comparison between standard DA and CADA.

Corollary 2. *(i) If $n \geq 3$, generically the allocations from DA-STB and DA-MTB are not PE. (ii) The equilibrium allocation of CADA is PE if all but one school is popular.*

Remark 1. *With finite students, the allocation from DA-STB is ex post Pareto efficient but is not OE. But as the number of students and school seats get large, the DA-STB allocation becomes OE in the limit (Che and Kojima, 2010). When the schools have intrinsic priorities, the DA-STB is not even ex post Pareto efficient (Abdulkadiroğlu, Pathak and Roth (2009)). Similarly, the CADA is not OE in the finite economy, but the inefficiency vanishes in the large economy.*[24]

The results so far give a sense of a three-way ranking of DA-MTB, DA-STB, and CADA. Specifically, if the allocation from DA-MTB is PE within $K \subset S$, then so is the allocation from

[24] OE implies ex post PE, therefore both DA-STB and CADA are ex post PE in large economies. However, the CADA allocation may fail to be ex post PE in the finite economy, since agents may miscoordinate on their targeting strategies. To see this suppose that there are four schools a, b, c, d each with one seat. There are four students; two of them are type-a with vNM values $v_a = 10, v_b = 7$ and $v_c = v_d = 0$, and last two students are type b with $v_a = 7, v_b = 10, v_c = v_d = 0$. There is an inefficient equilibrium in which type-a students target b and type-b students target a, so the former students have assignment $(0, 1/2, 1/4, 1/4)$ and the latter have $(1/2, 0, 1/4, 1/4)$. No agent can profitably deviate by targeting his favorite school since the odds of success would fall to $1/3$ whereas her odds of success at the second preferred school at the candidate equilibrium is $1/2$ (and $1/2$ of 7 is higher than $1/3$ of 10). Of course, there is also an efficient equilibrium in which type-i students target school $i = a, b$. It is easy to see that the inefficient equilibrium disappears as the economy gets large. For instance, if the economy doubles (i.e., each school has two seats, and there are four students of each type), the inefficient equilibrium disappears since a deviation by targeting one's favorite school leads to probability $2/5$ of assignment at that school, and $2/5$ of 10 is higher than $1/2$ of 7.

DA-STB, although the converse does not hold; and if the allocation from DA-STB is PE within $K' \subset S$, then so is the allocation from CADA, although the converse does not hold. Between the two DA algorithms, the DA-STB allocation is OE, whereas the DA-MTB allocation is not pairwise PE.

In particular, the CADA allocation is PE within a strictly bigger set of schools than the allocations from DA algorithms, if there are more than two popular schools. Unfortunately, this is not the case when all students have the same ordinal preference. This case, though special, is important since parents often tend to rank schools similarly. In this case, there is only one popular school in a CADA equilibrium, so Theorem 6 and Corollary 1-(i) do little to distinguish CADA from DA-STB. Nevertheless, we can find the CADA to be superior in a more direct way. To this end, let $\mathcal{V}^U := \{\mathbf{v} \in \mathcal{V} | v_1 > ... > v_n\}$.

Theorem 7. *Suppose all students have the same ordinal preferences in the sense $\mu(\mathcal{V}^U) = \mu(\mathcal{V})$. Every equilibrium allocation of CADA (weakly) Pareto dominates the allocation arising from DA-STB and DA-MTB.*

This result generalizes Example 1 discussed in the introduction. If all students have the same ordinal preferences, the DA algorithm with any random tie-breaking treat all students in the same way, meaning that each student is assigned each school with equal probability. Under CADA, the students can at least replicate this random assignment via targeting.[25]

5 Simulations

The theoretical results in the previous sections do not speak to the magnitude of efficiency gains or losses achieved by each mechanism. Here, we provide a numerical analysis of the magnitude via simulations. The numerical analysis also enables us to examine the effects of (coarse) school priorities on the CADA and the standard DA mechanisms.

In our numerical model, we have 5 schools, each with 20 seats and 100 students. The computational burden of computing Nash equilibrium of CADA limits our ability to run simulations with larger numbers. However, simulations with 5 schools, each with 20 seats and 100 students prove to be sufficient to produce comparative statics that is in line with theory. They also

[25]This Pareto dominance result also holds in a finite economy with incomplete information in the following sense. When all students have the same ordinal preferences in any symmetric Bayesian equilibrium of CADA, each type of student is weakly better off than she is under DA with any symmetric tie-breaking. The argument is the same as that of Theorem 1 of Abdulkadiroğlu, Che and Yasuda (2011), which shows that the Boston mechanism weakly Pareto dominates DA in such Bayesian settings. We shall remark on the Boston mechanism in Section 7.

suggest that our analytical results obtained from the continuum model seem to hold even when the capacity of each school is fairly small.

In our model, student i's vNM value for school a, $\tilde{v}_{ia}$, is given by

$$\tilde{v}_{ia} = \alpha u_a + (1-\alpha) u_{ia}$$

where $\alpha \in [0,1]$, u_a is common across students and u_{ia} is specific to student i and school a. For each α, we draw $\{u_a\}$ and $\{u_{ia}\}$ uniformly and independently from the interval $[0,1]$ to construct student preferences. Since we shall focus on Utilitarian welfare, it is important to normalize vNM utilities so that the findings are robust to their affine transformation. To this end, we normalize each student's vNM utilities by $v_{ia} = \zeta_a(\tilde{\mathbf{v}}_i) := \frac{\tilde{v}_{ia} - \min_{a'} \tilde{v}_{ia'}}{\max_{a'} \tilde{v}_{ia'} - \min_{a'} \tilde{v}_{ia'}}$. Under this normalization, the values of schools range from zero to one, with the value of the least preferred school set to zero and that of the most preferred to one. This normalization is invariant to affine transformation in the sense that $\zeta_a(\tilde{v}_{ia_1}, ..., \tilde{v}_{ia_5}) = \zeta_a(\theta \tilde{v}_{ia_1} + \beta, ..., \theta \tilde{v}_{ia_5} + \beta)$, for any $\theta \in \mathbb{R}_{++}, \beta \in \mathbb{R}$.

The students' preferences become similar to one another both ordinally and cardinally as α gets large. In the extreme case with $\alpha = 1$, students have the same cardinal (as well as ordinal) preferences. In the opposite extreme with $\alpha = 0$, students' preferences are completely uncorrelated. Given a profile of normalized vNM utility values, we simulate DA-STB and DA-MTB, compute a complete-information Nash equilibrium of CADA and the resulting CADA allocation. We repeat this computation 100 times each with a new set of (randomly drawn) vNM utility values for all values of α. In addition, we solve for a **first-best** solution, which is the utilitarian maximum for each set of vNM utility values. We then compute the average welfare under each mechanism, i.e., the total expected utilities realized under a given mechanism averaged over 100 draws (see the Supplementary Appendix for details).

In Figure 3, we compare the three mechanisms against the first best solution. We plot the welfare of each mechanism as the percentage of the welfare of the first best solution. Two observations emerge from this figure. First, the welfare generated by each mechanism follows a U-shaped pattern. Second, CADA outperforms DA-STB, which in turn outperforms DA-MTB at every value of α, and the gap in performance between CADA and the other mechanisms grows with α. All three mechanisms perform almost equally well and produce about 96% of the first-best welfare when $\alpha = 0$. In this case, students have virtually no conflicts of interests, and each mechanism more or less assigns students to their first choice schools. The welfare gain of CADA increases as α increases. This is due to the fact that competition for one's first choice increases as α increases (and students' ordinal preferences get similar to one another). In those instances, who gets her first choice matters. While DA-STB and DA-MTB determine this purely randomly, CADA does so based on students' messages. Intuitively, if a student's vNM value for a school increases, the likelihood of the student targeting that school in an equilibrium of CADA — therefore the likelihood of her getting into that school — increases. This feature of CADA contributes to its welfare gain. DA-STB and DA-MTB start catching up with CADA at $\alpha = 0.9$. In this case, students have almost the same cardinal preferences, so any matching is close to being ex ante efficient. At $\alpha = 0.9$, CADA achieves 95.5% of the first best welfare, whereas DA-STB achieves 92.2%.[26]

Figure 4 gives further insight into the workings of the mechanisms. It shows the percentage of students getting their first choices under each mechanism. First, DA-MTB assigns noticeably smaller numbers to first choices. This is due to the artificial stability constraints created by the use of multiple tie breaking, which also explains the bigger welfare loss associated with DA-MTB. The patterns for CADA and DA-STB are more revealing. In particular, both assign almost the same number of students to their first choices for each value of α. That is, whereas the poor welfare performance of DA-MTB is explained by the low number of students getting their first choices, the difference between the other two is explainable not by *how many* students, but rather by *which students*, are assigned their first choices.

This is illustrated more clearly by Figure 5, which shows the ratio of the mean utility of those who get their k-th choice under CADA to the mean utility of those who get their k-th choice under DA-STB at the realized matchings, for $k = 1, 2, 3$. Specifically, those who get their k-th choice achieve a higher utility under CADA than under DA-STB for each $k = 1, 2, 3$. The utility gain is particularly more pronounced for those assigned their second or third choices. This simply reflects the feature of CADA that assigns students based on their preference intensities: under CADA, those who have less to lose from the second- or third-best choices are more likely to target those schools, and are thus more likely to comprise such assignments.

Figure 6 shows that the number of oversubscribed schools is larger on average than the number of popular schools. Note that the average number of oversubscribed schools is larger than 2 at all values of α. Recall from Theorems 5 and 6 that DA-STB is generically never PE within a set of more than 2 schools, but that CADA is PE within the set of oversubscribed schools. Figure 6 thus shows the scope of efficiency achieved by CADA can be much higher than is predicted by Corollary 1. It is also worth noting that the average number of oversubscribed schools exceeds 3 for $\alpha \leq 0.4$. This implies that there are often 4 oversubscribed schools. At those instances, CADA achieves full Pareto efficiency (recall Theorem 6-(ii)).

In practice, some schools have (non-strict) intrinsic priorities. We thus study their impact on assignments numerically. To this end, we modify our model as follows: Each school has two

[26] At the extreme case of $\alpha = 1$, preferences are the same so every matching is efficient and the welfare generated by each mechanism is equal to the first best welfare.

priority classes, high priority and low priority. For each preference profile above, we assume that 50 students have high priority in their first choice and low priority in their other choices, 30 students have high priority in their second choice and low priority in their other choices, and 20 students have high priority in their third choice and low priority in their other choices.[27]

It is well known that standard mechanisms such as DA do not produce student optimal stable matching when schools have non-strict priorities. Erdil and Ergin (2008) have proposed a way to attain constrained ex post efficiency subject to respecting school priorities, via performing so-called stable improvement cycles after an initial DA assignment. We thus simulate this algorithm, referred to as DASTB+SIC, to see how it compares with the CADA.

In Figure 7, we compare CADA, DA-STB and DA-STB+SIC again measured as percentage of first-best welfare. Again, CADA outperforms DA-STB for all values of α. Since DA-STB+SIC is designed to achieve constrained ex post efficiency (while CADA and DA-STB are not), it is not surprising that the former does better when α is relatively small. In that case, students' ordinal preferences are sufficiently dissimilar that ordinal efficiency matters. As α gets large, however, ordinal efficiency becomes less relevant and cardinal efficiency becomes more important. For $\alpha \geq 0.5$, CADA catches up with DA-STB+SIC and outperforms it as α gets large. In particular, when α is close to 1, virtually all matchings are ex post efficient, so DA-STB+SIC has little bite. The cardinal efficiency still matters, and in this regard, CADA does better than the other mechanisms. This finding is noteworthy since parents are likely to have similar ordinal preferences in real-life choice settings. In those instances, CADA allocates schools more efficiently than other mechanisms in ex ante welfare.

6 Discussion

6.1 Enriching the Auxiliary Message

One can modify CADA to allow for richer auxiliary messages, perhaps at the expense of some practicality. For instance, the auxiliary message can include a rank order of schools up to $k \leq n$, with a tie broken in the lexicographic fashion according to this rank order: students targeting a school at a higher lexicographic component is favored by that school in a tie relative to those who do not target or target it at a lower lexicographic component. We call the associated CADA a **CADA of degree** k.

A richer message space could allow students to signal their relative preference intensities better, and this may lead to a better outcome (see Abdulkadiroğlu, Che and Yasuda (2008)

[27]This assumption is in line with the stylized fact about the Boston school system.

for an example). A richer message space need not deliver a better outcome, however. With more messages, students have more opportunities to express their relative preference intensities over different sets of schools. The increased opportunities may act as substitutes and militate each other. For instance, an increased incentive to self select at low-tier schools may lessen a student's incentive to self select at high-tier schools. This kind of "crowding out" arises in the next example.

Example 3. *There are 4 schools, $S = \{a, b, c, d\}$, and two types of students $\mathcal{V} = \{\mathbf{v}^1, \mathbf{v}^2\}$, with $\mu(\mathbf{v}^1) = 3$ and $\mu(\mathbf{v}^2) = 1$.*

	v_j^1	v_j^2
$j = a$	12	8
$j = b$	2	4
$j = c$	1	3
$j = d$	0	0

Consider first CADA of degree 1. Here, it is optimal for a type-$\mathbf{v}^1$ student to target a for any strategy profile of other students, so that type-$\mathbf{v}^1$ students target a. Then it is optimal for type-$\mathbf{v}^2$ students to target school b. In other words, the latter type of students self select into the second popular school in the unique equilibrium. The resulting allocation is $\phi^*(\mathbf{v}^1) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$ and $\phi^*(\mathbf{v}^2) = (0, 1, 0, 0)$. The expected utilities are $EU^1 = 4.33$ and $EU^2 = 4$. In fact, this allocation is PE.

Suppose now CADA of degree 2 is used. In the unique equilibrium, type-$\mathbf{v}^1$ students choose schools a and b as their first and second targets, respectively. Meanwhile, type-$\mathbf{v}^2$ students choose school a (instead of school b!) for their first target and school c for their second target. Here, the opportunity for type 2 students to self select at a lower-tier school (school c) blunts their incentive to self select at a higher-tier school (school b). The resulting allocation is thus $\phi^{**}(\mathbf{v}^1) = (\frac{1}{4}, \frac{1}{3}, \frac{1}{12}, \frac{1}{3})$ and $\phi^{**}(\mathbf{v}^2) = (\frac{1}{4}, 0, \frac{3}{4}, 0)$, which yield expected utilities of $\overline{EU}^1 = 3.75$ and $\overline{EU}^2 = 4.25$. This allocation is not PE since type-$\mathbf{v}^2$ students can trade probability shares of schools a and c in exchange for probability share of b, with type-$\mathbf{v}^1$ students.

Even though ϕ^* does not Pareto dominate ϕ^{**}, the former is PE whereas the latter is not. Further, the former is superior to the latter in the Utilitarian sense (recall that students' payoffs are normalized so that they aggregate to the same value for both types): the former gives aggregate utilities of 17, the highest possible level, whereas the latter gives 15.5.

This example suggests that the benefit from enriching the message space is not unambiguous. This is a potentially important point. In practice, expanding a message space may add a burden on the parents to be more strategically sophisticated,[28] which may be important to avoid for

[28]We do not make a theoretical argument here, as we do not have a theoretical framework to measure strategic

the success of a procedure. Hence, the example reinforces the appeal of the simple CADA (i.e., of degree 1).

6.2 Strategic Naivety

Since CADA involves some "gaming" aspect, albeit limited to tie-breaking, a natural concern is that not all families may be strategically competent. It is thus important to investigate how CADA will perform when some families are not strategically sophisticated. To this end, we consider students who are "naive" in the sense that they always target their most preferred schools in the auxiliary message and submit preference rankings truthfully. Targeting the most preferred school appears to be a simple and reasonable choice when a student is unsure about the popularity of alternative schools or is unclear about the role the auxiliary message plays in the assignment. Such a strategy will indeed be a best response for many situations, particularly if the first choice is distinctively better than the rest of the choices, so it could be a reasonable approximation of "naive" behavior. We assume that there is a positive measure of students who are naive in this way, and the others know the presence of these students and their behavior, and respond optimally against them. Surprisingly, the presence of naive students do not affect the main welfare results in a qualitative way.

Theorem 8. *In the presence of naive students, the equilibrium allocation of CADA satisfies the following properties: (i) The allocation is OE, and is thus pairwise PE. (ii) The allocation is PE within the set K of oversubscribed schools. (iii) If every student is naive, then the allocation is PE within $K \cup \{l\}$ for any undersubscribed school $l \in J := S \backslash K$.*

Theorem 8-(i) and (ii) are qualitatively the same as the corresponding parts of Theorem 6. Further, Lemma 5-(ii) remains valid in the current context, implying that any popular schools must be oversubscribed here as well. Hence, the same conclusion as Corollary 1 holds.

Corollary 3. *In the presence of naive students, the equilibrium allocation of CADA is PE within the set of popular schools.*

Strategy-proofness of DA ensures no change in its welfare performance. Therefore, the welfare comparison between CADA and DA in the presence of naive students does not change in a qualitative way.

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sophistication

6.3 CADA with "Exit Option"

The preceding subsection has seen that the main welfare property of CADA extends to the situation where some students behave naively. This does not mean, however, that naive students are not disadvantaged by the others who behave more strategically.[29] The CADA mechanism can be modified to provide an extra safeguard for those who are averse to strategic aspect of the game. This can be done by augmenting the message space to include an "exit option."

Specifically, the CADA with exit option (CADA-EO) involves the following three steps:

- **Step 1:** All students submit ordinal preferences, plus an "auxiliary message," naming a "target" school or specifying "exit."

- **Step 2:** A standard DA-STB is run (i.e., ignoring the auxiliary message). The students who specified "exit" in their auxiliary message are assigned the seats according to this procedure. Remove these students along with the seats they are assigned. The capacity of schools are reset to reflect the seats removed.

- **Step 3:** CADA is run with respect to the remaining seats and students based on the target messages. That is, target and regular priority lists are randomly generated, and the priorities of the remaining students at schools are determined based on the schools' intrinsic priorities, the two random lists and students' target messages, according to the rule described in Section 2. The DA is run based on the students' ordinal preferences and the schools' priories of students determined in this way.

Clearly, CADA-EO maintains the same feature of DA as described in Theorem 1. Most important, CADA-EO offers each student an option to "replicate" the same lottery of schools as she will obtain from DA-STB, by simply specifying "exit" in her auxiliary message. The following is immediate:

Theorem 9. *Fix an arbitrary school choice problem in the finite or large economy where schools have arbitrary (intrinsic) priorities over students. The allocation implemented in any equilibrium (either complete-information or Bayesian) of CADA-EO weakly Pareto dominates the allocation of DA-STB.*

Theorem 9 shows that one can easily modify CADA to ex ante Pareto-dominate the DA-STB—one of the current favorite mechanisms known so far. While it is important to know that there is a mechanism that does no worse and possibly do better for some students than

[29] Pathak and Sönmez (2008) formalize the sense in which the sophisticated students benefit from the Boston mechanism in comparison with the DA mechanism at the expense of the naive players. Meanwhile, Abdulkadiroğlu, Che and Yasuda (2011) suggest that some unsophisticated students may actually benefit from the presence of the sophisticated students.

the best current procedure, we do not necessarily favor CADA-EO over CADA. The reason is that, although CADA-EO is definitely a safer alternative when it comes to switching from the DA-STB, its benefit in terms of realizing cardinal efficiency is also limited in comparison with CADA. In particular, the desirable properties of CADA described in Theorem 6 may not obtain.

This point can be illustrated again using Example 2 in Section 4. Observe that the type-$\{\mathbf{v}^1, \mathbf{v}^2\}$ students are worse off from CADA in comparison with DA-STB. Specifically, their assignment is $\phi^{STB}(\mathbf{v}^1) = \phi^{STB}(\mathbf{v}^2) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ under DA-STB, but $\phi^*(\mathbf{v}^1) = \phi^*(\mathbf{v}^2) = (\frac{1}{2}, 0, \frac{1}{2})$ under CADA, and they prefer the former since they have the same probability share of a but higher share of b under the former, in comparison with the latter. Hence, even though the CADA allocation is PE and the DA-STB allocation is not, the former does not Pareto dominate the latter. Suppose now CADA-EO is employed instead. It is an equilibrium for all type $\mathbf{v}^3$ students to target b, and for all type-$\mathbf{v}^1$ or $\mathbf{v}^2$ students to "exit." This equilibrium thus produces the same allocation as DA-STB.[30]

6.4 Dynamic Implementation

As noted, the welfare benefit of CADA originates from the competitive markets it induces. Unlike the usual markets where there are explicit prices, however, in the CADA-generated markets, students' beliefs about the relative popularity of schools act as the prices. Hence, for the CADA to have the desirable welfare benefit, their beliefs must be reasonably accurate. In practice, students/parents' beliefs about schools are formed based on their reputations; thus, as long as the school reputations are stable, they can serve as reasonably good proxies for the prices. Nevertheless, students may not share the same beliefs, and their beliefs may not be accurate, in which case CADA procedure will not implement the CADA equilibrium precisely.

The CADA mechanism can be modified to implement the desired equilibrium more precisely. The idea is to allow students to dynamically revise their target choices based on the population distribution of choices, which is made public, until the number of students changing their choices fall under a certain threshold. We can implement the desired equilibrium via such best response dynamics by making choices final and find the CADA assignment *only* when the threshold is

[30] This is indeed the unique symmetric equilibrium in which some (positive measure of) agents invoke targeting. An outcome in which all students exit is always an equilibrium of any CADA-EO game, because unilateral deviation does not impact the random assignment of the deviating player. Second, some measure of students must invoke "exits" in equilibrium, or else type $\mathbf{v}^1$ and $\mathbf{v}^2$ would be targeting a and $\mathbf{v}^3$ would be targeting b, so $\mathbf{v}^1$ and $\mathbf{v}^2$ would benefit from exiting. Therefore, at least one type exits in equilibrium. If $\mathbf{v}^3$ does not exit, then $\mathbf{v}^3$ targets b and it is optimal for both $\mathbf{v}^1$ and $\mathbf{v}^2$ to exit. If $\mathbf{v}^3$ exits, it is optimal for both $\mathbf{v}^1$ and $\mathbf{v}^2$ to target a or to exit. In all cases, the equilibrium outcome coincides with the DA-STB outcome.

passed.

6.5 Excess Capacities and Outside Options

Thus far, we have made simplifying assumptions that the aggregate measure of students equal the aggregate capacities of public schools and that all students find each public school acceptable. These assumptions may not hold in reality. While public schools must guarantee seats to students, all the seats need not be filled. And some students may find outside options, such as home or private schooling, better than some public schools. One can relax these assumptions by letting the aggregate capacities to be (weakly) greater than n and by endowing each student with an outside option drawn from $[0, 1]$.[31] Extending the model in this way entails virtually no changes in the main tenet of our paper. All theoretical results continue to hold in this relaxed environment. A subtle difference arises since, with excess capacities, there may be more than one school with cutoff equal to one under DA-MTB, so its allocation may become PE within more pairs of schools. Nevertheless, Theorems 1-9 remain valid. For instance, the DA-STB allocation is ordinally efficient. The CADA allocation is ordinally efficient and Pareto efficient within oversubscribed, and thus popular, schools.

6.6 A Limit Foundation of the Continuum Economy Model

As mentioned earlier, real-life school choice problem involve a large but finite number of participants. A comparison of CADA with DA in such markets requires analysis of students' strategic behavior with regard to their targeting of schools under CADA. However, such an analysis is not tractable in the general finite economy model. Our modeling choice with non-atomic continuum of students and finitely many schools with mass capacities helps overcome such difficulties. Further, our continuum economy model is well founded as an approximation of large finite economy models with similar fundamental characteristics. Bodoh-Creed (2011) shows that the equilibrium of the continuum model obtains as the limit of equilibrium strategies of the analogous large finite model if each agent's payoff is continuous function of his own type, his action, the empirical distribution of other players and the state of the world. Since these latter properties are satisfied in our model, the equilibrium of our continuum model obtains as the limit of equilibrium strategies of the analogous large finite model. In this sense, our findings provide an approximation for otherwise intractable large school choice problems.

[31] This modeling approach implicitly assumes the outside options to have unlimited capacities, which may not accurately reflect the scarcity of outside option such as private schooling.

There is a growing literature on large matching markets (see Che and Kojima, 2010; Immorlica and Mahdian 2005; Kojima and Pathak, 2009; Kojima, Pathak and Roth 2010) studying consequential impact of market size on certain market trade-offs. Recently, Azevedo and Leshno (2012) and Azevedo (2011) study the set of stable matchings in a similar environment with continuum players. Our continuum economy model of the DA mechanism provides a framework for studying large markets.

7 Conclusion

In this paper, we have proposed a new deferred acceptance procedure, Choice-augmented DA (CADA), in which students are allowed, via signaling of their preferences, to influence how they are treated in a tie for a school.

There are other matching procedures that also allow participants to express their cardinal preferences. The Boston mechanism which was replaced with the DA by the BPS in 2005 is one such procedure. In the Boston mechanism, students also rank the schools, and each school assigns its seats according to the order students rank that school during registration: each school accepts first those who rank it first, using their own priorities or random lotteries to break ties, and accepts those who rank it second only when seats are available, and so forth. Under this mechanism, therefore, a student can increase her odds of assignment at a school by ranking that school highly. For instance, in Example 1, student 3 can ensure her sure assignment at school b by raking it at the top, if the other two rank b at the second. This feature allows the students to express their cardinal preferences. In fact, just like the CADA, the Boston mechanism implements the desirable assignment in that example as the unique equilibrium; students 1 and 2 have a dominant strategy of ranking the schools truthfully, and student 3 has a best response of (strategically) ranking school b as her first choice. In a companion paper (Abdulkadiroğlu, Che, Yasuda, 2011), we show that this benefit generalizes to any symmetric Bayesian equilibrium of Boston mechanism, which weakly Pareto dominates the DA with standard tie-breaking if all students have the same ordinal preferences and schools have no priorities.[32]

Despite this similarity, the Boston mechanism has a number of disadvantages compared with the CADA. First, as is clear from the example, its beneficial effect—the ability to express one's cardinal intensities—is achieved only as a result of manipulating her rankings. That is, not only does the Boston mechanism fail to be strategy-proof, its failure is needed for the agents

[32]Further, Miralles (2008) applies the arguments developed in this paper to show that a variant of the Boston mechanism that breaks ties at schools independently has a similar ex ante welfare property as the CADA.

to express cardinal preferences. Second, the strategic nature of Boston mechanism makes it susceptible to mistakes and miscoordination on the part of the participants. Even when the students can play equilibrium with full information about other participants, they may still coordinate on suboptimal stable matching (Ergin and Sönmez, 2006). Even more serious, and arguably more plausible, form of miscoordination is that students may not coordinate on any equilibrium play due to incomplete information and strategic uncertainty. In practice, students are unlikely to assess other students' preferences, their priorities, and their strategic responses, accurately, which can easily lead to non-equilibrium play. The consequences are both inefficiencies and lack of fairness (i.e., stability). Third, the demand for strategic play puts strategically naive participants at disadvantage against more sophisticated participants (Pathak and Sönmez, 2008). In the Boston mechanism, by not ranking a school as first choice, a student loses her priority at that school to those who rank it as first choice. For instance, a student with a neighborhood priority at a reasonably good school could lose her priority under the Boston mechanism if she did not rank it as first choice.

By contrast, CADA is ordinally strategy-proof. While CADA involves strategic plays, its scope is limited to targeting, and its influence is kept within a priority class. Although it is difficult to conceptualize, and measure in a principled way, the simplicity of strategic decision making involved in a mechanism, one sensible approach — a long-held one since Hurwicz (see Arrow (2009)) — is to use the dimensionality of the message space as a measure of informational burden facing the participants of a mechanism. In this regard, targeting involves a relatively simple and straightforward strategic decision. We thus believe that the scope for miscoordination is limited in CADA. Further, we provide a dynamic implementation of targeting game that facilitates strategic coordination of the students. While targeting requires strategic play, the CADA limits the harm from strategic mistakes. For instance, suppose a student targets her favorite school while she enjoys a neighborhood priority at her second-best school. In case she fails to get in her favorite school, she does not lose high priority at the second-best school unlike the Boston mechanism. Such a student may even benefit from her naive behavior under CADA if her neighborhood school is not oversubscribed by neighborhood students.

Another mechanism that incorporates cardinal welfare is the pseudo-market mechanism proposed by Hylland and Zeckhauser (1979).[33] This mechanism purports to install competitive

[33] A similar mechanism is also used in course allocation mechanisms (see Budish and Cantillon, 2011). Sönmez and Ünver (2010) imbed the DA algorithm in "course bidding" employed by some business schools. These two proposals differ in the application, however, as well as in the nature of the inquiry: we are interested in studying the benefit of adding a "signaling" element to the DA algorithm. By contrast, their interest is in studying the effect of adding ordinal preferences and the DA feature to course bidding.

In a broader sense, our paper is an exercise of mechanism design without monetary transfers, and in that markets for trading probability shares of alternative objects using a fictitious currency. Specifically, the mechanism endows each agent with a fixed budget in a fictitious currency, 100 tokens say, and allows the agents to spend their budget endowments to "buy" probability shares of alternative goods, and the price per unit probability of owning each good is then adjusted to clear the markets. For large markets, this mechanism admits a competitive equilibrium, which is ex ante efficient by the first welfare theorem.[34] Our result has a similar flavor. Indeed, the main contribution of our paper is to recognize that adding a signaling device as simple as targeting a school can have the same kind of "market-activating" effects as the pseudo-market mechanism. Although CADA does not generally attain full ex ante efficiency, the strategic environment is simple, and the strategic deliberation required for the agents is not so demanding; by contrast, formulating ones' cardinal utilities (instead of simply "acting on them") could be more onerous,[35] and the consequence of miscalculation on efficiency may be large. Also, competitive equilibria are computationally difficult to find (see McLennan (2011)), which further limits its practicality. Most important, school priorities are already imbedded in CADA, whereas the pseudo-market mechanism does not include priorities. This is an important distinction since priorities are a salient feature of school choice.

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sense is similar to the recent ideas of "storable votes" (Casella, 2005) and "linking decisions" (Jackson and Sonnenschein, 2007). Just like them, CADA "links" how a student is treated in a tie at one school to how she is treated in a tie at another school, and this linking makes communication credible. Clearly, applying the idea in a centralized matching context is novel and differentiates the current paper. There is a further difference. Jackson and Sonnenschein (2007) demonstrated the efficiency of linking when (linkable) decisions tend to infinity, relying largely on the logic of the law of large numbers. To our knowledge, the current paper is the first to characterize the precise welfare benefit of linking fixed (finite) number decisions (albeit with continuum of agents). Coles, Kushnir and Niederle (2011) introduce preference signaling in two-sided decentralized matching markets.

[34] In the finite economy, the agents do not have to act as price takers. More generally, for a finite economy, there is no strategy-proof mechanism that treats the agents with the same preferences equally and implements an ex ante efficient allocation (Zhou, 1990).

[35] As mentioned, a traditional way to measure the simplicity of a mechanism is to consider the dimensionality of the message space for its participants. From this perspective, CADA involves a lower-dimensional message space than the HZ's pseudo market mechanism in which the participants must submit their vMN values. Also, computer scientists recognize that the "humans have a hard time precisely assessing utilities," which led them to favor "comparison queries" to "direct evaluation queries" as a method of eliciting preferences (Sandholm and Boutilier (2006)).

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Appendix: Proofs of the main results

Proof of Theorem 1. (i) If student i has higher priority than student j at school s, then i is ranked higher than j at s at step 2 of CADA regardless of their targeting behavior and the tie breakers. Consequently, if i prefers s to her CADA assignment, j is not assigned s under CADA, since CADA is a DA. Therefore CADA eliminates justified envy. To the contrary, if there were a profile of student preferences, school preferences and targeting behavior at which some student could benefit from preference manipulation, the student would benefit from preference manipulation in the corresponding DA that is induced by the same profile, a contradiction with the strategy-proofness of DA (Dubins and Freedman, 1981; Roth 1982). Therefore CADA is strategy-proof with respect to students' ordinal preferences. (ii) Consider an arbitrary targeting behavior and a realization of the corresponding CADA assignment x. Suppose to the contrary that there is an individually rational assignment y that every student prefers over the CADA assignment. Note that x is the student optimal stable assignment and y is an individually rational assignment in the induced problem with strict school priorities. Then every student in the induced problem prefers y to x, which contradicts with Theorem 6 of Roth (1982). ▌

Proof of Lemma 3. For any $S' \subset S$ and $a \in S'$, let $m_a(S') := \mu(\{\mathbf{v}|v_a \geq v_b, \forall b \in S'\})$ be the measure of students who prefer school a the most among S'. The cutoffs of the schools are then defined recursively as follows. Let $\hat{S}^0 \equiv S$ $\hat{c}^0 \equiv 0$, and $\hat{x}_a^0 \equiv 0$ for every $a \in S$. Given $\hat{S}^0, \hat{c}^0$, $\{\hat{x}_a^0\}_{a\in S}, \dots,$ $\hat{S}^{t-1}, \hat{c}^{t-1}, \{\hat{x}_a^{t-1}\}_{a\in S}$, and for each $a \in S$ define

$$\hat{c}_a^t = \sup\left\{c \in [0,1] \,\middle|\, \hat{x}_a^{t-1} + m_a(\hat{S}^{t-1})\left(c - \hat{c}^{t-1}\right) < 1\right\}, \tag{3}$$

$$\hat{c}^t = \min_{s_i \in \hat{S}^{t-1}} \hat{c}_a^t, \tag{4}$$

$$\hat{S}^t = \hat{S}^{t-1} \setminus \{a \in \hat{S}^{t-1} | \hat{c}_a^t = c^t\}, \tag{5}$$

$$\hat{x}_a^t = \hat{x}_a^{t-1} + m_a(\hat{S}^{t-1})\left(\hat{c}^t - \hat{c}^{t-1}\right). \tag{6}$$

Each recursion step t determines the cutoff of school(s) given cutoffs $\{\hat{c}^0, ..., \hat{c}^{t-1}\}$. Students with draw $\omega > \hat{c}^{t-1}$ can never be assigned schools $S \setminus S^{t-1}$. For each school $a \in S^{t-1}$ with remaining capacity, a fraction $\hat{x}_a^{t-1}$ is claimed by students with draws less than $\hat{c}^{t-1}$, so only fraction $1 - \hat{x}_a^{t-1}$ of seats can be assigned to students with draws $\omega > \hat{c}^{t-1}$. If school a has the next highest cutoff, $\hat{c}^t$, then the remaining capacity $1 - \hat{x}_a^{t-1}$ must equal the measure of those students who prefer a the most among S^{t-1} and have drawn numbers in $[\hat{c}^{t-1}, \hat{c}^t]$. This, together with the fact that school a has cutoff $\hat{c}^t$, implies (3) and (4). The recursion definition implies (5) and (6).

The recursive equations uniquely determine the set of cutoffs $\{\hat{c}^0, ..., \hat{c}^k\}$, where $k \leq n$. The cutoff for school $a \in S$ is then given by $c_a := \{\hat{c}^t | \hat{c}_a^t = \hat{c}^t\}$. It clearly follows from (3) and (4) for $t = 1$ that $\hat{c}^1 > 0$. It also easily follows that $\hat{c}^k = 1$. Obviously $\hat{c}^k \leq 1$. We also cannot have $\hat{c}^k < 1$, or else there will be positive measure of students unassigned, which cannot occur since every student prefers each school to being unassigned, and the measure of all students coincides with the total capacity of schools.

Although it is possible for more than one school to have the same cutoff, this is not generic. If there are schools with the same cutoff, we must have $a \neq b \in \hat{S}^{t-1}$ for some t and $\hat{S}^{t-1}$ such that $\hat{c}_a^t = \hat{c}_b^t$, which entails a loss of dimension for $\mathfrak{m}$ within $\mathfrak{M}$. Hence, the Lebesgue measure of the set of $\mathfrak{m}$'s involving such a restriction is zero. It thus follows that generically no two schools have the same cutoff. ▌

Proof of Lemma 4. For each $a \in S$ and any $S' \subset S \setminus \{a\}$, let

$$m_a^{S'} := \mu(\{\mathbf{v} \in \mathcal{V} | v_b \geq v_a \geq v_c, \forall b \in S', \forall s_c \in S \setminus (S' \cup \{a\})\})$$

be the measure of those students whose preference order of school a follows right after schools in S'. (Note that the order of schools within S' does not matter here.) We can then define the conditions for cutoffs $\{\tilde{c}_1, ..., \tilde{c}_n\}$ under DA-MTB as the following system of simultaneous equations. Specifically, for any school $a \in S$, we must have

$$\tilde{c}_a \left(m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b) \right] \right) = 1. \tag{7}$$

The LHS has the measure of students admitted by school a. They consist of those students who prefer a most and have admittable lottery draws for a (i.e., $\omega_a \leq \tilde{c}_a$), and of those who prefer schools $S' \subset S \setminus \{a\}$ more than a but have bad draws for those schools but have an admittable draw for school a. In equilibrium, the cutoffs must be such that these aggregate measures equal one (the capacity of school a).

To show that there exists a set $\{\tilde{c}_1, ..., \tilde{c}_n\}$ of cutoffs satisfying the system of equations (7), let $\Upsilon := (\Upsilon_1, ..., \Upsilon_n) : [0, 1]^n \to [0, 1]^n$ be a function whose a's component is defined as:

$$\Upsilon_a(\tilde{c}_1, ..., \tilde{c}_n) = \min \left\{ \frac{1}{m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b) \right]}, 1 \right\},$$

where we adopt the convention that $\min\{\frac{1}{0}, 1\} = 1$.

Observe that self mapping $\Upsilon(\cdot)$ is a monotone increasing on a nonempty complete lattice. Hence, by the Tarski's fixed point theorem, there exists a largest fixed point $\mathbf{c}^* = (c_1^*, ..., c_n^*)$ such that $\Upsilon(\mathbf{c}^*) = \mathbf{c}^*$, and $\mathbf{c}^* \geq \tilde{\mathbf{c}}^*$ for any fixed point $\tilde{\mathbf{c}}^*$.

We now show that at any such fixed point $\tilde{\mathbf{c}}^*$,

$$\frac{1}{m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*)\right]} \leq 1, \tag{8}$$

for each $a \in S$. Suppose this is not the case for some i. Then, by the construction of the mapping, we must have $\tilde{c}_a^* = 1$. This means that all students are assigned some schools. Therefore, by pure accounting,

$$\sum_{a \in S} \tilde{c}_a^* \left(m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*) \right] \right) = n. \tag{9}$$

Yet, since (8) fails for some school,

$$\begin{aligned} &\sum_{a \in S} \tilde{c}_a^* \left(m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*) \right] \right) \\ &< \sum_{a \in S} \left(\frac{1}{m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*)\right]} \right) \left(m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*) \right] \right) = n, \end{aligned}$$

where the strict inequality follows since, for school c for which (8) holds, $\tilde{c}_c^* = \frac{1}{m_c^{\emptyset} + \sum_{S' \subset S \setminus \{s_c\}} m_c^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*)\right]}$ and, for school a for which (8) does not hold, $\tilde{c}_a^* = 1 < \frac{1}{m_a^{\emptyset} + \sum_{S' \subset S \setminus \{a\}} m_a^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b^*)\right]}$. This inequality contradicts (9). Since (8) holds for each $a \in S$, the fixed point $(\tilde{c}_1^*, ..., \tilde{c}_n^*)$ solves the system of equations (7). It is immediate from (7) that $\tilde{c}_a > 0, \forall a$. Further, there must exist a school $w \in S$ with $\tilde{c}_w = 1$, or else a positive measure of students are unassigned, which would violate (7). As before, it follows that the solutions to (7) are generically distinct.

To establish uniqueness, suppose to the contrary $\mathbf{c}^* > \tilde{\mathbf{c}}^*$: $c_b^* \geq \tilde{c}_b^*$ for all b and $c_a^* > \tilde{c}_a^*$ for some a. Let $w \in S$ be such that $\tilde{c}_w^* = 1$. Since $\mathbf{c}^* \geq \tilde{\mathbf{c}}^*$, $c_w^* = 1$. Since (7) must be satisfied for w under both cutoffs, we have

$$\left(m_w + \sum_{S' \subset S \setminus \{w\}} m_w^{S'} \left[\prod_{b \in S'} (1 - c_b) \right] \right) = \left(m_w + \sum_{S' \subset S \setminus \{a\}} m_w^{S'} \left[\prod_{b \in S'} (1 - \tilde{c}_b) \right] \right) = 1,$$

which holds if and only if $c_b = \tilde{c}_b$ for all b. ▌

Proof of Theorem 3. The proof is an application of Theorem 2 of Mas-Colell (1984). ▌

Proof of Lemma 5. Part (i) follows trivially since such a student can target that school and get assigned to it with probability one. To prove part (ii) consider any student of type $\mathbf{v}$, whose values are all distinct. There are μ-a.e. such $\mathbf{v}$. Suppose her most-preferred school $\pi_1(\mathbf{v}) =: a$ is undersubscribed and not a worst school. It is then her best response to target a, since doing so can guarantee assignment to school a for sure, whereas targeting some other school reduces her chance of assignment to that school. Hence, the student must be targeting a in equilibrium.

To prove part (iii), consider any $\mathbf{v}$ (with distinct values), such that $\pi_1(\mathbf{v}) \neq w$. Suppose first $\sigma_a^*(\mathbf{v}) > 0$ for some oversubscribed school a. It follows from the above observation that she must strictly prefer school a to all undersubscribed schools. Hence, she lists a ahead of all undersubscribed schools in her ordinal list. Whenever she targets school a, she can never place in any oversubscribed school other than a, so she will apply to school a with probability one. Suppose next $\sigma_b^*(\mathbf{v}) > 0$ for some undersubscribed school b. Then, the student must prefer b to all other undersubscribed schools, so she will apply to school b with probability one whenever she fails to place in any oversubscribed school she may list ahead of b in the ordinal list. Whenever she targets school b, she is surely rejected by all oversubscribed schools she may list ahead of b, so she will apply to b with probability one. We thus conclude that $\sigma^*(\mathbf{v}) = \bar{\sigma}^*(\mathbf{v})$ for μ-a.e. $\mathbf{v}$. ▮

Proof of Theorem 4: To prove part (i), let school b be such that $\tilde{c}_b = 1$. Hence, any students who prefer b to a can never be assigned a. Hence, the allocation does not admit any trading cycle within $\{a, b\}$, and is thus OE within $\{a, b\}$ (Lemma 1). The allocation is then PE within $\{a, b\}$ by Lemma 2-(iii).

To prove part (ii), take any two schools $\{a, b\}$, with $\tilde{c}_a, \tilde{c}_b < 1$. There is a positive measure of students whose first- and second-most preferred schools are a and b, respectively (call them "type-a"). Likewise, there is a positive measure of so-called "type-b" students whose first- and second-most preferred schools are b and a, respectively. A positive measure of type-a students draw (ω_a, ω_b) such that $\omega_a > \tilde{c}_a$ and $\omega_b < \tilde{c}_b$; and a positive measure of type-b students draw (ω'_a, ω'_b) with $\omega'_a < \tilde{c}_a$ and $\omega'_b > \tilde{c}_b$. Clearly, the former type students are assigned b and the latter to a, so both types of students will benefit from swapping their assignments. Part (ii) then follows since generically there is only one school with cutoff equal to 1 (Lemma 4). ▮

Proof of Theorem 5: To prove part (i), suppose $a \rhd^{\phi^{STB}} b$. Then, we must have $c_a < c_b$. Or else, any students who prefer school a to b can never be assigned to school b. This is because any such student will rank a ahead of b (by strategyproofness), so if she is rejected by a, her draw must be $\omega > c_a \geq c_b$, not good enough for b. Hence, if $a_1 \rhd^{\phi^{STB}} ... \rhd^{\phi^{STB}} a_k \rhd^{\phi^{STB}} a_1$, then $c_{a_1} < ... < c_{a_k} < c_{a_1}$, a contradiction. Hence, it is OE (and thus pairwise PE).

To prove part (ii), recall from Lemma 3 that the schools' cutoffs are generically distinct.

Take any set $\{a, b, c\}$ with $c_a < c_b < c_c$. Then, by the full support assumption, there exists a positive measure of $\mathbf{v}$'s satisfying $v_a > v_b > v_c > v_d$ for all $d \neq a, b, c$. These students will then have a positive chance of being assigned to each school in $\{a, b, c\}$, for their draws will land in the intervals, $[0, c_a]$, $[c_a, c_b]$ and $[c_b, c_c]$, with positive probabilities. Again, given the full support assumption, such students will all differ in their marginal rate of substitution among the three schools. Then, just as with the motivating example, one can construct a mutually beneficial trading of shares of these schools among these students. ▌

Proof of Theorem 6: Part (i) builds on part (ii), so it will appear last. Throughout, we let K and J be the sets of over- and under-subscribed schools.

Part (ii): Let $\sigma^*(\cdot)$ be an equilibrium and $\phi^*(\cdot)$ be the associated allocation. For any $\mathbf{v} \in \mathcal{V}$, consider an optimization problem:

$$[P(\mathbf{v})] \qquad \max_{\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}} \sum_{a \in S} v_a x_a \text{ subject to } \sum_{a \in K} p_a x_a \leq \sum_{a \in K} p_a \phi^*_a(\mathbf{v}),$$

where $p_a \equiv \max\{\int \overline{\sigma}^*_a(\tilde{\mathbf{v}}) d\mu(\tilde{\mathbf{v}}), 1\}$ and

$$\Delta^K_{\phi^*(\mathbf{v})} := \{(x_1, ..., x_n) \in \Delta \mid x_a = \phi^*_a(\mathbf{v}), \forall a \in S \setminus K\}$$

is the set of all assignments that may differ from $\phi^*(\mathbf{v})$ only in the probability shares of schools in K.

We first prove that $\phi^*(\mathbf{v})$ solves $[P(\mathbf{v})]$. This is trivially true for any type $\mathbf{v}$-student whose most preferred school is the worst school w. Then, by Lemma 5-(i), $\phi^*_w(\mathbf{v}) = 1$ and $x_a = \phi^*_a(\mathbf{v}) = 0, \forall a \in K$. So, $\phi^*(\mathbf{v})$ solves $[P(\mathbf{v})]$.

Hence, assume that $\pi_1(\mathbf{v}) \neq w$ in what follows. Fix any such $\mathbf{v}$, and fix any arbitrary $\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}$ satisfying the constraint of $[P(\mathbf{v})]$. We show below that the type $\mathbf{v}$ student can achieve the assignment $\mathbf{x}$ by adopting a certain targeting strategy in the CADA game, assuming that all other players play their equilibrium strategies σ^*.

To begin, consider a strategy called a in which she targets school $a \in S$ and also top-ranks it in her ordinal list but ranks all other schools truthfully. If type $\mathbf{v}$ plays strategy a, then she will be assigned to school a with probability

$$\frac{1}{\max\{\int \overline{\sigma}^*_a(\tilde{\mathbf{v}}) d\mu(\tilde{\mathbf{v}}), 1\}} = \frac{1}{p_a}.$$

If $a \in J$, this probability is one. If $a \in K$, then she will be rejected by school a with positive probability. If she is rejected, she will apply to other schools. Clearly, she will not succeed in getting into any schools in K, since they are oversubscribed. The conditional probabilities of getting assigned to schools J do not depend on which school in K she has targeted (and gotten turned down), due to our design whereby her non-target draw ω_R is independent of her target draw ω_T (recall footnote 22). For each b, let that conditional assignment probability be $\bar{\phi}_b^*(\mathbf{v})$ for type $\mathbf{v}$. Obviously, $\sum_{b\in J}\bar{\phi}_b^*(\mathbf{v})=1$.

Suppose the type $\mathbf{v}$ student randomizes by choosing "strategy a" with probability $y_a := p_a x_a$, for each $a\in K$, and with probability

$$y_b := \sigma_b^*(\mathbf{v}) + \left[\sum_{a\in K}(\alpha^*(\mathbf{v})-p_a x_a)\left(1-\frac{1}{p_a}\right)\right]\bar{\phi}_b^*(\mathbf{v}),$$

for each $b\in J$. Observe $y_b\geq 0$ for all $b\in S$. This is obvious for $b\in K$. For $b\in J$, this follows since the terms in the square brackets are nonnegative:

$$\begin{aligned}\sum_{a\in K}(\alpha^*(\mathbf{v})-p_a x_a)\left(1-\frac{1}{p_a}\right) &= \sum_{a\in K}(p_a\phi_a^*(\mathbf{v})-p_a x_a)\left(1-\frac{1}{p_a}\right)\\ &= \left[\sum_{a\in K}p_a(\phi_a^*(\mathbf{v})-x_a)\right]-\left[\sum_{a\in K}(\phi_a^*(\mathbf{v})-x_a)\right]=\sum_{a\in K}p_a(\phi_a^*(\mathbf{v})-x_a)\geq 0,\end{aligned}$$

where the first equality is implied by Lemma 5-(iii), the third equality holds since $\mathbf{x}\in\Delta^K_{\phi^*(\mathbf{v})}$ (which implies $\sum_{a\in K}x_a=\sum_{a\in K}\phi^*(\mathbf{v})$), and the last inequality follows from the fact that $\mathbf{x}$ satisfies the constraint of $[P(\mathbf{v})]$. Further,

$$\begin{aligned}\sum_{a\in S}y_a &= \sum_{a\in K}p_a x_a + \sum_{b\in J}\left[\sigma_b^*(\mathbf{v})+\left[\sum_{a\in K}(\alpha^*(\mathbf{v})-p_a x_a)\left(1-\frac{1}{p_a}\right)\right]\bar{\phi}_b^*(\mathbf{v})\right]\\ &= \sum_{a\in K}p_a x_a+\sum_{b\in J}\sigma_b^*(\mathbf{v})+\sum_{a\in K}\left[(\sigma_a^*(\mathbf{v})-p_a x_a)\left(1-\frac{1}{p_a}\right)\right]\left(\sum_{b\in J}\bar{\phi}_b^*(\mathbf{v})\right)\\ &= \sum_{a\in K}p_a x_a+\sum_{b\in J}\sigma_b^*(\mathbf{v})+\sum_{a\in K}\left[(\sigma_a^*(\mathbf{v})-p_a x_a)\left(1-\frac{1}{p_a}\right)\right]\\ &= \sum_{a\in K}\sigma_a^*(\mathbf{v})+\sum_{b\in J}\sigma_b^*(\mathbf{v})+\sum_{a\in K}(\phi_a^*(\mathbf{v})-x_a)=\sum_{a\in S}\sigma_a^*(\mathbf{v})=1.\end{aligned}$$

The third equality holds since $\sum_{b\in J}\bar{\phi}_b^*(\mathbf{v})=1$, the fourth is implied by Lemma 5-(iii), and the fifth follows since $\mathbf{x}\in\Delta^K_{\phi^*(\mathbf{v})}$.

By playing the mixed strategy $(y_1,...,y_n)$, the student is assigned to school $a\in K$ with probability

$$\frac{y_a}{p_a}=x_a,$$

and to each school $b \in J$ with probability

$$\begin{aligned}
& y_b + \left[\sum_{a\in K} y_a \left(1 - \frac{1}{p_a}\right)\right] \bar{\phi}_b^*(\mathbf{v}) \\
&= \sigma_b^*(\mathbf{v}) + \left[\sum_{a\in K} (\sigma_a^*(\mathbf{v}) - p_a x_a) \left(1 - \frac{1}{p_a}\right)\right] \bar{\phi}_b^*(\mathbf{v}) + \left[\sum_{a\in K} p_a x_a \left(1 - \frac{1}{p_a}\right)\right] \bar{\phi}_b^*(\mathbf{v}) \\
&= \sigma_b^*(\mathbf{v}) + \left[\sum_{a\in K} \sigma_a^*(\mathbf{v}) \left(1 - \frac{1}{p_a}\right)\right] \bar{\phi}_b^*(\mathbf{v}) = \overline{\sigma}_b^*(\mathbf{v}) + \left[\sum_{a\in K} \overline{\sigma}_a^*(\mathbf{v}) \left(1 - \frac{1}{p_a}\right)\right] \bar{\phi}_b^*(\mathbf{v}) \\
&= \phi_b^*(\mathbf{v}) = x_b.
\end{aligned}$$

In other words, the type $\mathbf{v}$ student can achieve any $\mathbf{x} \in \Delta^K_{\phi^*(\mathbf{v})}$ that satisfies $\sum_{a\in K} p_a x_a \leq \sum_{a\in K} p_a \phi_a^*(\mathbf{v})$ by playing a certain strategy available in the CADA game. Since every feasible $\mathbf{x}$ can be mimicked by a strategy available in the equilibrium of CADA, $\phi^*(\cdot)$ is a best response for type $\mathbf{v}$, and since it satisfies the constraints of $[P(\mathbf{v})]$, $\phi^*(\cdot)$ must solve $[P(\mathbf{v})]$.

Moreover, since μ is atomless and $[P(\mathbf{v})]$ has a linear objective function on a convex set, $\phi^*(\mathbf{v})$ must be the unique solution to $[P(\mathbf{v})]$ for a.e. $\mathbf{v}$.

We prove the statement of the theorem by contradiction. Suppose to the contrary that there exists an allocation $\phi(\cdot) \in \mathcal{X}^K_{\phi^*}$ that Pareto dominates $\phi^*(\cdot)$. Then, for a.e. $\mathbf{v}$, $\phi(\mathbf{v})$ must either solve $[P(\mathbf{v})]$ or violate its constraints. For a.e. $\mathbf{v}$, the solution to $[P(\mathbf{v})]$ is unique and coincides with $\phi^*(\mathbf{v})$. This implies that for a.e. $\mathbf{v}$,

$$\sum_{a\in K} p_a \phi_a(\mathbf{v}) \geq \sum_{a\in K} p_a \phi_a^*(\mathbf{v}). \tag{10}$$

Further, for ϕ to Pareto-dominate ϕ^*, there must exist a set $A \subset \mathcal{V}$ with $\mu(A) > 0$ such that each student $\mathbf{v} \in S$ must strictly prefer $\phi(\mathbf{v})$ to $\phi^*(\mathbf{v})$, which must imply (since $\phi^*(\mathbf{v})$ solves $[P(\mathbf{v})]$)

$$\sum_{a\in K} p_a \phi_a(\mathbf{v}) > \sum_{a\in K} p_a \phi_a^*(\mathbf{v}), \ \forall \mathbf{v} \in S. \tag{11}$$

Combining (10) and (11), we get

$$\sum_{a\in K} p_a \int \phi_a(\mathbf{v}) d\mu(\mathbf{v}) > \sum_{a\in K} p_a \int \phi_a^*(\mathbf{v}) d\mu(\mathbf{v}). \tag{12}$$

Now since $\phi(\cdot) \in \mathcal{X}$, for each $a \in S$,

$$\int \phi_a(\mathbf{v}) d\mu(\mathbf{v}) = 1 = \int \phi_a^*(\mathbf{v}) d\mu(\mathbf{v}).$$

Multiplying both sides by p_a and summing over K, we get

$$\sum_{a\in K} p_a \int \phi_a(\mathbf{v}) d\mu(\mathbf{v}) = \sum_{a\in K} p_a \int \phi_a^*(\mathbf{v}) d\mu(\mathbf{v}),$$

which contradicts (12). We thus conclude that ϕ^* is Pareto optimal within K.

Part (iii): Consider the following maximization problem for every $\mathbf{v} \in \mathcal{V}$:

$$[\overline{P}(\mathbf{v})] \qquad \max_{\mathbf{x}\in\Delta} \sum_{a\in S} v_a x_a \text{ subject to } \sum_{a\in K} p_a x_a \leq 1.$$

When we have only one undersubscribed school, say b, then its assignment is determined by $x_b = 1 - \sum_{a\in K} x_a$. Therefore, an assignment $\mathbf{x} \in \Delta$ is feasible in CADA game if (and only if) the constraint of $[\overline{P}(\mathbf{v})]$ holds.

Now consider the following maximization problem:

$$[\overline{P}'(\mathbf{v})] \qquad \max_{\mathbf{x}\in\Delta} \sum_{a\in S} v_a x_a \text{ subject to } \sum_{a\in K} p_a x_a \leq \sum_{a\in K} p_a \phi^*_a(\mathbf{v}).$$

Since $\phi^*(\cdot)$ solves a less constrained problem $[\overline{P}(\mathbf{v})]$ and is still feasible in $[\overline{P}'(\mathbf{v})]$, it must be an optimal solution for $[\overline{P}'(\mathbf{v})]$. The rest of the proof is shown by the same argument as in Part (ii).

Part (i): The argument in the text already established that the allocation cannot admit a trading cycle that includes both oversubscribed and unsubscribed schools. It cannot admit a trading cycle comprising only oversubscribed schools, since the allocation is PE within these schools, by Part (ii), making it OE within the schools, by Lemma 2-(ii). It cannot admit a trading cycle comprising only undersubscribed schools, since the logic of Theorem 5-(i) implies that it is OE within undersubscribed schools. Since the allocation cannot admit any trading cycle, it must be OE. ▌

Proof of Theorem 7: Consider first a DA algorithm with any random tie-breaking. Since all students submit the same ranking of the schools, they are assigned to each school with the same probability $1/n$. In other words, the allocation is $\phi^{DA}(\mathbf{v}) = (\frac{1}{n}, ..., \frac{1}{n})$ for all $\mathbf{v}$.

Consider now CADA algorithm and an associated equilibrium σ^*. Then, a fraction $\alpha^*_a := \int \sigma^*_a(\mathbf{v})d\mu(\mathbf{v})$ of students target $a \in S$ in equilibrium. The equilibrium induces a mapping $\varphi^* : S \mapsto \Delta$, such that a student is assigned to school b with probability $\varphi^*_b(a)$ if she targets a.

Since the capacity of each school is filled in equilibrium, we must have, for each $b \in S$,

$$\sum_{a\in S} \alpha^*_a \varphi^*_b(a) = 1. \tag{13}$$

That is, a measure α^*_i of students target a, and a fraction $\varphi^*_b(a)$ of those is assigned to school b. Summing the product over all a then gives the measure of students assigned to b, which must equal its capacity, 1.

Consider a student with any arbitrary $\mathbf{v} \in \mathcal{V}$. We show that there is a strategy she can employ to mimic the random assignment ϕ^{DA}. Suppose she randomizes by targeting school a

with probability

$$y_a := \frac{\alpha_a^*}{\sum_b \alpha_b^*} = \frac{\alpha_a^*}{n}.$$

Then, the probability that she will be assigned to any school k is

$$\sum_b y_b \varphi_k^*(b) = \sum_b \frac{\alpha_b^*}{n} \varphi_k^*(b) = \frac{1}{n},$$

where the second equality follows from (13). That is, she can replicate the same ex ante assignment with the randomization strategy as $\phi^{DA}(\mathbf{v})$. Hence, the student must be at least weakly better off under CADA. ▌

Figure 3: Welfare as Percentage of First Best

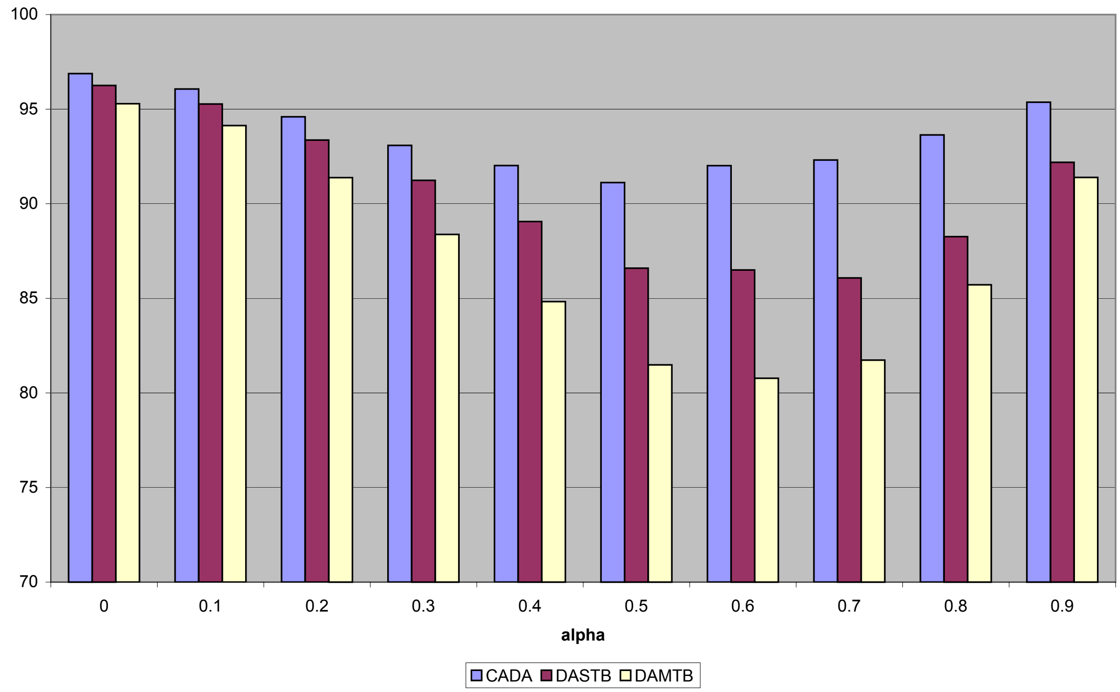

Figure 4: Percentage of Students Getting Their First Choice

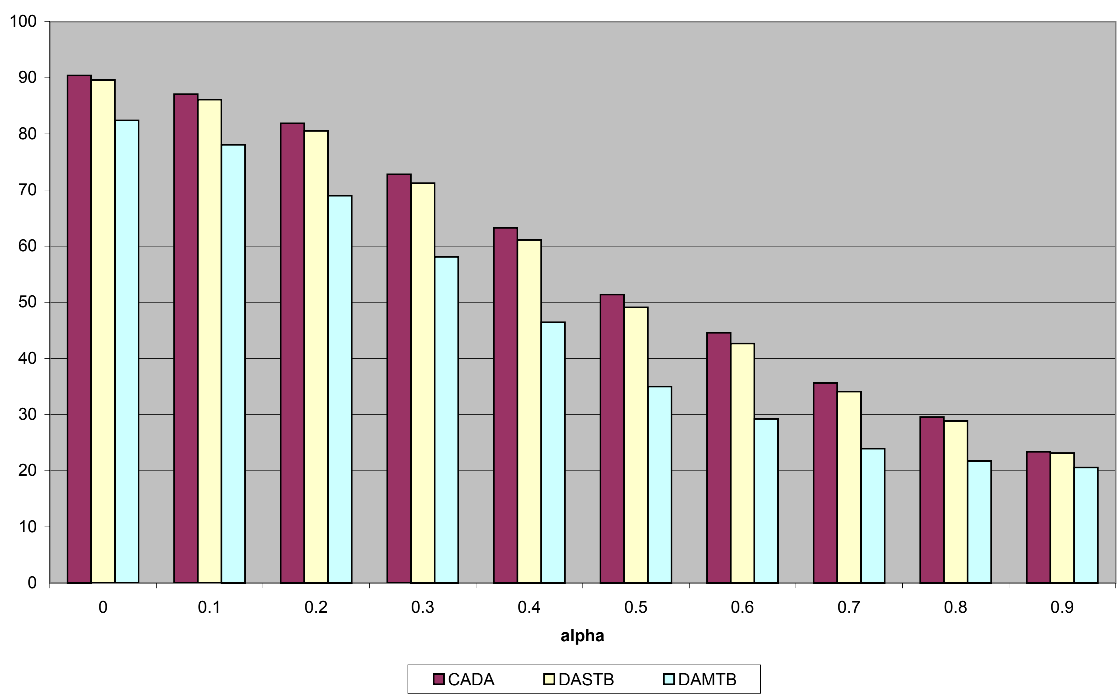

Figure 5: Average Utility of Receivers of kth Choice, CADA vs DASTB

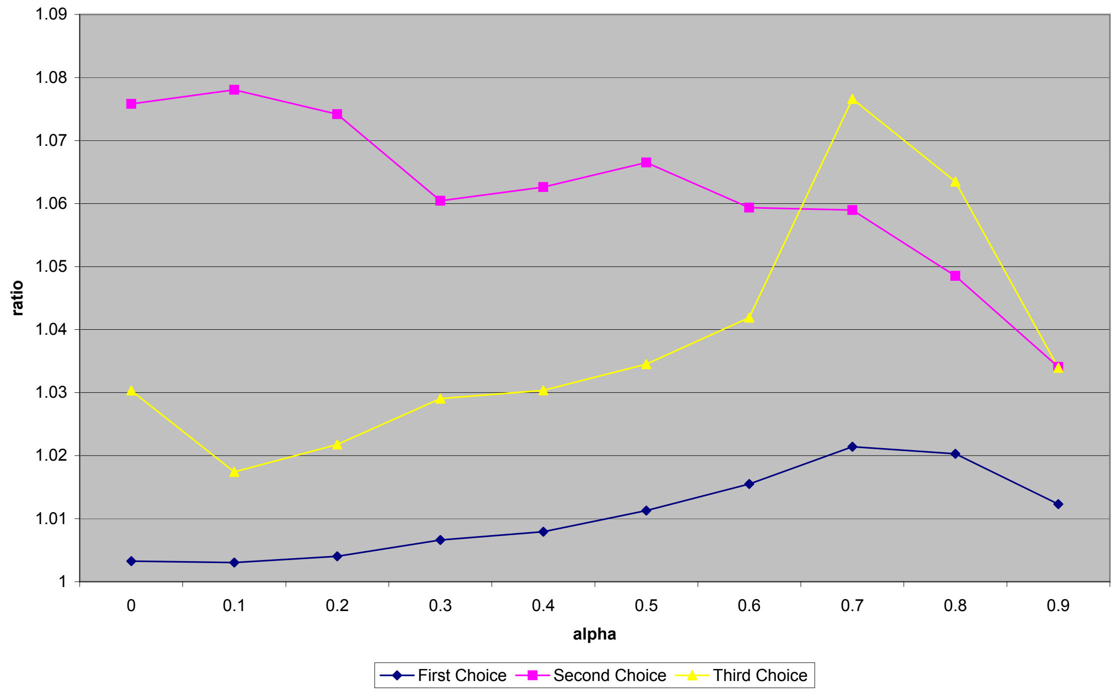

Figure 6: Average Number of Popular Schools and Oversubsribed Schools

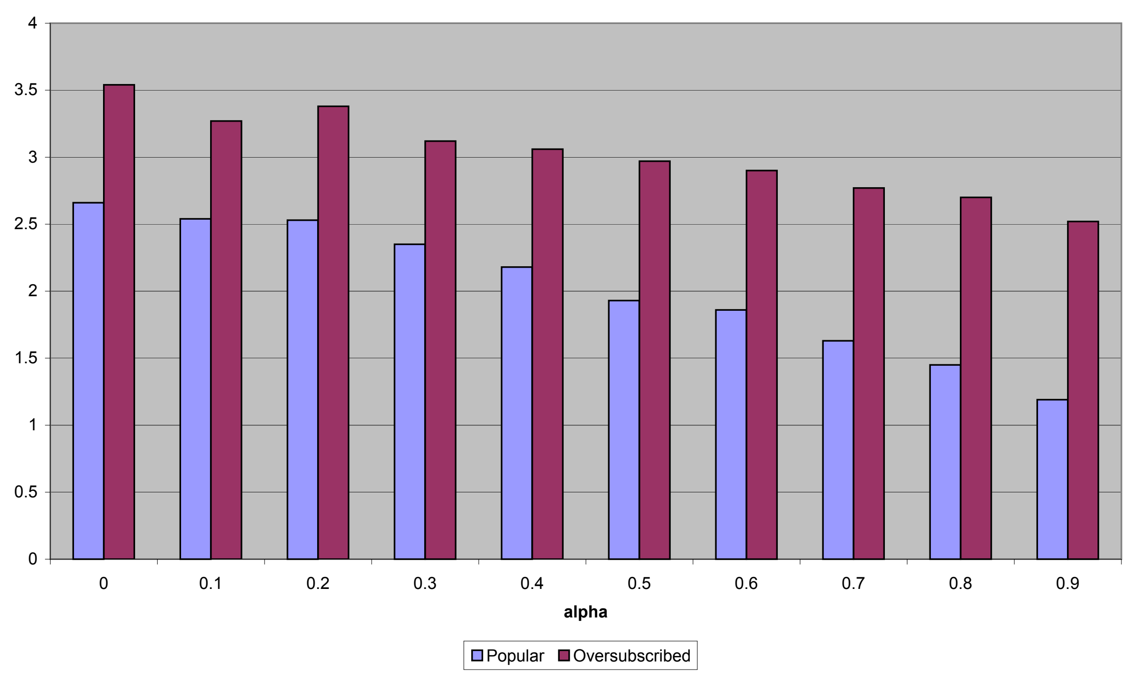

Figure 7: Welfare as Percentage of First Best - with priorities