Informed Momentum Trading versus Uninformed "Naive" Investors Strategies

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Abstract

We construct a zero-net-worth uninformed "naive investor" who uses a random portfolio allocation strategy. We then compare the returns of the momentum strategist to the return distribution of naive investors. For this purpose we reward momentum profits relative to the return percentiles of the naive investors with scores that are symmetric around the median. The score function thus constructed is invariant and robust to risk factor models. We find that the average scores of the momentum strategies are close to zero (the score of the median) and statistically insignificant over the sample period between 1926 and 2005, various sub-sample periods including the periods examined in Jegadeesh and Titman (1993 and 2001). The findings are robust with respect to sampling or period-specific effects, tightened score intervals, and the imposition of maximum-weight restrictions on the naive strategies to mitigate market friction considerations.

JEL: G11; G12; G14

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1 Introduction

There has been extensive research seeking to understand why the momentum strategies of buying the recent winner stocks and selling short the recent loser stocks generate profits as documented by Jegadeesh and Titman (1993 and 2001). The risk-based approach, models risk factors or dynamic risk exposures related to economic and firm fundamentals. Many studies demonstrate that momentum strategies, although reducing the exposure to the market risk by undertaking the long-short positions, do not hedge risk in all dimentions and are far from risk-free (e.g. Berk, Green and Naik (1999), Grundy and Martin (2001), Johnson (2002), Pastor and Stambaugh (2003), Avramov and Chordia (2006), Chordia and Shivakumar (2006), Sagi and Seasholes (2007) and Liu and Zhang (2008), among others). Some studies examine the momentum profits after taking into account of transaction costs (Lesmond, Schill and Zhou (2004), Korajczyk and Sadka (2004)) and liquidity (Sadka (2006)). Most studies, however, show that model alpha of momentum strategies remains significant[1].

The approach using asset-pricing models requires correct specifications of risk factors and the dynamics of time-varying coefficients. If there is an omitted factor in a model, the estimated alpha (and factor betas) suffer from omitted variable bias. In this paper, instead of investigating the potentially unidentified factors, we propose a different approach which removes the necessity of modeling risk to measure the performance of the momentum strategies.

Our method rewards and penalizes the momentum strategist for using the past return information in the formation of the strategies. To this end, we construct the "naive investors" who use no information and weight risky assets randomly. The question we ask is, compared

[1] The theory and empirical evidence from behavioral approach advocate that the deviations from rational behavior can result in momentum (See, e.g. Chan, Jegadeesh and Lakonishok (1996), Barberis, Shleifer and Vishny (1998), Hong and Stein (1999), Grinblatta and Han (2005), Hvidkjaer (2006) and Chui, Titman and Wei (2010).

to the "naive investors", by how much do we reward or penalize the momentum strategist for his/her efforts?

Imagine a stock market where there are both momentum investors (MI, hereafter) who observe the past price pattern and "naive investors" (NI, hereafter) who do not. The traders who pursue the *zero net-worth* momentum strategies must first observe past stock returns in order to quantitatively form portfolios of the winner (top 10% past returns) and the loser (bottom 10% past returns) stocks and then execute intensive long-short trading with overlapping strategy positions.

The NIs, on the contrary, are agnostic about the return generating process, and hence do not use any information. The NIs are unsophisticated and thus do not understand how to sell short stocks[2]. They therefore randomly form long positions in the N risky assets. The NIs invest in the same feasible set of stocks as that for constructing the momentum strategies and finance their investments in stock portfolios by borrowing at the risk-free rate, using the one-month Treasury bill rate as a proxy. Thus, the NI's strategies consist of both long (the risky assets) and short (the risk-free asset) positions, thereby creating an initial zero net-worth position. The excess portfolio returns from the naive strategies are thus the profits of the *zero net-worth* strategies of the naive investor.

To form a random portfolio, the uninformed NI chooses with equal chances any positive or zero weight for each of the feasible stocks at the beginning of each month t. The weights $\mathbf{\Phi}$ of this portfolio, summing to unity, are thus a non-negative vector of random drawings from an uniform distribution[3]. At the end of the period t, when asset returns $\mathbf{r}$ are realized the NI liquidates his/her portfolio and gets a return of $\mathbf{r}'\mathbf{\Phi}$. Since the portfolio weights are random variables, the portfolio returns are also random variables. Therefore we get a probability

[2] Jones and Lamont (2002), for example, describe the difficulties of short sales including the risks, costs, legal and institutional restrictions, and the need of sufficient stock supply from investors who are willing to lend.

[3] We generate the portfolios using Monte Carlo method which is described in detail in Section 2.

distribution of returns of the NI at the end of period t. The NI then uses the same method to reform and hold a portfolio for the next period $t+1$. We thus get the time-series of return distributions of the NI.

Now consider the sophisticated MI who, unlike the NI, uses past return information to form the momentum strategies $\mathbf{F}_P$ at period t and gets the profit $\mathbf{r}'\mathbf{F}_P$. Our method is to compare the profits $\mathbf{r}'\mathbf{F}_P$ of momentum strategies that buy the winners and sell short the losers to the quintiles of the return distribution of the simple strategy of the NI. In each period the MI gets a reward from the set of {2, 1, 0, -1 or -2}. If the profit $\mathbf{r}'\mathbf{F}_P$ is above the 80th percentile of the return distribution of the NI, we assign a reward of 2 units. Similarly, if it is above the 60th percentile but below the 80th percentile, an award of 1 unit is assigned for the MI. Thus, the reward decreases with the quintiles. Likewise, below the 40th percentile and the 20th percentile of the NI return distribution the MI is penalized by being awarded negative rewards of -1 and -2, respectively. Between the 40th percentile and the 60th percentile the reward is zero[4]. We then generate a time-series of rewards over a period of T years.

Note that by construction the median of the NI's profits is always awarded zero, and that the rewards/penalties of the NI are uniformly distributed with a probability of 1/5. Therefore, the expected reward of the NI is zero. Our hypothesis is that if the MI's strategy is better than the NI's strategy, then he/she should, on average, get a positive reward for the efforts over this $T-$year period. We test this hypothesis over different $T-$year horizons.

We call this function which assigns rewards to the MI by comparing $\mathbf{r}'\mathbf{F}_P$ to the NI return distribution a score function. We show (in Theorem 1) and prove (in the Appendix) that our method is appropriate because the score function is invariant under any common risk

[4]Note that the reward function does not have to be a five point function or even symmetric. Users can change the parameters of this function.

factors in addition to scaling returns by volatility. That is, the scores of the risk-adjusted returns of factor models are the same as those of the raw returns. Thus, the main advantage of our method is that the score function does not require us to identify the source of risk and to estimate factor loadings.

The second advantage of our method is that the score function is robust against return outliers, and thus is robust against unexpected booms or busts of a group of companies. Further, we can specify the level of robustness by estimation design. This advantage is particularly useful since we find that there are extreme values of returns and profits of the momentum and the naive strategies. In contrast, most moment-based estimation techniques are sensitive to outliers. For example, OLS has a breakdown point of 0% in that a change of a single observation can change the parameter estimates.

Moreover, our score function is also useful as it provides robust statistical tests based on it's analytical and statistical properties: (i) The score function is invariant under common affine transformation, i.e., that an overall increase or decrease in asset returns or a jump in return volatility does not change the score of a strategy. Thus, the momentum strategies do not receive rewards when everyone else in the market does as well; likewise, they are not punished during a period of an overall market crash. (ii) The score function rewards and penalizes a strategy, which can be directly used for performance management.

We also derive the asymptotic properties of the performance measure given a set of asset returns, and then perform a t-test based on the asymptotic distribution of the score function. If the momentum strategies outperform the naive strategies, the average score of the momentum profits should be significantly positive.

We use the monthly equity data of the NYSE, AMEX and NASDAQ over the sample period between 1926 and 2005, various sub-sample periods and 100 randomly selected ten-year periods. In order to provide further detailes on the anatomy of the momentum trading, we separately analyze the returns on the winner and the loser portfolios, and next look into the momentum profits from buying the winner and selling the loser portfolios. We find that the winner portfolio has a small average score, positive and statistically significant. In contrast, although the losers generate negative returns that are statistically significant, the losers have a negative average score, but statistically insignificant. This is because the return differences between the losers and the NI's have high variability and are small on average.

Strikingly, the average score of the momentum profits from buying the winners and selling the losers is close to zero. It is important to note that the scores of the momentum profits are not equal to the winners' scores minus the losers' scores as the scores are not linearly additive. The momentum strategies take both sides of the extreme return positions, and thus take up the tail risk. By doing so, they face large dispersions in the strategy profits. The momentum profits are almost equally likely to either go higher than the 80th percentile or drop below the 20th percentile of the profits of the naïve investors strategies, and hence, receive either the highest or the lowest scores at different points in time. As a result, the scores of the momentum profits offset each other. Overall results show that the momentum strategies do not outperform the portfolios of the naïve investors.

Furthermore, in order to mitigate period-specific sampling considerations we simulate the distributions of the average scores of the winner and the loser portfolios as well as the momentum profits by re-sampling with replacements for 100 times. In each of the 100 randomly chosen ten-year period we use all feasible stocks in the sample during that period and then construct the empirical distribution of the average scores. Our simulations confirm the previous results.

Finally, since the naive investor might not be able to enjoy all the possible strategies due to market frictions, we impose certain limits on the weight of a stock. This results in evaluating the momentum strategy over a relatively restricted return space than the unlimited one[5]. Since by construction, the naive investor is short in the risk-free asset to establish the long position in stocks, we first set a maximum weight of 10% on each stock. We also set a maximum weight of 10% on any stock in the smallest decile given their high transaction costs. For robustness checks, we also apply a more detailed 11-point score system based on the unrestricted return space and those with weight restrictions. These results are quantitatively similar and do not change our conclusions.

Overall, we find that over the long-run the rewards and penalties cancel out and that the momentum strategist is no better than a simple "naive" randomizer. Our findings are important for a number of reasons. First, the evidence suggests that by 50% chance the naive strategy would not perform worse than the momentum trading that is long the winners and short the losers. From a practical point of view, asset managers who charge fees and pursue the momentum strategies would have only 50% chance of beating the naive strategies. Employing the information of past stock returns to form the momentum strategies does not seem to be beneficial as it requires costly and intensive trading. The construction of the long-short strategy position is further complicated by the risks, costs and regulation restrictions on short sales of stocks.

Second, the positive average score of the winner portfolio shows that by 60% of the time the winner portfolio outperforms the naive strategies. Our evidence suggests that taking a long position in the winner stocks which is financed at the risk-free rate can receive a rather small, but positive reward. This, of course, does not take into account of the costs of transactions, and of acquiring and analyzing return information.

The rest of the paper is structured as follows. Section 2 defines the strategies of the naive investors. In Section 3 we construct a score function and describe the empirical tests. Section

[5] We are grateful to the anonymous referee for this valuable comment and the useful suggestions that make the analysis more interesting.

4 presents results and Section 5 concludes. The Appendix contains proofs of theorems, statistical properties of the score function and derivations of statistical tests not presented in the text.

2 The Naive Investors' Strategies

Define an investor's portfolio strategy $\mathbf{F}$ over the asset set A which consists of N stocks that are feasible for trading at a given time[6] as

$$\mathbf{F} : \Omega \to W(A)$$

where Ω is the information available at the time of investing, $W(A)$ is the vector of portfolio weights on A chosen by an investor. The excess portfolio returns from strategy $\mathbf{F}$ is then given by $\mathbf{r}'\mathbf{F}$ where $\mathbf{r}$ is the vector of returns on the feasible asset set in excess of the risk-free rate. Notice that $\mathbf{F}_W$, the strategy that selects winners, $\mathbf{F}_L$ the strategy that selects losers as well as the momentum strategy $\mathbf{F}_P = (\mathbf{F}_W - \mathbf{F}_L)$ are in the set of $W(A)$. For the momentum investors, the information set Ω is the past returns which they use to form Winner and Loser portfolios.

We define a naive investor (NI) as an agent who has an uninformed prior on asset returns and does not change his portfolio formation decisions through acquiring information and learning from own past investment outcomes. Since the NI does not use any information, in each period, he randomly forms a vector of non-negative weights adding up to unity to allocate his wealth to the set of feasible assets.

[6] We supress the time-t subscript here to simplify notation.

Formally we define the set of all possible strategies of the NI, $\mathbf{\Phi}$, as

$$\mathbf{\Phi} : \{\} \rightarrow W_{+}(A)$$

where $\{\}$ denotes the empty information set employed by the NI. $W_{+}(A)$ is the vector of portfolio weights on the given feasible asset set A. This weight vector defines the feasible portfolio set of the NI,

$$W_{+}(A) = \left\{ \mathbf{w} : (w_i) \text{ s.t. } 0 \leq w_i \leq 1 \text{ and } \sum w_i = 1, \; i = 1, 2, ..., N \right\} \tag{1}$$

where w_i is the weight on stock $i \in A$. The feasible portfolios set is of $N - 1$ dimension because of the restriction that the weights must sum to unity. In general this is also known as an N dimensional unit simplex. $W_{+}(A)$ forms an uniform distribution over the simplex. For example, Figure 1 illustrates that the feasible portfolio set with three assets is a two dimensional triangle, the vertices being (1, 0, 0), (0, 1 ,0) and (0, 0, 1).

[Insert Figure 1]

Also notice that $\mathbf{F}_W$ and $\mathbf{F}_L$ are in this set, i.e., $\{\mathbf{F}_W, \mathbf{F}_L\} \in W_{+}(A)$. We derive analytical expressions for the weights in Theorem A1 (see Appendix) which gives a method for generating the weight distribution of the NI's strategy uniformly over the unit simplex.

We emphasize that the random allocation by the NI means that the weights are randomly drawn from an uniform distribution over a simplex, and hence, do not necessarily arrive at an equally weighted portfolio. For example, in the 3-asset case the chance to allocate all amount into only one asset with weights of (1, 0, 0) is the same as the chance to equally split the investment amount with weights of (1/3, 1/3, 1/3) because the NI randomly chooses weights (see Figure 1). In the Appendix we demonstrate the properties of the special case where the NI chooses the same weight accross all feasible stocks to form an equally weighted portfolio.

Let the cumulative distribution of the profits of the NIs' strategies conditional on asset returns, $G\left(q|\,\mathbf{r}\right)$, be given by

$$G\left(q|\,\mathbf{r}\right) = \Pr\left(\mathbf{r}'\boldsymbol{\Phi} \leq q|\,\mathbf{r}\right).$$

Given the profit distribution we can then generate its percentiles q_k such that $G\left(q_k\,|\,\mathbf{r}\right) = \frac{k}{100}$, $k \in \{1, .., 100\}$. In practice we do not need to compute the profit distribution since the weights of the naive strategies can be simulated using Theorem A1.

To illustrate the density of the profits (given in Theorem A2 in the Appendix) of the NIs' portfolios, we use the 3-asset case as an example where the weights on the assets are generated by Figure 1. Figure 2 plots the profit density $g\left(q|\,\mathbf{r}\right)$ of the NI's when the returns on the 3 assets are, respectively, -1%, 1% and 2%. It shows that the first quintile point, q_{20}, is 0.11%, the second point, q_{40}, is 0.56%, ...etc.

[Insert Figure 2]

3 The Score Function

We define a score function for a strategy $\mathbf{F}$. Our objective is to evaluate the performance of $\mathbf{F}$ with respect to the NIs' strategies. We give a score to a strategy based on the relative performance of the excess portfolio returns $\mathbf{r}'\mathbf{F}$ against the percentiles of the profit distribution of the naive strategies.

Definition 1 *The score given to a strategy* $\mathbf{F}$ *is defined as*

$$S(\mathbf{F}:\mathbf{r}) = \sum_{k=1}^{K} s_k I\left[q_{k-1} \leq \mathbf{r}'\mathbf{F} < q_k\right]$$

where I *is an indicator which is 1 if the excess return of the strategy,* $\mathbf{r}'\mathbf{F}$*, falls between* q_{k-1} *and* q_k *such that* $G(q_k \mid \mathbf{r}) = \frac{k}{K}$*, and the strategy* $\mathbf{F}$ *receives a score of* s_k*;* I *is zero otherwise.*

In the empirical tests later we designate the scores to be increasing with the percentiles of profits. We then assign scores ranging from -2 to 2 with one unit apart in ascending order to the quintile intervals. Specifically, our score function for a strategy $\mathbf{F}$, $S(\mathbf{F}:\mathbf{r})$, with $K = 5$ is:

$$S(\mathbf{F}:\mathbf{r}) = \begin{cases} s_{20} = -2, & -\infty \leq \mathbf{r}'\mathbf{F} < q_{20} \\ s_{40} = -1, & q_{20} \leq \mathbf{r}'\mathbf{F} < q_{40} \\ s_{60} = 0, & q_{40} \leq \mathbf{r}'\mathbf{F} < q_{60} \\ s_{80} = 1, & q_{60} \leq \mathbf{r}'\mathbf{F} < q_{80} \\ s_{100} = 2, & q_{80} \leq \mathbf{r}'\mathbf{F} < \infty \end{cases} \tag{2}$$

where returns $\mathbf{r}$, is the monthly excess stock returns and $\mathbf{F}$ is a strategy of the MI.

In the example of the 3-asset case, Figure 2 shows the quintile regions and their respective scores when $\mathbf{r} = (-1\%, 1\%, 2\%)$. Note that from (A6) the score of the median strategy is always zero, (i.e. $S\left(\frac{1}{N}\mathbf{1}:\mathbf{r}\right) = 0$) when the score function is symmetric around zero. That is, the score of the portfolio strategy which equally weights all the feasible assets is identically equal to zero.

3.1 Analytical Properties of the Score Function

Assume that the cross-section of excess returns are characterized by a factor model:

$$\mathbf{r}_t = \boldsymbol{\alpha}_t + \mathbf{B}_t \mathbf{x}_t + \boldsymbol{\varepsilon}_t,\ \boldsymbol{\varepsilon}_t \sim \left(0,\, \sigma_t^2 \mathbf{I}_{N_t}\right),\ t = 1, ..., T \tag{3}$$

where $\boldsymbol{\alpha}_t = [\alpha_{1t}, ., \alpha_{it}, .., \alpha_{N_t t}]$, $\mathbf{B}_t = [\boldsymbol{\beta}_{1t}, .., \boldsymbol{\beta}_{it}, ..., \boldsymbol{\beta}_{Nt}]$ is the vector of factor loadings, $\mathbf{x}_t's$ are common risk factors, σ_t^2 is the cross-sectional variance at time t and N_t is the total number of assets at time t. For a particular strategy $\mathbf{F}_t$ using equation (3) we get the usual factor model

$$r_{F,t} = \alpha_{F,t} + \mathbf{x}_t' \beta_{F,t} + e_t,\ e_t \sim \left(0,\, \sigma_{F,t}^2\right) \tag{4}$$

where $\alpha_{F,t} = \boldsymbol{\alpha}_t' \mathbf{F}_t$, $\beta_{F,t} = \mathbf{B}_t' \mathbf{F}_t$, $r_{F,t} = \mathbf{r}_t' \mathbf{F}_t$ and $\sigma_{F,t}^2 = \sigma_t^2 \mathbf{F}_t' \mathbf{F}_t$.

The following theorem states that the score function for measuring the performance of a strategy is invariant under common risk factors, that is, the scores of the risk-adjusted returns of a strategy are the same as those of the raw returns. This property is important because, instead of identifying the 'right factors' and measuring the risk-adjusted returns, we can concentrate on raw returns for comparison purpose.

Theorem 1 *Let the excess return generating process be given by the factor model (3). For any strategy* $\mathbf{F} \in W(A)$ *the score of the risk-adjusted return is the same as that of the excess return,*

$$S\left(\mathbf{F} : \frac{\mathbf{r} - \mathbf{B}\mathbf{x}}{\sigma}\right) = S\left(\mathbf{F} : \mathbf{r}\right).$$

The theorem also gives an additional practical advantage over traditional methods of estimating the risk factor model. If there are omitted factors $\mathbf{x}_t$ in a model of (4), the estimates of $\alpha_{F,t}$ will be biased. This has led to considerable research in identifying the right factors, for example, the factors of the market (the CAPM), the SMB and HML (Fama-

French (1993)), liquidity (Pastor and Stambaugh (2003)), macroeconomic risk (e.g., Liu and Zhang (2008)) and momentum (e.g., Avramov and Chordia (2006)). Theorem 1 implies that we can avoid the identification of risk factors and hence, the estimation of factor loadings $\beta_{F,t}$.

Theorem 2 *The score function has a breakdown point*[7] *of* $\frac{100}{K}\%$ *when* $q_1 = -\infty$ *and/or* $q_K = +\infty$.

Theorem (2) shows that our score function is robust against outliers and structural shifts in the return data. By construction, only when a fraction of $\frac{100}{K}\%$ of the data taking arbitrary values (such as outliers or data shifts) would the score function change. In our empirical analysis later, we choose $K = 5$ i.e a break down point of 20%.

In contrast, conventional moment-based estimators are sensitive to changes in sample observations in that the point estimate of a model will change even if only one firm is removed from the sample. For example, if we estimate model (4) using OLS method, a bankruptcy of just one firm can change the value of the estimate of $\alpha_{F,t}$. Indeed, moment-based regression methods have the same problem. Least Squares Median estimator has a breakdown point of 50%, but lacks precision, and also typically fares even worse than OLS for cases with high leverage points[8]. Bounded influence methods have a high breakdown point but they effectively remove a large proportion of observations.

Part a) of the following theorem shows that the strategy of shorting the loosers for our symmetric 5 point strategy function receives a score of $S(-\mathbf{F}_L : \mathbf{r}) = -S(\mathbf{F}_L : \mathbf{r})$.

[7] Breakdown point of a statistic is the smallest fraction of "bad" data (outliers or data grouped at the extreme of a tail) the statistic can tolerate without taking on values arbitrarily far away from uncontaminated statistic (for details see: Huber and Ronchetti (2009))

[8] Outliers with respect to the predictors are called leverage points (for details see: Belsley, Kuh and Welsch (1980))

Theorem 3 *a) If the score function is symmetric i.e.* $s_k = -s_{K-k+1}$, *for* $k = 1, .., K$ *then*

$$S(-\mathbf{F} : \mathbf{r}) = -S(\mathbf{F} : \mathbf{r})$$

b) If $s_{k-1} \leq s_k$ *for all* k, *for two strategies* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *where* $S(\mathbf{F}_1 : \mathbf{r}) \leq S(\mathbf{F}_2 : \mathbf{r})$, *we have*

$$S(\mathbf{F}_1 : \mathbf{r}) \leq S(\alpha\mathbf{F}_1 + (1-\alpha)\mathbf{F}_2 : \mathbf{r}) \leq S(\mathbf{F}_2 : \mathbf{r})$$

for all $\alpha \in (0, 1)$.

Since the score function in a step-function of strategy returns, it is non linear. Thus, scores of the momentum profits from buying the winners and selling short the losers are not always equal to the winners' scores minus the losers' scores, i.e., $S(\mathbf{F}_P : \mathbf{r}) \neq S(\mathbf{F}_W : \mathbf{r}) - S(\mathbf{F}_L : \mathbf{r})$.[9]

3.2 Sample and Empirical Tests

We use monthly equity data of the NYSE, AMEX and NASDAQ files from the Center for Research in Security Price (CRSP) for the period between January 1926 and December 2005. Our tests focus on the representative momentum strategies that form equally weighted portfolios by sorting stocks on their past 6-month compounded returns and hold portfolios for 6 months. We exclude all stocks with prices below $5 at portfolio formation as in Jegadeesh and Titman (1993). This defines our feasible asset set A. At the end of each month, the stocks within the top 10% of past returns comprise the winner portfolio and stocks within the bottom 10% of past returns comprise the loser portfolio. The overlapping momentum strategies thus consist of six strategies with each starting one month apart.

[9] As a counter example, suppose $0 < S(\mathbf{F}_1 : \mathbf{r}) < S(\mathbf{F}_2 : \mathbf{r})$, then $\mathbf{r}'\mathbf{F}_1 < \mathbf{r}'\mathbf{F}_2$. Without loss of generality let $\mathbf{r}'\mathbf{F}_1 = 0$, then $S(\mathbf{F}_1 + \mathbf{F}_2 : \mathbf{r}) = S(\mathbf{F}_2 : \mathbf{r}) \neq S(\mathbf{F}_1 : \mathbf{r}) + S(\mathbf{F}_2 : \mathbf{r})$.

Portfolios are initially equally weighted at the time of formation and are held for six months without rebalancing during the holding period[10]. We compute monthly excess portfolio returns and the profits to momentum strategies using single-period returns as in Liu and Strong (2008). The monthly portfolio returns from the overlapping strategies are averages of the six strategies as in Jegadeesh and Titman (1993).

One of the reasons for developing a score function to compare the momentum strategies against the entire distribution of the strategies of NIs is that a standard t-test is not robust against return outliers. Indeed, over the whole sample period the kurtosis of the momentum profits is 33. The Jarque-Bera test rejects the normality assumption for the momentum profits (the p-value is essentially zero).

We generate the distribution of the cross-section of excess portfolio returns. Using the result in Theorem (A1) we construct, each month, a cross-section of excess returns of 1,000 portfolios for the NIs and obtain the quintile points of the excess return distribution. We then assign scores as described in equation (2).

To illustrate the score distribution we plot Figure (3) which shows in box charts the monthly profit distribution of the NIs' strategies for the period from 1990 to 1998 examined by Jegadeesh and Titman (2001). The bottom box displays the 20th to the 40th percentiles of the excess returns of the NIs strategies; the green box in the middle displays the 40th to the 60th percentiles; and the top box displays the 60th to the 80th percentiles.

[Insert Figure 3 here]

Using the centered excess returns $\mathbf{r}_c = \mathbf{r} - \bar{r}\mathbf{1}$ we plot the centered box charts in Figure 4. Corollary A4 shows that the score function is affine invariant in returns, i.e., $S(\mathbf{F} : \mathbf{r}) = S(\mathbf{F} : \mathbf{r}_c)$. One significant observation is in order, the distribution of the centered returns

[10] In the case when a stock is delisted during the holding period, the liquidating proceeds are reinvested in the remaining stocks in the portfolio.

is stable, and hence is the score function. We also plot the figures (unreported for brevity, but available upon request) for the periods from 1965 to 1989 of Jegadeesh and Titman (1993), and the period from 1999 to 2005, respectively. The patterns in these figures are qualitatively the same and show that the distributions of the centered returns are stable.

[Insert Figure 4 here]

The empirical average score of the strategy $\mathbf{F}_t$ is given by:

$$\widehat{\mu}\left(\{\mathbf{F}_t : \mathbf{r}_t\}\right) = \frac{1}{T}\sum_{t=1}^{T} S_t\left(\{\mathbf{F}_t : \mathbf{r}_t\}\right).$$

The mean of the average scores $\widehat{\mu}\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right)$ using the results from (A8) when the score function is given by (2) or any symmetric score function is given by:

$$E\left[\widehat{\mu}\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right)\right] = 0.$$

We use the score function (2) to evaluate the momentum strategies. The winner and the loser portfolios and the momentum strategies receive a score of zero if their excess returns or profits fall within the 40th and the 60th percentiles of the excess return distribution of the NIs strategies; a score of 1 for falling within the 60th and the 80th percentiles; and a score of 2 for going higher than the 80th percentile. The negative scores are given vice versa. We then use the score function to evaluate the momentum profits and the excess returns on the winner and the loser portfolios. Appendix C gives details for the statistical tests of the scores.

Our goal is to find out whether the scores of momentum profits are significantly higher than those of the NIs' strategies. Therefore we test the null hypothesis $H_0 : E[S\left(\mathbf{F}_P \colon \mathbf{r}\right)] = 0$ against the alternative $H_A : E[S\left(\mathbf{F}_P \colon \mathbf{r}\right)] > 0$. We evaluate the score of the momentum strategies every month during the T evaluation periods and then calculate the average score, $\widehat{\mu}\left(\{\mathbf{F}_{P_t} : \mathbf{r}_t\}\right)$. Therefore a test of whether the momentum strategies outperform the NIs strategies is to test whether the average score of the momentum strategies is significantly positive. We perform a test based on the t-test given by (A9) in the Appendix. We also run the tests for the winner $H_0 : E[S(\mathbf{F}_W : \mathbf{r})] = 0$ (against $H_A : E[S(\mathbf{F}_W : \mathbf{r})] < 0$). We noted before that shorting the losers will gives positive scores (see Theorem 3 a)). Therefore we test whether the score of the loser portfolios is negative i.e. $H_0 : E[S(\mathbf{F}_L : \mathbf{r})] = 0$ against $H_A : E[S(\mathbf{F}_L : \mathbf{r})] < 0$.

4 Results

Table 1 presents the average scores of the winner, the loser portfolios and the momentum profits. Panel A shows, for the whole sample period, that the winner portfolio has a small, but positive and statistically significant average score of 0.11; the loser portfolio has a negative and statistically significant average score of -0.09. The average score of the momentum profits is 0.02 and statistically insignificant, showing that the momentum strategies do not outperform the zero net-worth strategies of the NIs.

[Insert Table 1 here]

Figure 5 shows the relative frequencies of the scores. In Panel A for the whole sample period, the scores of the winner, the loser portfolios and the momentum profits tend to be at the extreme ends of either positive or negative 2. The excess returns on the winner and loser portfolios are more likely to receive positive and negative extreme scores, respectively. In contrast, the chances for the momentum profits to receive either positive or negative 2 are very close (0.468 versus 0.483), thereby offsetting each other. As a result, the average score of the momentum profits is close to zero, which corresponds to the results in Table 1.

[Insert Figure 5 here]

4.1 Over Sub-Sample Periods

We then perform the analysis for various sub-sample periods. Panel B of Table 1 shows that over the period between 1965 and 1989 as in Jegadeesh and Titman (1993), both the average scores of the winner and the loser portfolios are very low and statistically insignificant. The average score of the momentum profits is essentially zero and statistically insignificant. The results in Panels C, D and E for the sample periods of Jegadeesh and Titman (2001), the period between August 1929 and March 1933 during the great depression as designated by the NBER, and the period between January 1995 and March 2000 for the dot-com bubble, respectively, show very similar patterns. Overall, the average scores of the momentum profits are close to zero in all the periods considered.

Panels B and C of Figure 5 show respectively, the score frequencies for the periods of Jegadeesh-Titman (1993 and 2001). The scores of the winner, the loser portfolios and the momentum profits exhibit very similar patterns as in Panel A. Panels D and E show, respectively, the score frequencies for the periods during the great depression and the dot-com bubble. Comparing to the other periods examined, during the great depression period the cases of extremely positive and negative returns on the winner portfolio occur less frequently while the loser portfolio is more likely to generate extreme losses. Therefore, the momentum strategies that buy winners and sell short losers are more likely to receive a score of +2. During the dot-com bubble the loser portfolio seems to become less likely to generate extreme losses. Again, the momentum profits have extreme scores on both ends, offsetting each other.

To understand this phenomenon, in Figure 6 we plot the box charts for the momentum profits for the period between 1990 and 1998.[11] The blue lines above the top boxes show

[11] To avoid ambiguity in the graph we only show the charts over this short period for clear illustration. The

the magnitudes of the momentum profits going higher than the 80th percentiles of the distributions of the excess returns of the NIs strategies. The red lines below the bottom boxes show the magnitudes of the momentum profits being lower than the 20th percentiles. An observation is clear that the momentum strategies incur high levels of profits and loses from time to time, but overall, are likely to offset each other.

[Insert Figure 6 here]

This phenomenon is better illustrated in Figure 7 where we plot the box charts for the momentum profits using the distribution of centered excess returns $\mathbf{r}_c$ which subtract $\overline{r}$ from $\mathbf{r}$. Note that the magnitudes of the momentum profits displayed here are the same as those in Figure 6.[12] Interestingly, we find that there are very few periods of momentum profit runs i.e. consecutive months of large positive profits, highlighting the risky nature of the momentum trading.

[Insert Figure 7 here]

4.2 100 Randomly Selected 10-Year Periods

We design a simulation experiment to see whether the momentum strategies outperform the naive strategies in any given period and any given set of assets. For each run, we select an experiment period of 120 months with the starting month randomly chosen between July 1926 and December 1995. We then use all sample stocks in that period and the portfolio formation methods described earlier to form the momentum strategies and the strategies of the NIs. The set of sample stocks that are used to construct NIs portfolios is the same as

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results (available upon request) for the whole sample period and other sub-sample periods are qualitatively very similar and do not change the conclusions.

[12]
$$\mathbf{r}_c'\mathbf{F}_P = (\mathbf{r}-\overline{r}\mathbf{1})'(\mathbf{F}_W - \mathbf{F}_L) = \mathbf{r}(\mathbf{F}_W - \mathbf{F}_L) \text{ since } \mathbf{1}'\mathbf{F}_W = \mathbf{1}'\mathbf{F}_L = 1.$$

that of the momentum portfolios. We compute average excess returns of the momentum portfolios and the monthly profit distributions of the zero net-worth naive strategies over the 120 months.

In a randomly selected sample we give a score to the momentum strategies, i.e., $\widehat{\mu}\left(\{\mathbf{F}_{jt} : \mathbf{r}_t\}\right)$, $j = W, L, P$ in each month and then compute an average score over 120 months in the sample. We sample 100 times with replacements and then generate 100 average scores for each of the winner and the loser portfolios as well as the momentum profits.

The winner portfolio tends to receive positive average scores, while the average scores of the loser portfolio are positively skewed and tend to be negative. We test, in each sample, whether the average scores of the momentum profits are positive. Specifically, we test the null hypothesis of $H_0 : E[S\left(\mathbf{F}_P \colon \mathbf{r}\right)] = 0$ against the alternative of $E[S\left(\mathbf{F}_P \colon \mathbf{r}\right)] > 0$ using the 120 monthly scores of the momentum profits. At the 5% level 82% of the samples accept the null. Hence, the naive diversifier is almost as good as the momentum strategist. The results demonstrate that the momentum strategies do not outperform the strategies of the NIs in the randomly selected samples, consistent with the results in the earlier sections.

4.3 Robustness Checks

4.3.1 Imposing Weight Restrictions on the NI's Portfolio

We impose some restrictions on the naive investors' strategies in order to examine the robustness of our results in a more realistic setting than the unrestricted one we examined earlier. The first restriction is that the weight on each stock in the naive investor's portfolio cannot exceed a maximum of 10%. Specifically, the feasible set is now given by:

$$W_{+}^{(1)}\left(A\right) = \left\{\mathbf{w} : 0 \leq w_i \leq 0.1 \text{ for all } i \in A \text{ and } \sum w_i = 1\right\}. \tag{5}$$

Consequently, the chance for the naive investor to place an excessively large weight on a stock that experiences an extreme return during the holding period is eliminated, leading to a shrinkage in the tails of the profit distribution of the naive strategies. Therefore, those momentum returns falling within the range between the 80th and the 20th percentiles in the previous analysis are now possible to fall outside of this range, and hence receive extreme scores on both ends.

Notice that, however, since most of the momentum returns in the previous analysis are already outside of the range between the 80th and the 20th percentiles, they continue to either drop below the 20th percentile or go higher than the 80th percentile of the naive profit distribution after applying for the restriction. Again, we find that the momentum strategies receive extreme scores at different points in time, and cancel out each other accross time.

Part I of Table 2 reports the results. Over the whole sample period both the winner and the loser portfolios have rather small, but statistically significant average scores. The extreme positive and negative scores for the momentum profits offset each other. Thus, the average score of the momentum profits is close to zero and statistically insignificant. Overall results for various sub-periods, again, confirm our findings reported earlier.

[Insert Table 2 here]

Given the typically high transaction costs of small size stocks, we exam the second restriction that the weight of the naive investor's portfolio on a stock whithin the smallest size decile of the feasible asset set cannot exceed a maximum of 10%. This implicitly implies an assumption that the naive investor has the information on firm size. Formally we define the set of all possible strategies of the NI, $\mathbf{\Phi}$, as

$$\mathbf{\Phi} : \{MV_i,\ i \in A\} \rightarrow W_{+}^{(2)}(A)$$

where MV_i is the market value of asset i. The weight vector that defines the feasible portfolio set of the NI is

$$W_+^{(2)}(A) = \left\{ \mathbf{w} : 0 \leq w_i \leq 0.1 \text{ for all } i \text{ s.t. } MV_i \leq \; q_{10}(MV) \text{ and } \sum w_i = 1 \right\} \qquad (6)$$

where $q_{10}(MV)$ is the bottom decile of the market values of all feasible stocks.

We randomly choose weights $\mathbf{w}$, as in (1) and use an Acceptance-Rejection algorithm [13] to restrict our weights in the feasible sets defined in (5) and (6). Overall results reported in Part II of Table 2 confirm our findings over the whole sample period and various sub-periods. Again, we find that the momentum strategies receive extreme scores at different points in time, and cancel out each other accross time.

4.3.2 Using 11 Points Score Function

We repeat the empirical tests using 11 score points of the return distribution by assigning scores ranging from -5 to +5 with one unit apart in ascending order of equal intervals of 9.1 (i.e. 100/11) percentiles. Specifically, our score function for a strategy $\mathbf{F}$, $\widetilde{S}(\mathbf{F} : \mathbf{r})$, with $K = 11$ can be written compactly as:

$$\widetilde{S}(\mathbf{F} : \mathbf{r}) = \sum_{k=1}^{11} (k-6)\, I\left[q_{\frac{100*(k-1)}{11}} \leq \mathbf{r}'\mathbf{F} < q_{\frac{100*k}{11}} \right] \qquad (7)$$

where q_α is $\alpha-$percentile of the NI's returns and $\mathbf{r}'\mathbf{F}$ is the return of the MI. It is worth to note that the choice of odd-number points for a score function ensures that the median of the naive strategies always gets zero score.

Part I of Table 3 presents the results. The only difference in results from using a 11-point score system is that both the winner and the loser portfolios now have insignificant average

[13] See Robert and Casella (2004) for details.

scores over the whole sample period. This is a result of increasing the variance of the scores because the scores now can take 11 points instead of 5 points in the earlier analysis.

[Insert Table 3 here]

Figure 5 further gives the relative frequencies of the scores over various periods we examined. The overall pattern reveals that the scores of the winner portfolio, the loser portfolio and the momentum profits tend to be at the extreme ends of either positive or negative 5, providing further evidence that momentum returns often locate at the tails of the naive return distribution. Again, the extreme positive and negative scores for the momentum profits offset each other, resulting in an average score close to zero.

[Insert Figure 8 here]

Parts II and III of Table 3 report the results, respectively, for the analyses of applying the first restriction of a maximum weight of 10% on each stock and the second restriction of a maximum weight of 10% on small size stocks. In all cases, the winner and the loser portfolios show small (of opposite signs) and statistically insignificant average scores. The average scores of momentum profits are close to zero and statistically insignificant. Overall, the results from applying such restrictions are very similar to those of the case with unrestricted weights and do not change our conclusions.

5 Conclusions

In this paper we evaluate the profits of the momentum strategies which use past return information against the cross-section of the profit distribution of the zero net-worth strategies of "naive investors" (NIs) who do not use any information. We define a score function which is invariant under different risk factors, and by which to give scores to the momentum strategies relative to the profits of the population of the NIs. We thus do not need to specify the risk factors underlying the momentum effect. We find that average scores of the momentum strategies are close to zero and statistically insignificant over the sample period between 1926 and 2005, various sub-sample periods including the periods examined in Jegadeesh and Titman (1993 and 2001). The findings are robust with respect to sampling or period-specific effects in our simulations where we randomly select 10 years for 100 times. Our overall results are also robust to the use of a more detailed score system, or maximum weight restrictions on stocks. The preponderance of the evidence suggests that professional asset managers who pursue the momentum strategies would have only 50% chance of beating the naive strategies.

Appendix

A Proofs of Theorems

Theorem A1 *The joint distribution of the portfolio weights of the NI is uniform and is given by* $\mathbf{\Phi} = (\phi_1, ..., \phi_i, ..., \phi_N)$ *where the marginal distribution of the weight on stock* i *is* $\phi_i = \frac{E_i}{\sum E_i}$*, and* $E_i \sim iid$ ExponentialDist(1) *such that the density of* $\mathbf{\Phi}$ *is*

$$
\begin{aligned}
h(\mathbf{w}) &= N!, \ \ \textit{if } \mathbf{w} \in W_+(A); \\
&= 0, \ \ \textit{otherwise}
\end{aligned}
$$

Proof of Theorem. A1: First note that $\sum_{i=1}^{N} \phi_i = 1$ and $0 \leq \phi_i \leq 1$ for all $i = 1, ...N$. We prove that the joint density of $\mathbf{\Phi}$ is a constant.

Let $E = \sum_{i=1}^{N} E_i$ and the joint density of E_i 's is: $\exp\left(-\sum_{i=1}^{N} e_i\right)$. Therefore the joint density of $(E_1, .., E_i, ..E_N, E)$ is given by :

$$\exp\left(-\sum_{i=1}^{N} e_i\right) I\left[\sum_{i=1}^{N} e_i = e\right].$$

Consider the reverse transformation $E_i = E\phi_i$. The Jacobian of the transformation is E^N. Thus the joint density of $(\mathbf{\Phi}, E)$ is $ce^N \exp(-e)$, where c is a normalizing constant. Integrating out e and normalizing the constant gives $h(\mathbf{w}) = N!$. ∎

Corollary A1 *The cross-sectional mean and the projection median, PM,* [14] *of $\mathbf{\Phi}$ are given by*

$$E[\mathbf{\Phi}] = PM(\mathbf{\Phi}) = \frac{1}{N}\mathbf{1} \tag{A1}$$

where $\mathbf{1}$ is a vector of ones with length N.

Corollary A2 *The variance-covariance matrix of $\mathbf{\Phi}$ is*

$$E[\mathbf{\Phi}\mathbf{\Phi}'] = \Sigma_{\mathbf{\Phi}} = \frac{1}{N(N+1)}\left(\mathbf{I} - \frac{1}{N}\mathbf{1}\mathbf{1}'\right). \tag{A2}$$

Corollary A3 *The excess portfolio return from a naive strategy $\mathbf{\Phi}$ is given by $\mathbf{r}'\mathbf{\Phi}$. Following from (A1), the cross-sectional mean and the median of the excess portfolio returns of the NI strategies are*

$$E[\mathbf{r}'\mathbf{\Phi}] = M(\mathbf{r}'\mathbf{\Phi}) = \frac{1}{N}\sum_{i=1}^{N} r_i = \overline{r}$$

Hence, $\overline{r}$ is effectively the equally weighted excess return of the feasible asset set. From (A2)

[14] See: Zuo (2003) for a recent discussion.

the variance of the excess portfolio returns of the NI strategies is

$$Var\left(\mathbf{r}'\mathbf{\Phi}\right) = \frac{1}{N+1}\sum_{i=1}^{N}\left(r_i - \overline{r}\right)^2.$$

Theorem A2 *Let g be the density of the profits of the NIs' strategies such that $G\left(q|\,\mathbf{r}\right) = \int_{-\infty}^{q} g\left(q|\,\mathbf{r}\right)dz$, where $\mathbf{r}$ is a vector of excess returns. We have:*

$$g\left(q|\,\mathbf{r}\right) = \sum_{k=1}^{N}\frac{\left(r_k - q\right)^{N-1}}{\prod\limits_{l=1,l\neq k}^{N}\left(r_l - r_k\right)}$$

Proof of Theorem. A2: We can write

$$G\left(q|\,\mathbf{r}\right) = \Pr\left(\sum_{i}^{N} r_i E_i \leq q\sum_{i}^{N} E_i\right) = \Pr\left(\sum_{i}^{N}\left(r_i - q\right)E_i \leq 0\right) \tag{A3}$$

The density of $Y = \sum_i^N \left(r_i - q\right)E_i$ at $y = 0$ follows from equations (27) and (28) of Biswal et. al. (1998, pages 593-594) by substituting $\lambda_{ij} = 1$ and $x_k = \left(r_i - q\right)$. ■

Proof of Theorem. 1: Note that

$$G\left(q:\mathbf{r}\right) = \Pr\left(\mathbf{r}'\mathbf{\Phi} \leq q\right) = \Pr\left(\frac{\mathbf{r}'\mathbf{\Phi} - \left(\boldsymbol{\alpha} + \mathbf{Bx}\right)'\mathbf{F}}{\sigma} \leq \frac{q - \left(\boldsymbol{\alpha} + \mathbf{Bx}\right)'\mathbf{F}}{\sigma}\right).$$

then

$$S\left(\mathbf{F}:\mathbf{r}\right) = \sum_{k=2}^{K} s_k I\left[q_{k-1} \leq \mathbf{r}'\mathbf{F} < q_k\right]$$

$$= \sum_{k=1}^{K} s_k I\left[\frac{q_{k-1} - \mathbf{x}'\mathbf{B}'\mathbf{F}}{\sigma} \leq \frac{\mathbf{r}'\mathbf{F} - \mathbf{x}'\mathbf{B}'\mathbf{F}}{\sigma} < \frac{q_k - \mathbf{x}'\mathbf{B}'\mathbf{F}}{\sigma}\right] = S\left(\mathbf{F}:\frac{\mathbf{r} - \mathbf{Bx}}{\sigma}\right).$$

■

As a special case we get the following corollary.

Corollary A4 *The score function is invariant under common affine transformation i.e., for any $\sigma \in R^{+}$ and $\mu \in R$, the score of affine transformations of the excess returns remains the same.*

$$S\left(\mathbf{F}:\sigma\mathbf{r}+\mu\mathbf{1}\right)=S\left(\mathbf{F}:\mathbf{r}\right).$$

Proof of Theorem. 2: It is easy to see that since we are using k^{th} percentiles, N/K returns in $\mathbf{r}$ needs to be arbitrarily large to change the score function. ■

Proof of Theorem. 3: a) Notice that $\Pr(\mathbf{r}'\mathbf{\Phi} \leq q_a\left(\mathbf{r}'\mathbf{\Phi}\right))=\alpha \Longleftrightarrow \Pr(-\mathbf{r}'\mathbf{\Phi} \leq -q_a\left(\mathbf{r}'\mathbf{\Phi}\right))=1-\alpha$, implying that $-q_a\left(\mathbf{r}'\mathbf{\Phi}\right)=q_{1-a}\left(-\mathbf{r}'\mathbf{\Phi}\right)$. Now because of the symmetry of the scores we have:

$$\begin{aligned}
S\left(-\mathbf{F}:\mathbf{r}\right) &= \sum_{k=1}^{K} s_k I\left[q_{k-1}\leq -\mathbf{r}'\mathbf{F}<q_k\right] \\
&= \sum_{k=1}^{K} s_k I\left[-q_{k-1}\geq \mathbf{r}'\mathbf{F}>-q_k\right] \\
&= \sum_{k=1}^{K} -s_{K-k} I\left[q_{K-k}\leq \mathbf{r}'\mathbf{F}<q_k\right]=-S\left(-\mathbf{F}:\mathbf{r}\right).
\end{aligned}$$

b) Note that $S\left(\mathbf{F}:\mathbf{r}\right)=\sum_{k=1}^{K} s_k I\left[q_{k-1}\leq \mathbf{r}'\mathbf{F}<q_k\right]$ is an increasing function of $\mathbf{r}'\mathbf{F}$, and is both quasi-concave and quasi-convex in $\mathbf{r}'\mathbf{F}$. Hence, if $S\left(\mathbf{F}_1:\mathbf{r}\right)\leq S\left(\mathbf{F}_2:\mathbf{r}\right)$, then $\mathbf{r}'\mathbf{F}_1\leq \mathbf{r}'\mathbf{F}_2$

$$\begin{aligned}
&S\left(\mathbf{F}_1:\mathbf{r}\right)=\sum_{k=1}^{K} s_k I\left[q_{k-1}\leq \mathbf{r}'\mathbf{F}_1<q_k\right] \\
&\leq \sum_{k=1}^{K} s_k I\left[q_{k-1}\leq \mathbf{r}'\left[\alpha\mathbf{F}_1+\left(1-\alpha\right)\mathbf{F}_2\right]<q_k\right] \\
&\leq \sum_{k=1}^{K} s_k I\left[q_{k-1}\leq \mathbf{r}'\mathbf{F}_2<q_k\right]=S\left(\mathbf{F}_2:\mathbf{r}\right).
\end{aligned}$$

■

B Statistical Properties of the Score Function

Given an excess return vector r, the probability of strategy F receiving a score s_k is

$$p_k \equiv \Pr\left(S\left(\mathbf{F}:\mathbf{r}\right)=s_k\right) \equiv \Pr\left(q_{k-1} \leq \mathbf{r}'\mathbf{F} < q_k\right)$$

Thus, the expected score of the strategy is

$$E[S\left(\mathbf{F}:\mathbf{r}\right)] = \mu_S\left(\mathbf{F}:\mathbf{r}\right) = \sum_{k=1}^{K} s_k p_k\left(\mathbf{F}:\mathbf{r}\right) = \mathbf{s}'\mathbf{p}.$$

where $\mathbf{p} = \left[p_1\left(\mathbf{F}:\mathbf{r}\right), ..p_k\left(\mathbf{F}:\mathbf{r}\right).., p_K\left(\mathbf{F}:\mathbf{r}\right)\right]$ is the vector of probabilities that strategy F receives the designated scores $\mathbf{s} = (s_1, ..s_k.., s_K)$ for its returns falling in the corresponding return ranges. The variance of the scores of strategy F is

$$Var\left(S\left(\mathbf{F}:\mathbf{r}\right)\right) = \sigma_S^2\left(\mathbf{F}:\mathbf{r}\right) = \sum_{k=1}^{K}\left[s_k\right]^2 p_k\left(\mathbf{F}:\mathbf{r}\right) - \left[\mu_S\left(\mathbf{F}:\mathbf{r}\right)\right]^2. \quad \text{(A4)}$$

The mean and variance of the scores for the NI strategy $\mathbf{\Phi}$ are, respectively:

$$\mu_S\left(\mathbf{\Phi}:\mathbf{r}\right) = \frac{1}{K}\sum_{k=1}^{K} s_k, \text{ and } \sigma_S^2\left(\mathbf{\Phi}:\mathbf{r}\right) = \frac{1}{K}\sum_{k=1}^{K}\left[s_k\right]^2 - \left[\mu_S\left(\mathbf{\Phi}:\mathbf{r}\right)\right]^2 \quad \text{(A5)}$$

Normalizing the expected score of the NI strategy to be zero by choosing appropriate $\mathbf{s}$, the variance is then simply $\sigma_S^2\left(\mathbf{\Phi}:\mathbf{r}\right) = \frac{1}{K}\sum_{k=1}^{K}\left[s_k\right]^2$. The median of the score of the NI strategy is

$$M(S\left(\mathbf{\Phi}:\mathbf{r}\right)) = s_m, \text{ such that } q_{m-1} \leq \overline{r} < q_m. \quad \text{(A6)}$$

Recall that the median strategy is feasible as the equally weighted portfolio $\frac{1}{N}1$ and thus always has a score of s_m, i.e., $S\left(\frac{1}{N}\mathbf{1}:\mathbf{r}\right)=s_m$. The variance of the equally weighted portfolio is zero, i.e.,$Var\left(S\left(\frac{1}{N}\mathbf{1}:\mathbf{r}\right)\right)=0$.

C Tests of Statistical Significance of the Scores

We test the null hypothesis $H0: E[S(\mathbf{F}:\mathbf{r})]=E[S(\mathbf{\Phi}:\mathbf{r})]$ against the alternative $H1$: $E[S(\mathbf{F}:\mathbf{r})]>E[S(\mathbf{\Phi}:\mathbf{r})]$. The empirical average score of any strategy $\mathbf{F}_t$ is given by:

$$\widehat{\mu}\left(\{\mathbf{F}_t:\mathbf{r}_t\}\right)=\frac{1}{T}\sum_{t=1}^{T}S\left(\{\mathbf{F}_t:\mathbf{r}_t\}\right).$$

In order to construct a test statistic for the purpose under the null hypothesis we find the sampling distribution of the average score under the null i.e. $\widehat{\mu}\left(\{\mathbf{\Phi}_t:\mathbf{r}_t\}\right)$ where $\mathbf{\Phi}_t$ and $\mathbf{r}_t$ are the naive strategy and the vector of excess stock returns at period $t=1,...,T$. The likelihood function of the scores of the naive strategies is a multinomial distribution:

$$L\left(\tau_1,...,\tau_K:\{\mathbf{\Phi}_t,\mathbf{r}_t\}\right)=c\prod_{k=1}^{K}p_k^{\tau_k}$$

where $\tau_k=\sum_{t=1}^{T}I\left[q_{k-1,t}\leq\mathbf{r}_t'\mathbf{\Phi}_t<q_{k,t}\right]$ is the number of times the naive strategy return $\mathbf{r}_t'\mathbf{\Phi}_t$ falls between $q_{k-1,t}$ and $q_{k,t}$ where $q_{k,t}$ is the k/K percentile of the excess return distribution of the naive strategy at period t, such that $\sum\tau_k=T$ and c is a normalizing constant.

The maximum likelihood estimator (MLE) for p_k is then given by: $\widehat{p}_k=\frac{\tau_k}{T}$. The asymptotic distribution of $\widehat{p}_k$ is

$$\sqrt{T}\widehat{p}_k\overset{asy}{\sim}N\left(p_k,p_k\left(1-p_k\right)\right),\ k=1,...,K.$$

The variance and covariance of $\widehat{p}_k$ are

$$var\left(\widehat{p}_k\right) = \frac{p_k\left(1-p_k\right)}{T} \text{ and } Cov\left(\widehat{p}_k, \widehat{p}_l\right) = -\frac{p_k p_l}{T}.$$

Therefore, from (A5), the empirical mean of the scores is

$$\widehat{\mu}_S\left(\{\mathbf{F}_t : \mathbf{r}_t\}\right) = \sum_{k=1}^{K} s_k \widehat{p}_k$$

The expectation of $\widehat{\mu}_S\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right)$ and the sample variance of $\widehat{\mu}_S\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right)$ are

$$E[\widehat{\mu}_S\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right)] = \sum_{k=1}^{K} s_k p_k \text{ and } Var\left(\widehat{\mu}_S\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right)\right) = \frac{\sigma_S^2\left(\mathbf{\Phi}_t\text{:}\, \mathbf{r}\right)}{T}. \tag{A7}$$

From the discussion above, the asymptotic properties of the performance measure for naive strategy $\mathbf{\Phi}_t$ given a vector of asset returns $\mathbf{r}_t$ at period t is

$$\sqrt{T}\widehat{\mu}\left(\{\mathbf{\Phi}_t : \mathbf{r}_t\}\right) \overset{asy}{\sim} N\left(\mu_S\left(\mathbf{\Phi} : \mathbf{r}\right),\ \sigma_S^2\left(\mathbf{\Phi} : \mathbf{r}\right)\right). \tag{A8}$$

Therefore the difference in performance between strategy $\mathbf{F}$ and the NI strategy $\boldsymbol{\phi}$ can be measured by $\widehat{\mu}\left(\{\mathbf{F}_t : \mathbf{r}_t\}\right) - \mu\left(\mathbf{\Phi} : \mathbf{r}\right)$. One can compute a t-ratio using (A8) for testing the null hypothesis that strategy F performs the same as the NI strategy Φ as

$$t\left(\mathbf{F}\right) = \sqrt{T}\frac{\widehat{\mu}\left(\{\mathbf{F}_t : \mathbf{r}_t\}\right) - \mu_S\left(\mathbf{\Phi} : \mathbf{r}\right)}{\sqrt{\sigma_S^2\left(\mathbf{F} : \mathbf{r}\right)}}. \tag{A9}$$

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Table 1

Average Scores of the Momentum Strategies

The score function assigns 5 score points ranging from -2 to +2 with one unit apart in ascending order to the quintile intervals of the profits of the naïve investor strategies. This table presents the mean scores of the momentum strategies for the whole sample period (Panel A), the sub-sample periods of Jagadeesh-Titman (1993 and 2001) (Panels B and C), the periods during great depression (Panel D) and the dot-com bubble (Panel E). 'Winner' and 'Loser' are, respectively, the excess returns on the winner and the loser portfolios. 'Momentum Profits' represent the profits from buying long the winner and selling short the loser portfolios. *t*-statistics are in parentheses. * and ** denote statistical significance at the 10% and 5% levels, respectively.

Panel A. The Whole Sample: 01-1927 to 07-2005			
	Winner	Loser	Momentum Profits
Average Score	0.107	-0.091	0.017
	(1.775**)	(-1.503*)	(0.267)
Panel B. Jagadeesh-Titman (1993): 01-1965 to 02-1989			
Average Score	0.120	-0.116	0.051
	(1.112)	(-1.090)	(0.448)
Panel C. Jaggadeesh-Titman (2001): 01-1990 to 02-1998			
Average Score	0.132	-0.089	0.030
	(0.747)	(-0.486)	(0.154)
Panel D. Great Depression: 08-1929 to 03-1933			
Average Score	0.105	-0.073	0.114
	(0.380)	(-0.247)	(0.397)
Panel E. Dot-Com Bubble: 01-1995 to 02-2000			
Average Score	0.126	-0.061	0.039
	(0.526)	(-0.255)	(0.152)

Table 2

Five Score Points, Restriction of 10% Maximum Weight on Each Stock or on Small Size Stocks

The naïve investors' strategies are subject to: (I) a restriction that any stock in the feasible asset set has a maximum weight of 10%, and (II) another restriction that any stock in the smallest size decile has a maximum weight of 10%. The score function assigns 5 score points ranging from -2 to +2 with one unit apart in ascending order to the quintile intervals of the profits of the naïve investor strategies. *t*-statistics are in parentheses. * and ** denote statistical significance at the 10% and 5% levels, respectively.

	I. 10% Maximum Weight on Each Stock			II. 10% Maximum Weight on Small Size Stocks		
Panel A. The Whole Sample: 01-1927 to 07-2005						
	Winner	Loser	Momentum Profits	Winner	Loser	Momentum Profits
Average Score	0.107	-0.091	0.017	0.108	-0.090	0.018
	(1.761**)	(-1.497*)	(0.271)	(1.785**)	(-1.484*)	(0.274)
Panel B. Jagadeesh-Titman (1993): 01-1965 to 02-1989						
Average Score	0.119	-0.116	0.051	0.121	-0.116	0.051
	(1.103)	(-1.088)	(0.448)	(1.118)	(-1.087)	(0.448)
Panel C. Jaggadeesh-Titman (2001): 01-1990 to 02-1998						
Average Score	0.133	-0.091	0.030	0.135	-0.087	0.030
	(0.755)	(-0.495)	(0.154)	(0.766)	(-0.473)	(0.154)
Panel D. Great Depression: 08-1929 to 03-1933						
Average Score	0.105	-0.073	0.114	0.105	-0.073	0.114
	(0.380)	(-0.247)	(0.397)	(0.380)	(-0.247)	(0.397)
Panel E. Dot-Com Bubble: 01-1995 to 02-2000						
Average Score	0.126	-0.065	0.039	0.129	-0.061	0.039
	(0.526)	(-0.267)	(0.152)	(0.540)	(-0.255)	(0.152)

Table 3

Eleven Score Points, With and Without Maximum Weight Restrictions

The score function assigns 11 score points ranging from -5 to +5 with one unit apart in ascending order of equal intervals of 9.1 (i.e. 100/11) percentiles of the profits of the naïve investor strategies. *t*-statistics are in parentheses.

	I. Unrestricted			II. 10% Maximum Weight on Each Stock			III. 10% Maximum Weight on Small Size Stocks		
Panel A. The Whole Sample: 01-1927 to 07-2005									
	Winner	Loser	Momentum Profits	Winner	Loser	Momentum Profits	Winner	Loser	Momentum Profits
Average Score	0.122	-0.104	0.019	0.122	-0.105	0.019	0.122	-0.104	0.020
	(0.815)	(-0.696)	(0.120)	(0.813)	(-0.698)	(0.121)	(0.812)	(-0.694)	(0.122)
Panel B. Jagadeesh-Titman (1993): 01-1965 to 02-1989									
Average Score	0.136	-0.131	0.059	0.136	-0.131	0.059	0.135	-0.131	0.059
	(0.506)	(-0.496)	(0.210)	(0.504)	(-0.495)	(0.207)	(0.503)	(-0.493)	(0.211)
Panel C. Jaggadeesh-Titman (2001): 01-1990 to 02-1998									
Average Score	0.152	-0.103	0.033	0.151	-0.102	0.034	0.151	-0.103	0.033
	(0.352)	(-0.227)	(0.068)	(0.348)	(-0.225)	(0.070)	(0.346)	(-0.227)	(0.068)
Panel D. Great Depression: 08-1929 to 03-1933									
Average Score	0.114	-0.091	0.128	0.114	-0.095	0.128	0.116	-0.093	0.128
	(0.167)	(-0.126)	(0.179)	(0.166)	(-0.133)	(0.179)	(0.171)	(-0.130)	(0.179)
Panel E. Dot-Com Bubble: 01-1995 to 02-2000									
Average Score	0.141	-0.072	0.041	0.139	-0.067	0.041	0.141	-0.070	0.041
	(0.237)	(-0.121)	(0.065)	(0.234)	(-0.113)	(0.065)	(0.236)	(-0.118)	(0.065)

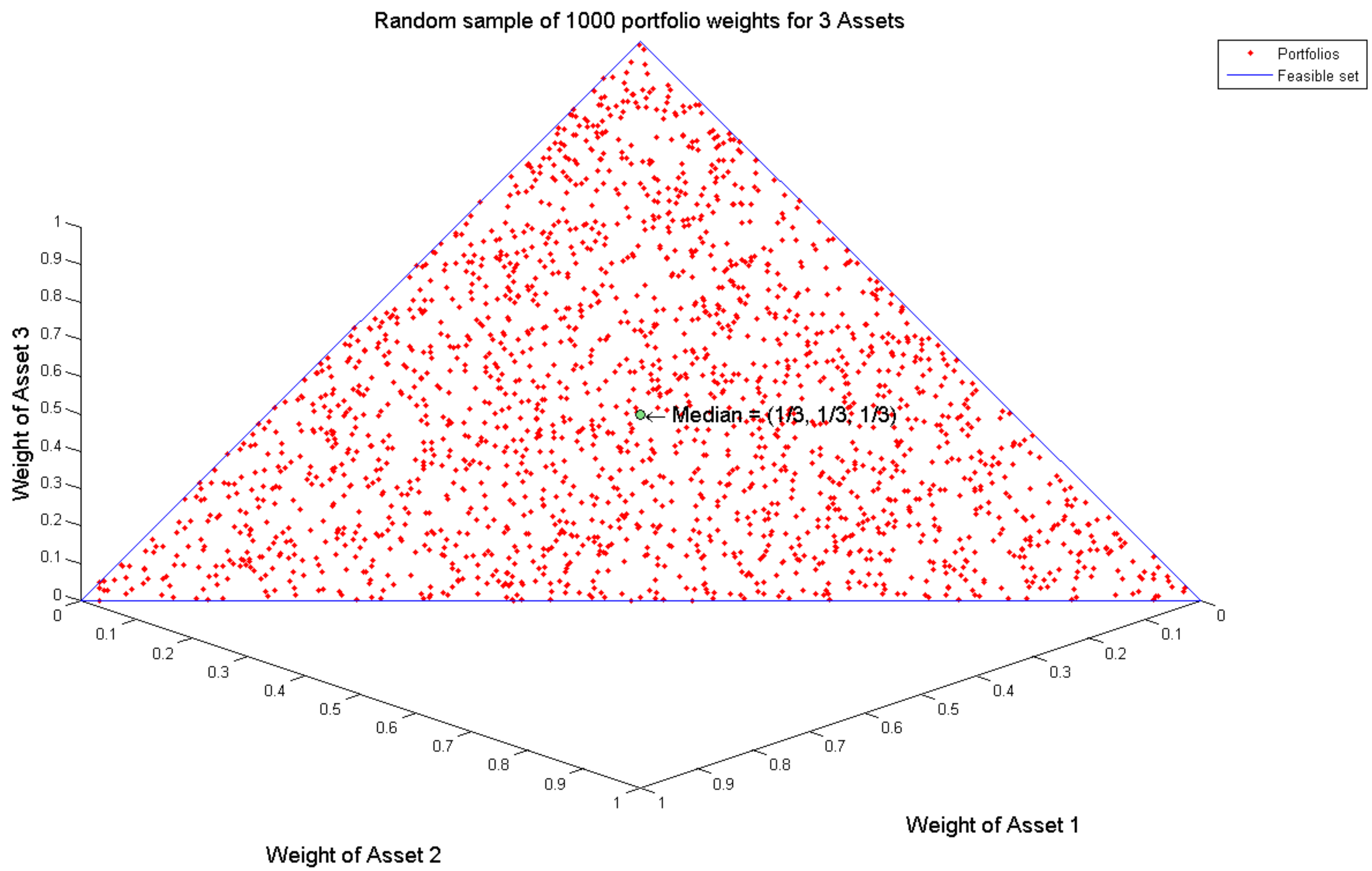

Figure 1: Random Sample of 1000 portfolio weights when $N = 3$.

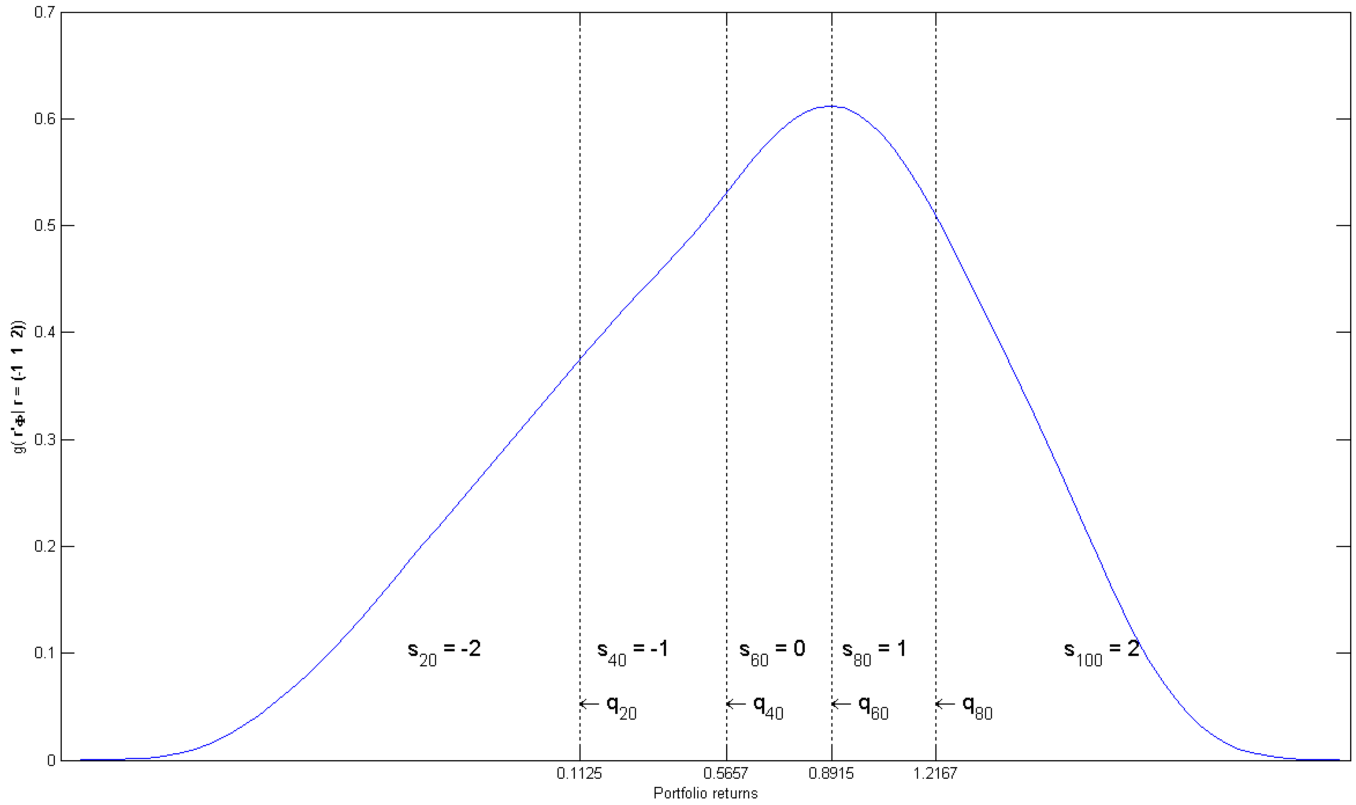

Figure 2: Density of the NIs' profits $g(\mathbf{r}'\mathbf{\Phi}|\,\mathbf{r})$ in percentage given $\mathbf{r} = (-1, 1, 2)$ and $N = 3$.

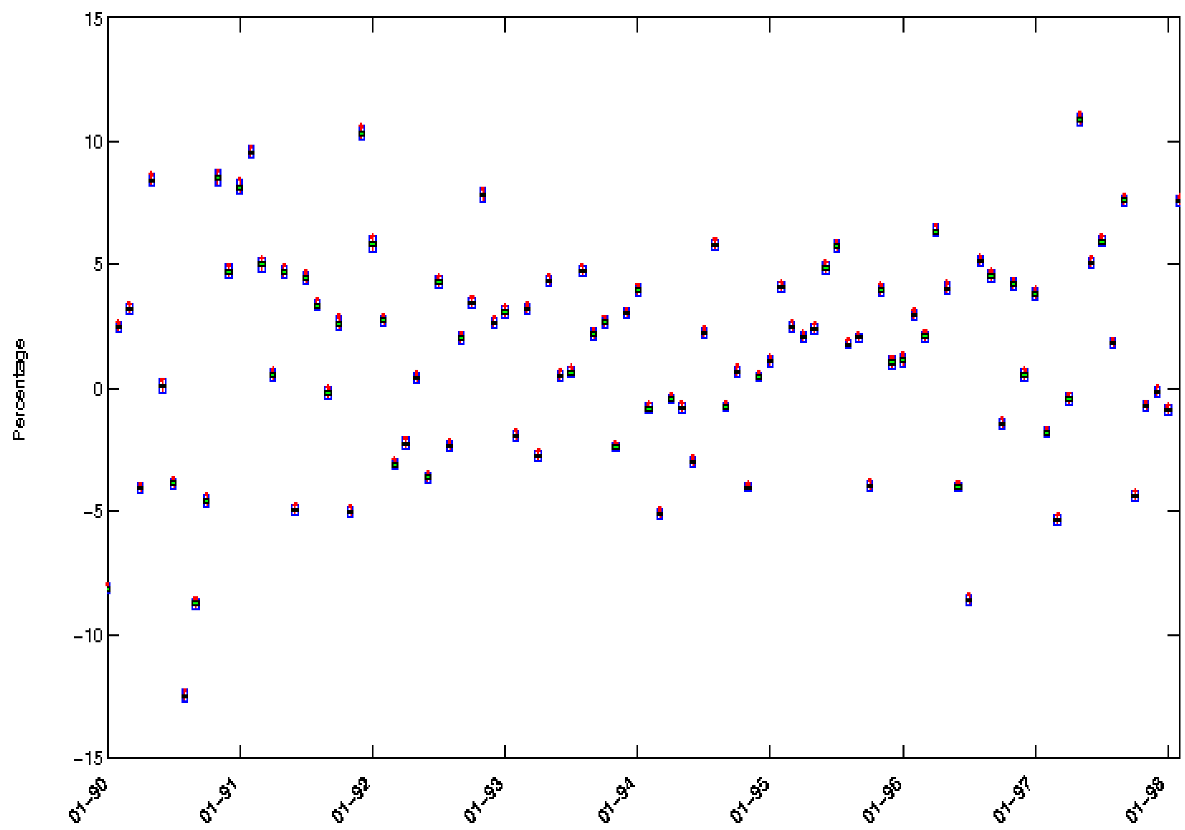

Figure 3: **The monthly profit (in %) distributions of the naive strategies from 1990 to 1998.** The distributions are displayed in box charts. The bottom box shows the 20th to the 40th percentiles of the profits of the naive investors strategies; the green box in the middle displays the 40th to the 60th percentiles; and the top box displays the 60th to the 80th percentiles.

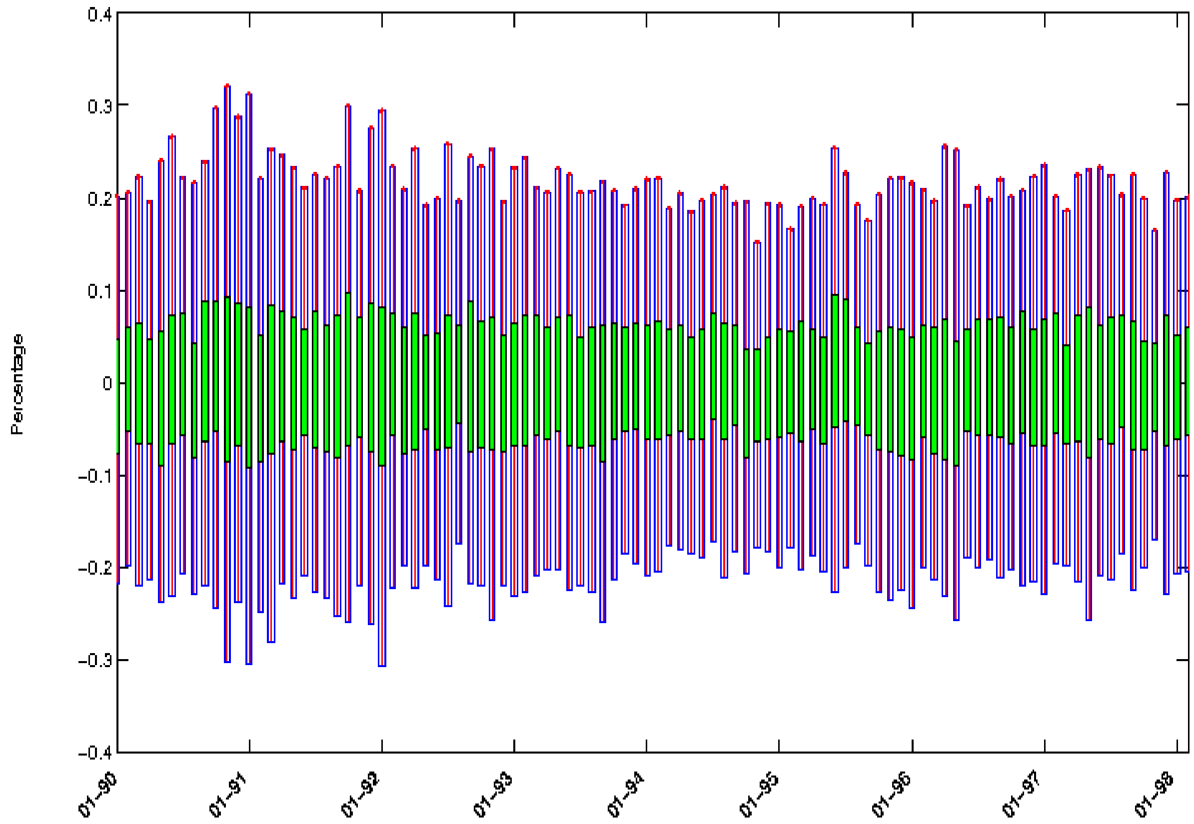

Figure 4: **The centered monthly profit (in %) distributions of the naive investors strategies from 1990 to 1998.** The distributions are centered by their medians and displayed in box charts. The bottom box shows the 20th to the 40th percentiles of the profits of the naive investors strategies; the green box in the middle displays the 40th to the 60th percentiles; and the top box displays the 60th to the 80th percentiles.

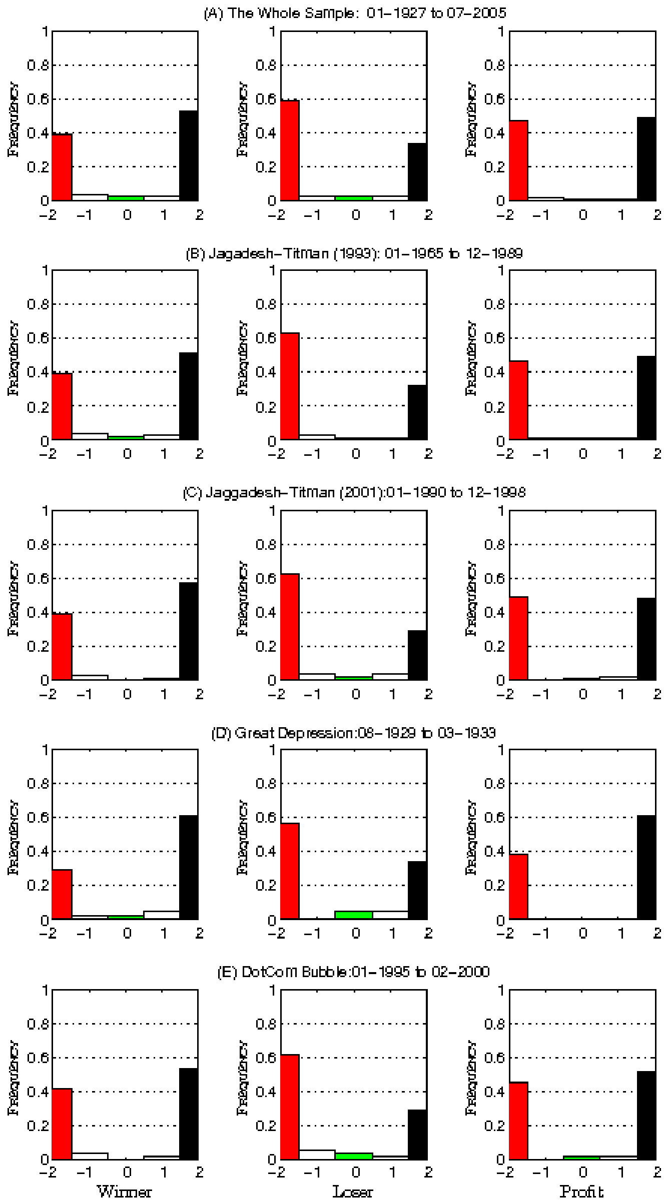

Figure 5: **The distributions of 5 scores of the winner (1st column), the loser (2nd column) portfolios and the momentum profits (3rd column).**
Panels A, B, C, D and E display the results for the whole sample period, the sub-sample periods of the J-T (1993), J-T (2001), the Great Depression and the DotCom bubble respectively.

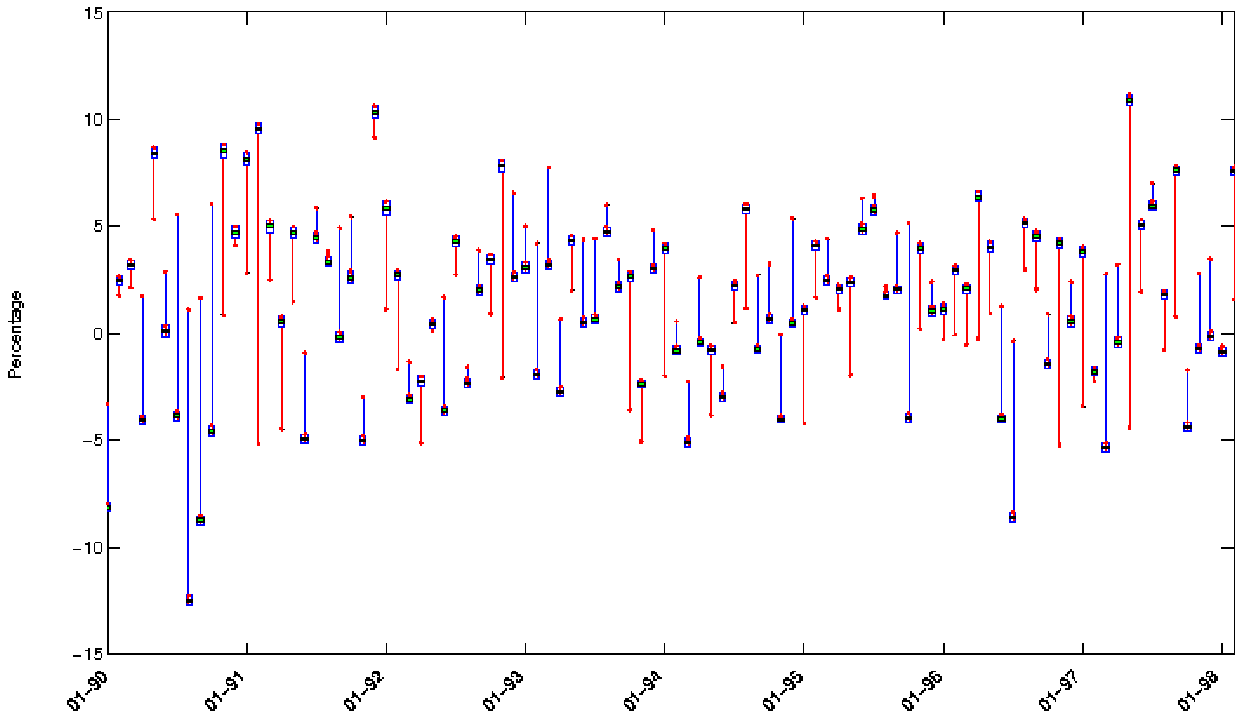

Figure 6: **The monthly return (in %) distributions of the momentum profits: JT(2001): 1990-1998.** The distributions are displayed in box charts. The bottom box shows the 20th to the 40th percentiles of the profits of the naive investors strategies; the green box in the middle displays the 40th to the 60th percentiles; and the top box displays the 60th to the 80th percentiles. The deviations of the momentum profits above the top 80th percentile of the profits of the naive strategies are in blue; and the deviations of the momentum profits below the bottom 20th percentile of the profits of the naive strategies are in red.

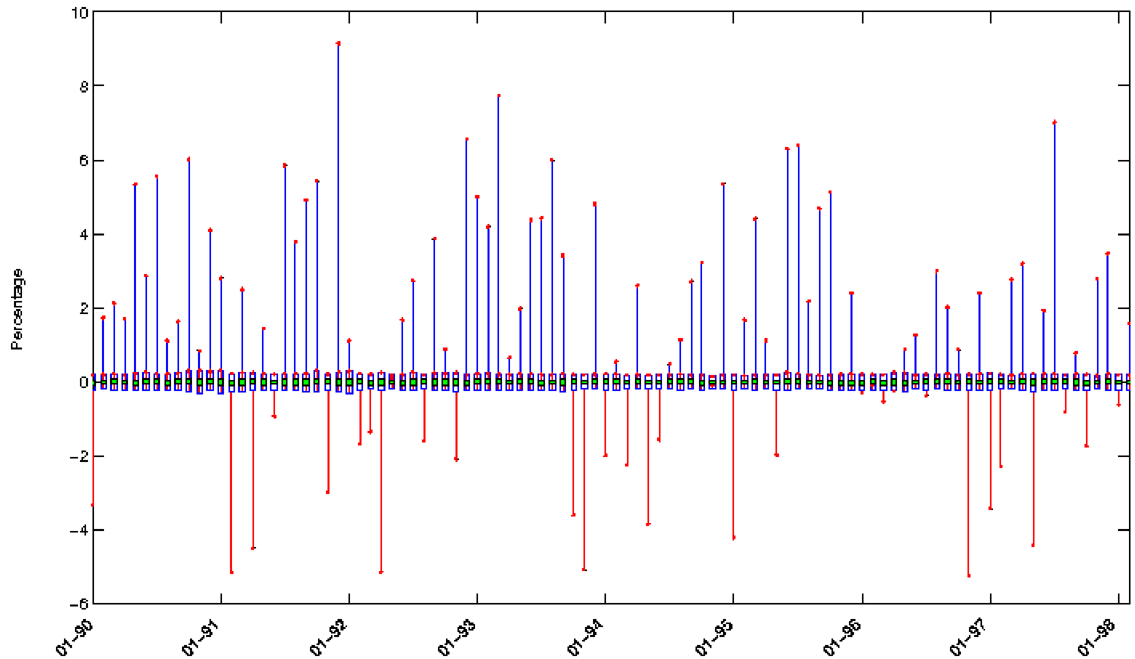

Figure 7: **The centered monthly return (in %) distributions of the momentum profits: JT(2001): 1990-1998.** The distributions are centered by the median profits of the naive investors strategies and displayed in box charts. The bottom box shows the 20th to the 40th percentiles of the profits of the naive investors strategies; the green box in the middle displays the 40th to the 60th percentiles; and the top box displays the 60th to the 80th percentiles. The deviations of the momentum profits above the top 80th percentile of the profits of the naive investors strategies are in blue; and the deviations of the momentum profits below the bottom 20th percentile of the profits of the naive investors strategies are in red.

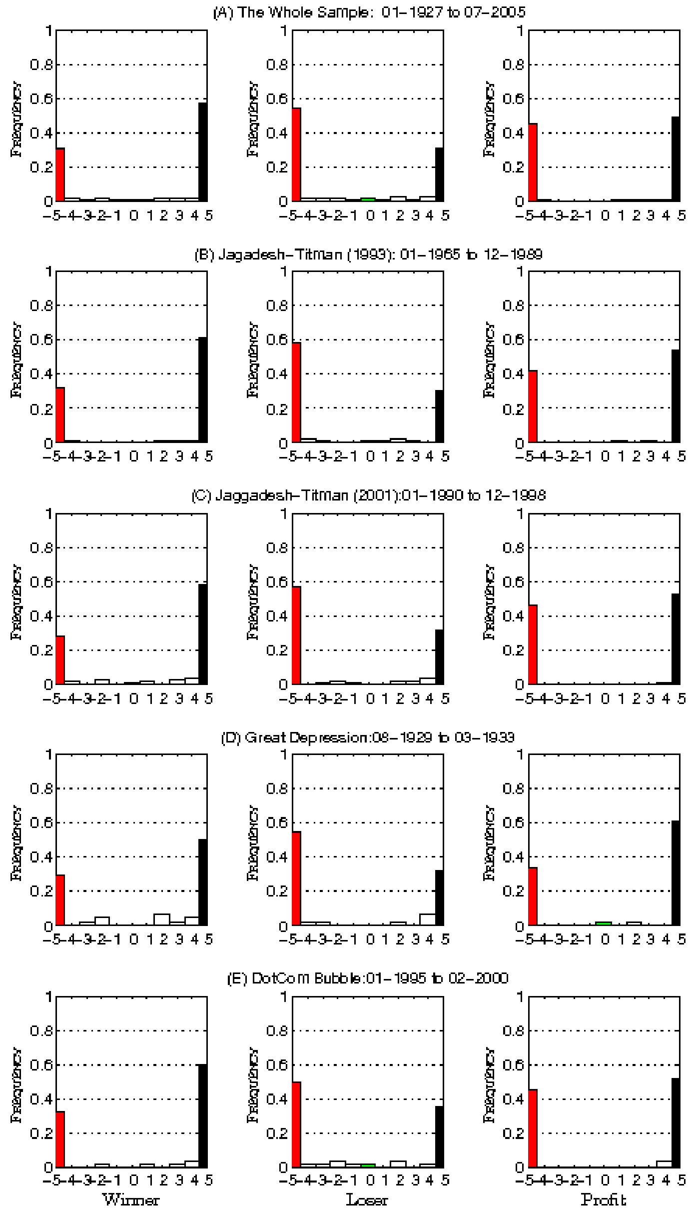

Figure 8: **The distributions of 11 scores of the winner (1st column), the loser (2nd column) portfolios and the momentum profits (3rd column).**
Panels A, B, C, D and E display the results for the whole sample period, the sub-sample periods of the J-T (1993), J-T (2001), the Great Depression and the DotCom bubble respectively.