HW TIP!
https://www.monash.edu/rlo/assignment-samples/science/science-writing-a-lab-report

Szydagis

VON NEUMANN'S MONTE CARLO INTEGRATION AND RELATED

HW3 Help: Rand 3D Angles How-To

I try, but can't always predict the HW questions, so that's what Tuesday morns are for

- https://mathworld.wolfram.com/SpherePointPicking.html
- Assuming phi means polar, theta azimuthal
  - X = r cos(phi) sin(theta) where phi [0,2pi) random
  - Y = r sin(phi) sin(theta) Naming conventions change!
  - Z = r cos(theta) where theta between [0,pi] radians
- BUT, cosine of theta not theta randomized
  - Why? Has to do with uniform solid angle arguments
  - Use sin^2 + cos^2 = 1 in order to derive sine from cosine
- Elastic collision? (Fixed energy & angle) What to do: https://en.wikipedia.org/wiki/Elastic_collision again is the right answer; you can ignore (complex) and do random angle, half-*E*
- MFP: diff types of exp. Mine v. 1/MFP (C++ default)
- While loop: do NOT stop at 0, but *>* it! You can use – instead of /

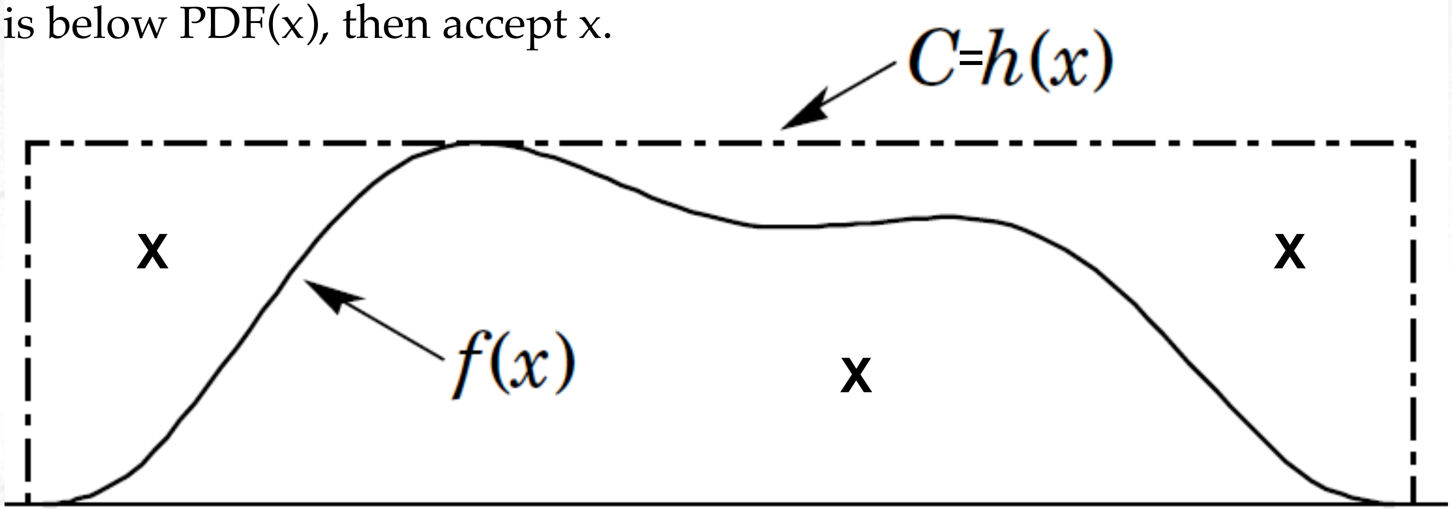
Time permitting: the C glossary doc.

Accept-Reject method

(Von Neumann method)

If the PDF* we wish to sample from is bounded both in x and y, then we can use the "Accept-Reject" method to select random numbers from it, as follows:

- Pick x and y uniformly without in the range of the PDF.
- If y is below PDF(x), then accept x.



The main advantage of this method is its **simplicity**, and given modern computers, one does not care much about efficiency. However, it requires **boundaries**!

* PDF = Point Distribution Function

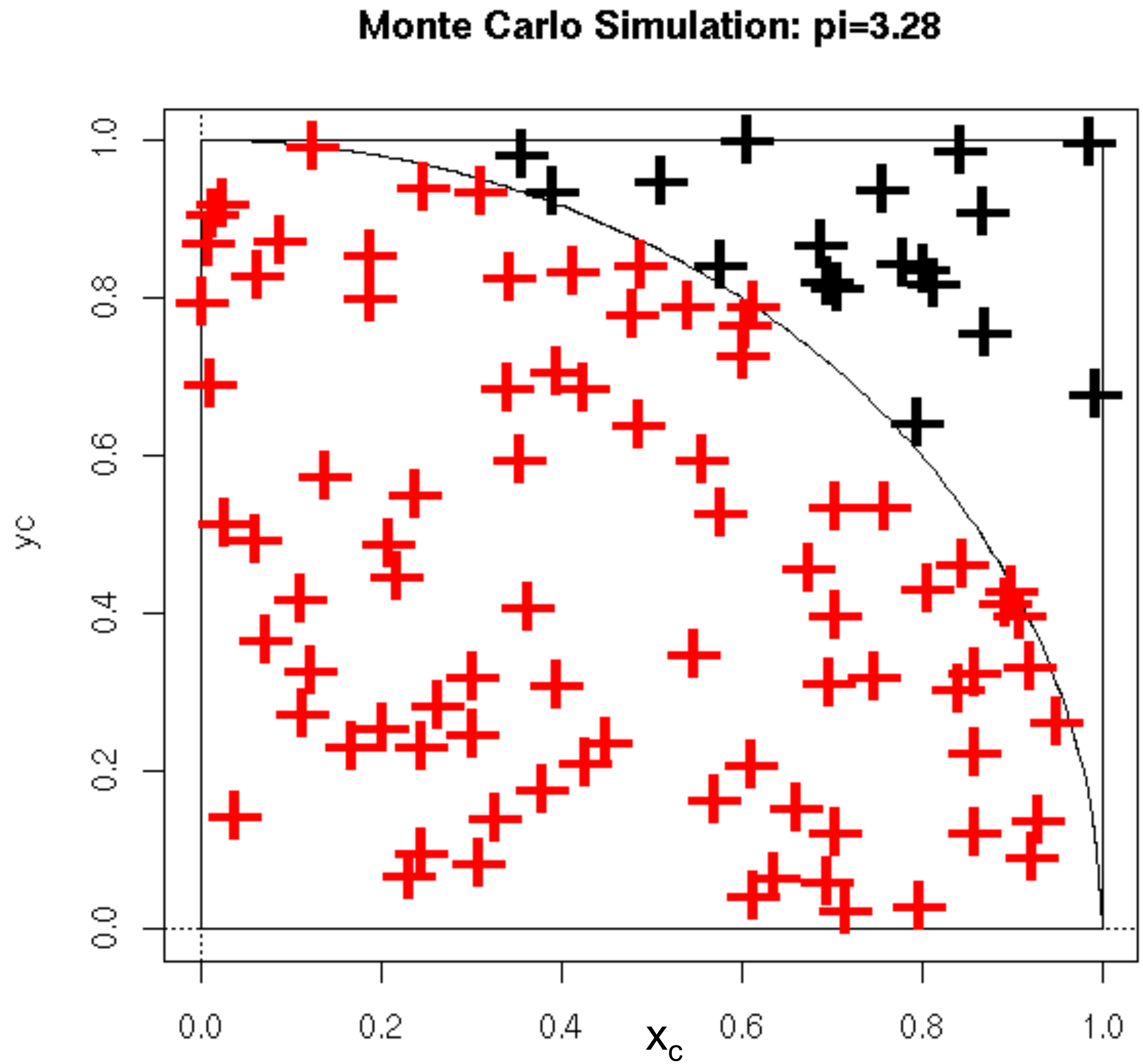
https://www.nbi.dk/~petersen/Teaching/Stat2015/Week3/AS2015_1130_MonteCarlo.pdf

Calculating Pi

Why quarter unit circle? Because then 1D *function* (monotonic too)

This is called the Hit-and-Miss method (Von Neumann method):

- Find min and max in both x and y.
- Generate uniform random numbers (x,y) in these ranges.
- Accept x, if y < f(x).
- Reject x, if y > f(x). >= ?



It is remarkable accuracy (though not precision) for such a relatively small number of darts throws! (at right)

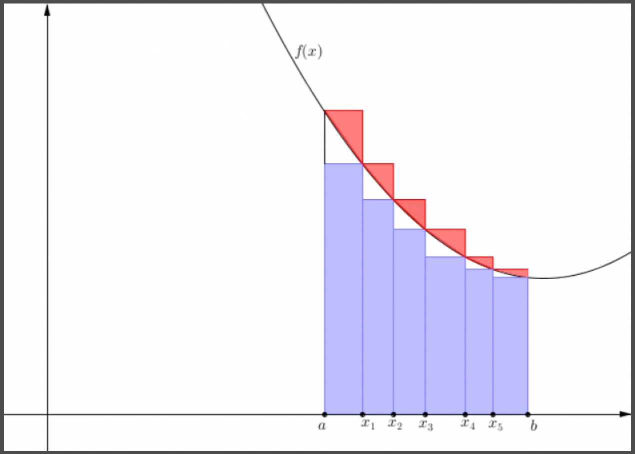
Goes as 1/sqrt(N) and thus wins over all other methods (*e.g.* trapezoidal integration) eventually

Higher Dimensions

- Your homework #4 will be for 1-D integration (single-valued function of y versus x)
- A circle pi example (completing circle from last slide all the way around) is more 2-D
- We can extend this to 3-D: vol of (unit+) sphere
  - Bounding cube (or rectangular prism more generally) instead of a bounding box
  - Can also be used to find pi, but only if you know spherical volume formula first ☺ And, this is slower
  - Download “sphere.cpp” sample code from site
- Very powerful: method will work for any shape
  - As #trials goes to inf, though may converge slowly

Contrast: Blast From the Past

- Good old-fashioned Riemann integration
  - Left-hand "rule"
  - Right-hand
  - Midpoint
  - Trapezoid
- Simpson's rule
  - 3/8-version, higher-order polynomials



DOWNLOAD "integrator.cpp" from the course website

- Adaptive step size
- Taylor, Maclaurin, Laurent, or similar series
  - Expand first, then integrate that (easier?)
- What does Mathematica do? Richardson extrapolation compared to Simpsons-like

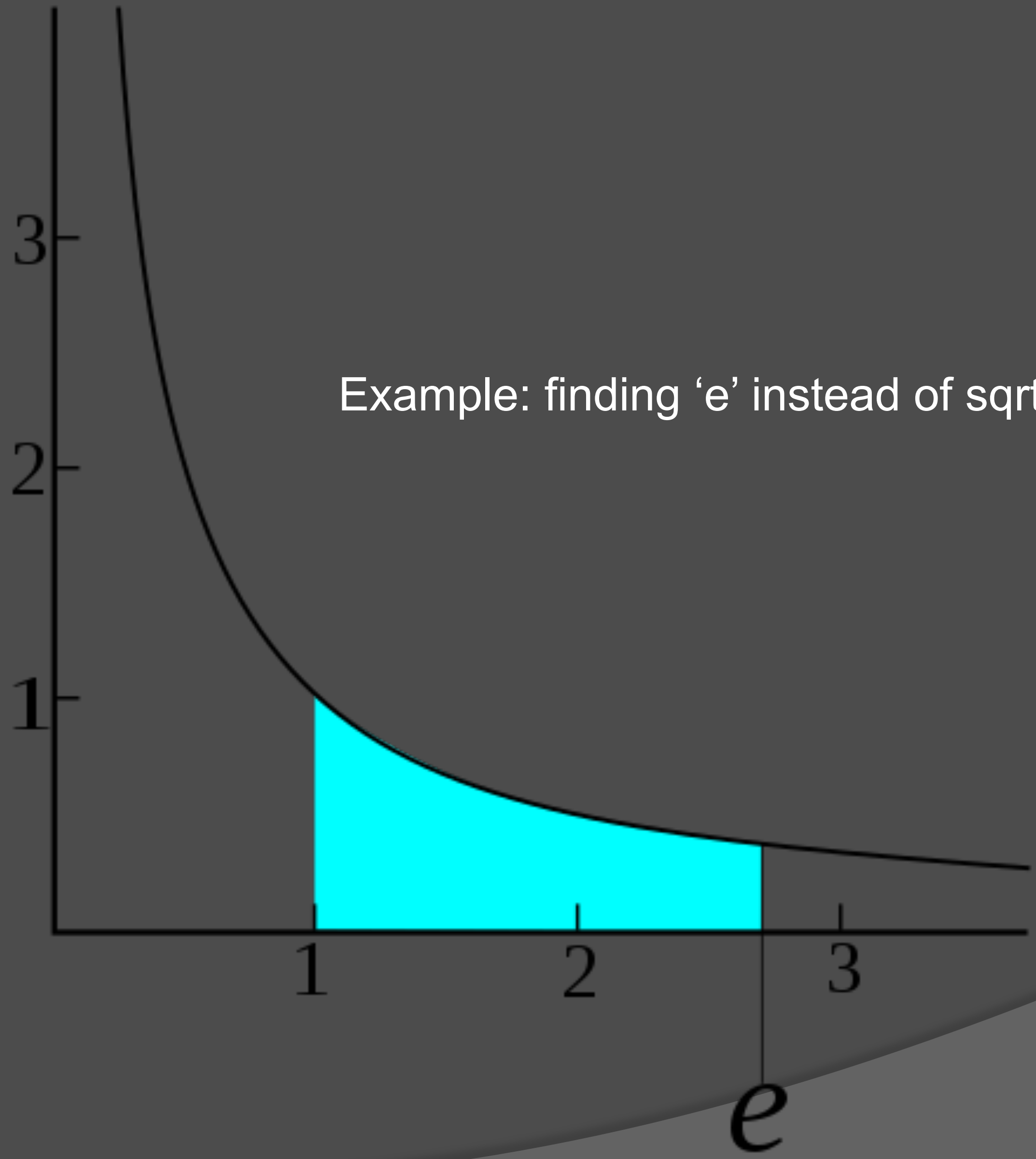
Integrator dot CPP

- We will pick integrands that canNOT be integrated analytically
  - And pick some reasonable finite bounds ‘a,’ ‘b’ (no singularities YET)
- We will compare these different methods and pick the fastest + most accurate
  - N=? to match online answer(s)?
  - Richardson: I accidentally re-derived myself!

- If difference between convergent numerical result from the main part of our code and the right answer (or a pi or ‘e’ trick) differs by less than 10^-7, then we can safely call that the analytical answer (???)

HW #4: Integration. Due Feb. 22

- Integrate the area (should come out to exactly root 2) under the curve 0.5/x^1.5 from 1-2 to find a precise and accurate value as fast as possible
  - The analytical answer is 1-1/sqrt(2), so manipulate it
- Use Monte Carlo integration first, but then...
- Check your answer by taking the analytic function
  - And integrating that with one (your choice) of the more “regular” methods (e.g. a Riemann sum or Simpson’s)
- 2 PLOTS: A v. N or step size. Show convergence
  - Answer the question: which method converges faster?
  - So, final answer is 1 number this time, and only 2 plots
- Bonus: Come up w/ a better bounding box shape



Example: finding ‘e’ instead of sqrt(2)

Backup Slides +Helpful Links

- Can cross-check numerical answers with
  - http://www.wolframalpha.com/widget/widgetPopup.jsp?p=v&id=29c546473e1c796d6076bb18901b15e7&i0=4133000%20*%20n%5E-0.491&i1=1&i2=3000000&podSelect=&showAssumptions=1&showWarnings=1
- Cross-check the full, analytical answers with Mathematica, or for free with
  - http://www.wolframalpha.com/widgets/view.jsp?id=a787670f0f1047d7fbe288763c55ba14
  - https://www.wolframalpha.com/calculators/integral-calculator/ (Google: does not have to be Wolfram)

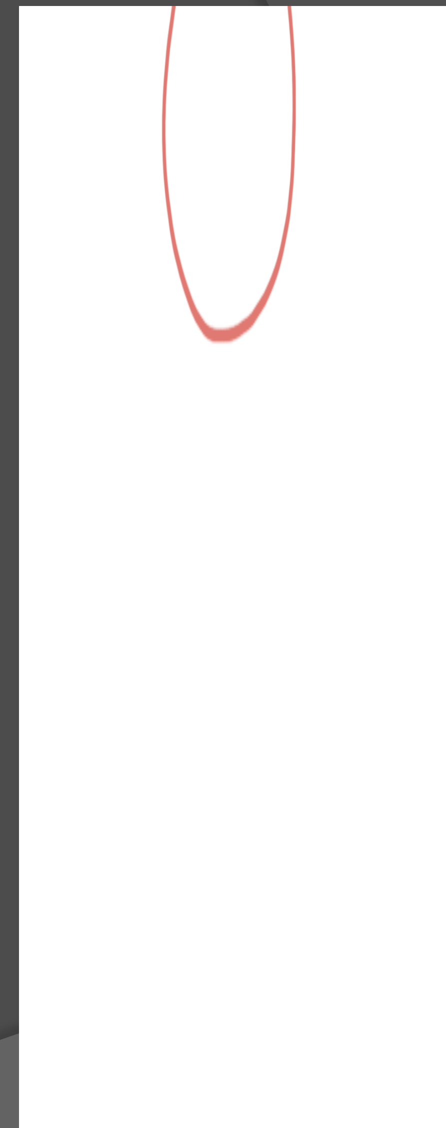
Can X2-check everything analytically (and/or with ~Mathematica online) but not main point.

Improper Integrals: Infinite Series

- Can do infinite sequences or sums, as effectively on your own as with com software
  - And with complete control and freedom to customize
- Convergence and divergence (not rigorous)
- summer.cpp (get it? Ba dum ching)
  - We will explore several different interesting infinite series to find their sums
  - Code just uses plain addition (no fancy tricks!!!)
  - Exploration: small tweaks will be able to change convergence rate, or even cause divergence!
- Let’s apply to physics now, beyond “just math”

The Importance of Infinite Series

- Occur all of the time in physics, including in quantum mechanics, especially in QFT; string theory!
- Though examples still exist in stat mech/therm & classic mechanics
  - Consider bouncing ball problem: infinite on paper! Damped pendulum, coin spin.
  - Total gravitational force on an object in universe from all others? BHs in-spiral?
- Astronomy and astrophysics
  - Approaching edge of a finite universe in a non-trivial cosmological topology
  - Falling into a black hole from outside
- *Numerical recipes for integration!*

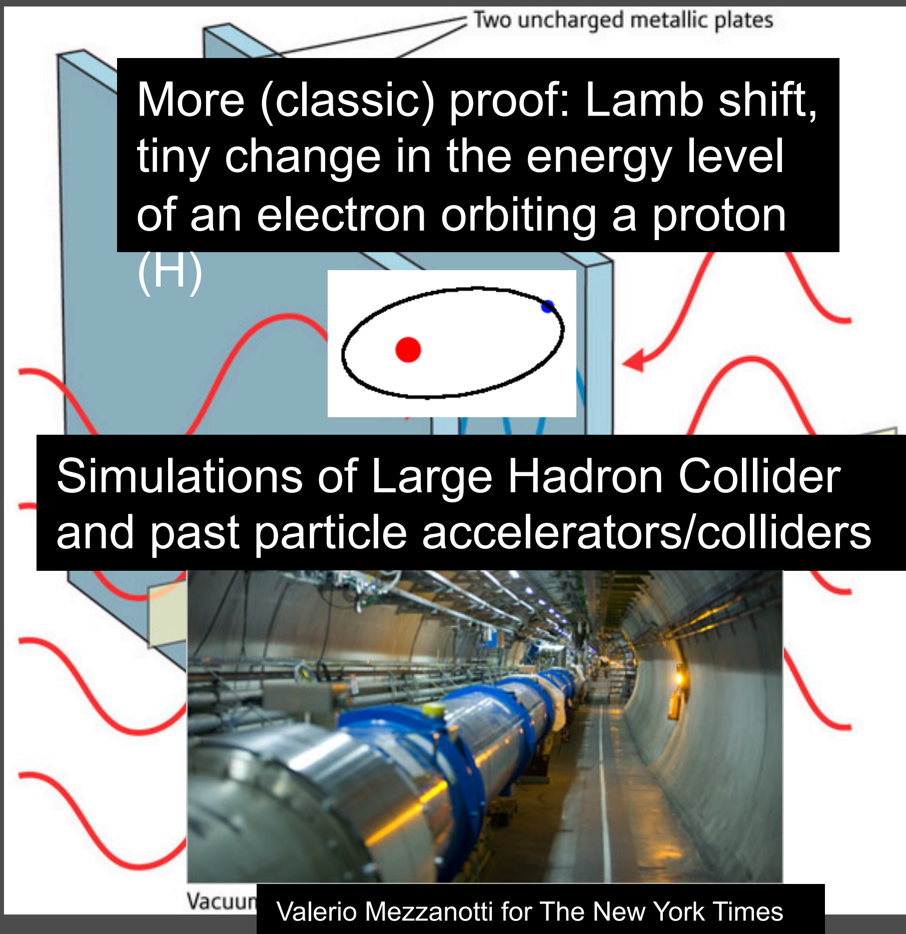


distance bouncing ball goes = ?

Let's Have Some Fun With This

- Let's explore (seeing what converges fastest or not)
  - 1/n^p, where p is a positive power that is not necessarily integer. Look for pattern in convergence; n all odd, all even
  - 1/n^n, 1/n!, 1/log(n), 1/a^n a>1, 1/exp(n), 1/sin(n). Which converge and which don't? (Same question for above)
  - Same as above but with (-1)^n in numerator instead of 1.
    - With (-1)^n is, naturally, known as an alternating series
  - Fibonacci sequence: derive the golden ratio (of ~1.6)
- Look for integer multiples or fractions of pi or *e*…
  - Can potentially beat Mathematica in finding an analytical solution to a particularly trying (incorrigible) series that breaks our sharp, spear-tip pointed analyses repeatedly, in a non-rigorous (guess-and-check/empirical) way
  - Though one that can still lead to some deep physics at some point, potentially. Integer multiples, fractions of π, *e* (e.g., $\pi^4/512$). Can use loops to automatically scan...

A Different Kind of Force: Casimir



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The Casimir Effect

- Can borrow a little energy from vacuum, and it doesn't even have to be a temporary loan (think short distance scales)
  - "Virtual" particles have real effects (can carry a force)
- Plates initially uncharged develop calculable electrical potential difference & then attract each other (diff electric charges, – and + or +/-)
  - Very small effect, difficult to measure, but we've done it

1 + 2 + 3 + 4 + … = – 1 / 12 ?!

- Riemann zeta function (but this is only *one* of a great many but, remarkably, *consistent* ways of “proving” this “fact” to be true
- Analytical continuation (other examples include factorial and gamma function, as well as scale factor vs. time in an empty anti-de Sitter space in cosmology)
- Casimir force: nature *knows* about the infinite energy and SUBTRACTS it out

Proofs

Via analytic continuation, one can show that

$$\zeta(-1) = -\frac{1}{12}$$

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n},$$

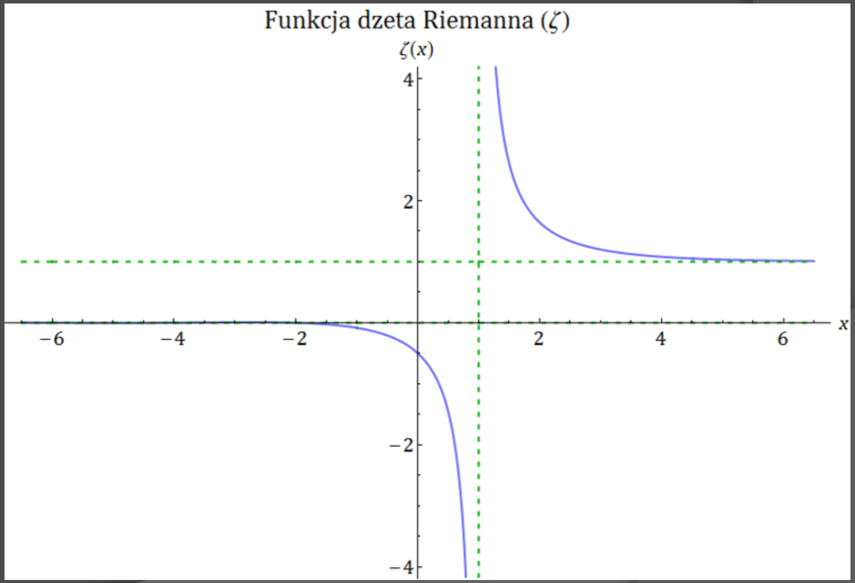
$$\zeta(0) = -\frac{1}{2};$$

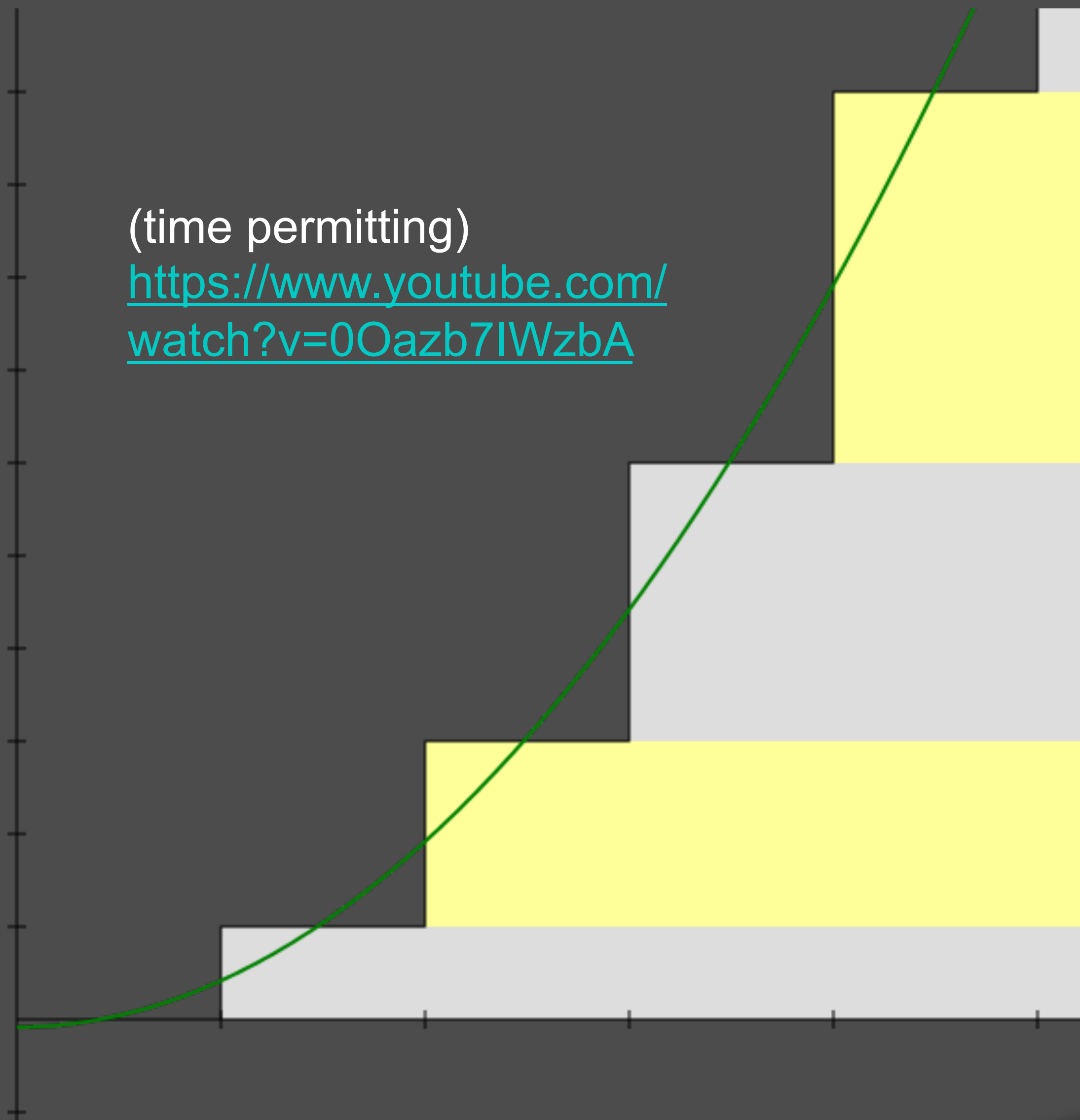
$$\zeta(1/2) \approx -1.4603545$$

- $\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$

The first few values are:

- $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (the Basel problem)
- $\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$
- $\zeta(6) = \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}$





(time permitting)
https://www.youtube.com/watch?v=0Oazb7IWzbA

$$\begin{aligned} c &= 1+2+3+4+5+6+\cdots \\ 4c &= \quad\;\; 4 \quad\;\; +8 \quad\;\; +12+\cdots \\ -3c &= 1-2+3-4+5-6+\cdots \end{aligned}$$

$$-3c = 1-2+3-4+\cdots = \frac{1}{(1+1)^2} = \frac{1}{4}$$

Dividing both sides by -3, one gets $c = -1/12$.

Another way of finding the constant is as follows — 41
Let us take the series $1+2+3+4+5+$&c. Let C be its con-stant. Then $c = 1+2+3+4+$&c
$\therefore 4c = \quad 4 \quad +8 \quad +$&c
$\therefore -3c = 1-2+3-4+$&c $= \frac{1}{(1+1)^2} = \frac{1}{4}$
$\therefore c = -\frac{1}{12}$.