

Transforms of PDFs

Here we consider what happens to a probability density function as we perform a change of variables : $f: x \rightarrow y$

Recall that a pdf is a density function, so the probability that a parameter has a value between x and $x + \delta x$ is

$$p(x|I)dx$$

This should equal the probability that its transformed value is between $f(x)$ and $f(x + \delta x)$.

$$p(x|I)dx = p(f(x)|I)df(x) = p(y|I)dy$$

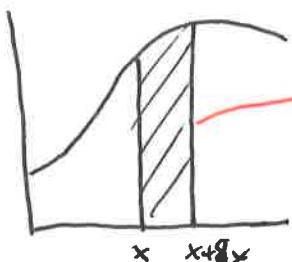
More precisely

$$p(x|I) = p(y|I) \left| \frac{dy}{dx} \right|$$

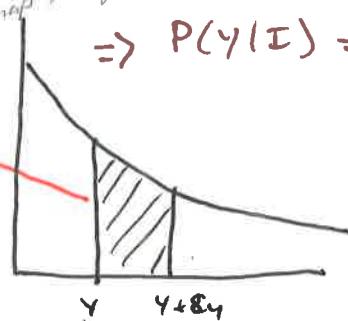
Jacobian of
the transformation

To find it
represents a ratio of
lengths even when positive
increments in x map to negative ones for y

$$\Rightarrow p(y|I) = p(x|I) \left| \frac{dy}{dx} \right|^{-1}$$



AREAS
EQUAL



y $y + \delta y$

Example of Change of Variable

$$\text{Let } p(x|I) = \begin{cases} C & \text{if } x_0 \leq x \leq x_1 \\ 0 & \text{otherwise} \end{cases}$$

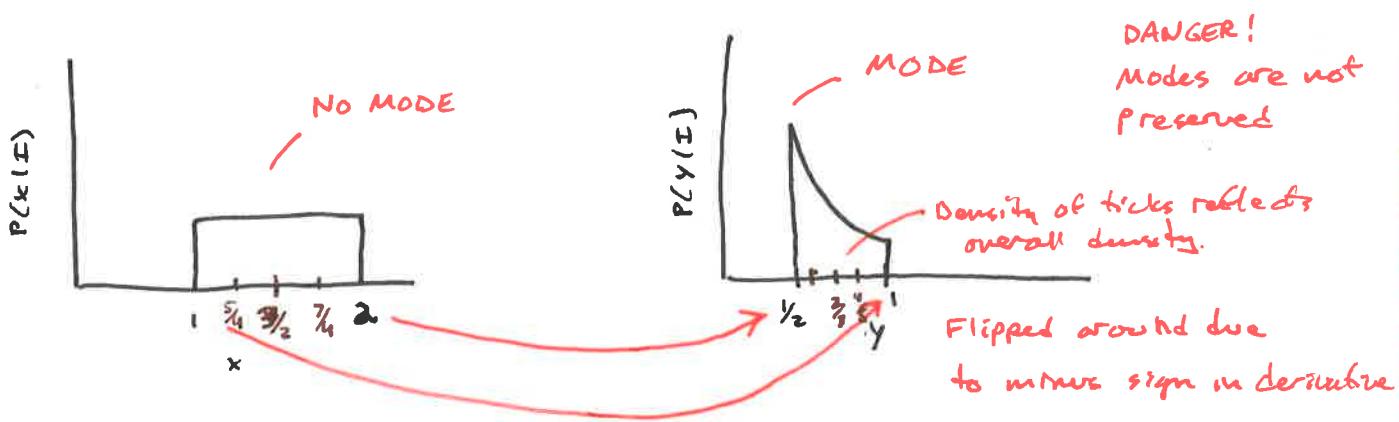
$$\text{Let } f(x) = \frac{1}{x}$$

$$p(y|I) = p(x|I) \left| \frac{dx}{dy} \right|^{-1}$$

$$= C | -x^{-2} |^{-1}$$

$$= C x^2$$

$$= \frac{C}{y^2}$$



Most probable value of x DOES NOT map to the most probable value of y .

ONE OF THE MOST COMMON MISTAKES

Sampling from a Gaussian

Box-Muller Method

$$p(y) dy = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Consider x_1 and x_2 with $p(x_1) = p(x_2) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

and the transformation

$$y_1 = \sqrt{-2 \ln x_1} \cos 2\pi x_2$$

$$y_2 = \sqrt{-2 \ln x_1} \sin 2\pi x_2$$

$$\Rightarrow x_1 = \exp \left[-\frac{1}{2} (y_1^2 + y_2^2) \right]$$

$$x_2 = \frac{1}{2\pi} \tan^{-1} \frac{y_2}{y_1}$$

$$\text{Now } p(y_1, y_2) dy_1 dy_2 = p(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| dy_1 dy_2$$

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = - \left[\frac{1}{\sqrt{2\pi}} e^{-y_1^2/2} \right] \left[\frac{1}{\sqrt{2\pi}} e^{-y_2^2/2} \right]$$

Each y is independently distributed according to a Gaussian!

Transformation Method

Consider x as having a pdf that is uniform over the unit interval.

$$\Rightarrow p(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow p(x) dx = \begin{cases} dx & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall that

$$|p(y) dy| = |p(x) dx|$$

$$\Rightarrow p(y) = p(x) \left| \frac{dx}{dy} \right|$$

$$\text{Consider } y = -\ln x \Rightarrow x = e^{-y}$$

$$\frac{dx}{dy} = -e^{-y} \Rightarrow \left| \frac{dx}{dy} \right| = e^{-y}$$

$$\Rightarrow p(y) = 1 \cdot e^{-y} = e^{-y}$$

$$\Rightarrow p(y) dy = e^{-y} dy$$

Take $x = \text{rand}();$
 $y = -\log(x);$

y is now
exponentially
distributed!

Transformation Method cont.

So how do we choose the transformation to get the desired pdf?

$$p(y) = p(x) \left| \frac{dx}{dy} \right|$$

our desired pdf

Clearly we need $\frac{dx}{dy} = f(y)$

$$\Rightarrow dx = f(y) dy$$

$$x = \int f(y) dy \equiv F(y)$$

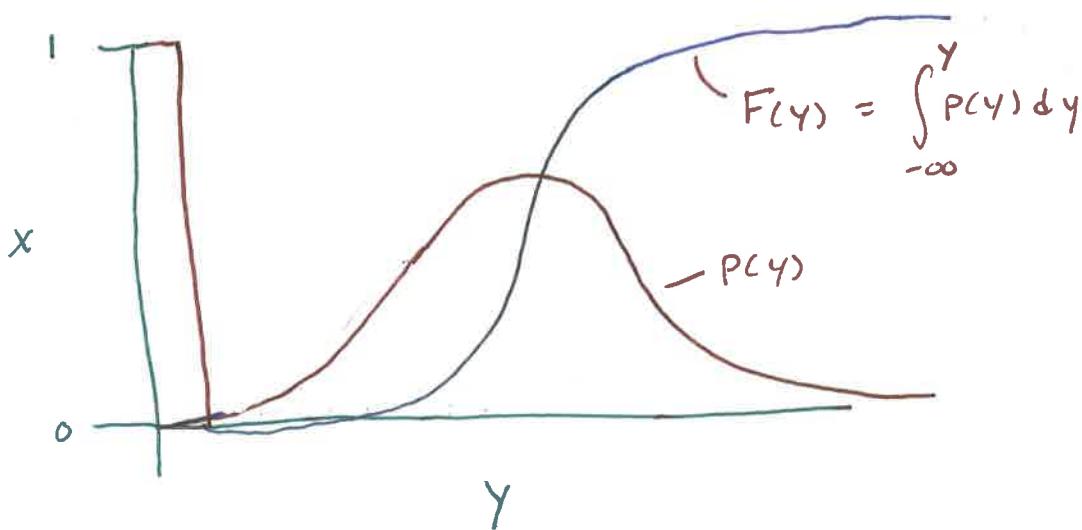
where $F(y)$ is
the indefinite
integral of $f(y)$

How to we obtain y ?

$$y = F^{-1}(x)$$

Two potential difficulties: Is $f(y)$ integrable?
Can $F(y)$ be inverted?

GRAPHICALLY, here is the soln



Sampling from a Gaussian cont.

To obtain a ^{two} Gaussian distributed parameter:

- sample x_1 and x_2 from the unit interval uniformly.
- $y_1 = \sqrt{-2 \ln x_1} \cos 2\pi x_2$
- $y_2 = \sqrt{-2 \ln x_1} \sin 2\pi x_2$

Since trigonometric functions take some computing effort, this is faster...

- sample x_1 and x_2 from the unit square
- ↳ - if they fall outside the unit circle, try again
- $R^2 = V_1^2 + V_2^2$ is uniformly distributed ^{on $[0,1]$} and can be used for x_1 (Note R is not unif. dist.)
- $\theta = \tan^{-1} \frac{V_2}{V_1}$ is uniformly distributed on $[0, 2\pi]$

~~so $\frac{1}{2\pi} \tan^{-1} \frac{V_2}{V_1}$ is uniformly dist. on $[0, 1]$~~

and can be used as $2\pi x_2$

$$-$$
 Now $\sin 2\pi x_1 = \frac{V_2}{\sqrt{R^2}}$

$$\cos 2\pi x_1 = \frac{V_1}{\sqrt{R^2}}$$

More Complex Transformations

See Sivia 3.6

Note that sometimes you want to transform from two variables to one variable.

The Jacobian is not sufficient, since you are performing a marginalization at some point to reduce the dimensionality of the space.

This can be handled with delta functions...

Say that $z = f(x, y)$

And given $p(x, y)$, you want $p(z)$

Then write

$$p(z) dz = \iint_{x,y} p(x, y, z) dx dy dz$$

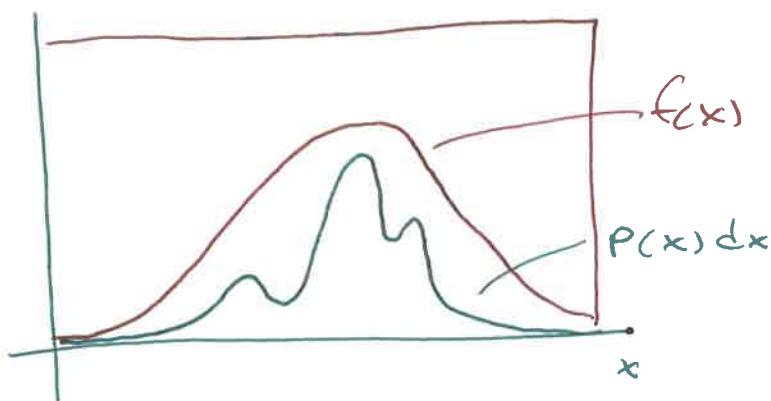
$$= \iint_{x,y} p(z|x, y) p(x, y) dx dy dz$$

But $p(z|x, y) = \delta(z - f(x, y))$

$$p(z) dz = \iint_{x,y} \delta(z - f(x, y)) dx dy dz$$

Rejection Method

Consider sampling from a complex $p(x) dx$



Imagine sampling from within the rectangular area uniformly ... EASY!

If the sample falls within the area under $p(x)$ then keep it, else try again.

Since the area under the density curve is proportional to the probability, the samples will be dist. according to $p(x)$.

This method suffers from a lot of rejects.
WASTEFUL COMPUTATIONS!

Instead choose a comparison function $f(x)$
st. $f(x) > p(x) dx$

To sample from $f(x)$ uniformly, choose a uniform sample from the range x and choose a uniform sample between $[0, f(x)]$ in y . You could also choose y to be $[0, 1]$ and compare to the ratio $p(x) / f(x)$.