Specialized Macdonald polynomials, quantum K-theory, and Kirillov-Reshetikhin modules

Cristian Lenart

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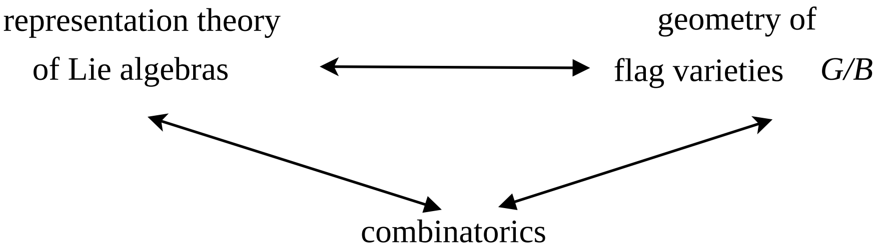
representation theory
of Lie algebras ⟷ geometry of
flag varieties G/B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- type C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}$;
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- exceptional types: E_6, E_7, E_8, F_4, G_2.

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Chevalley generators: $\mathfrak{g}_{\alpha_i} = \mathbb{C} E_i \, , \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C} F_i \, .$

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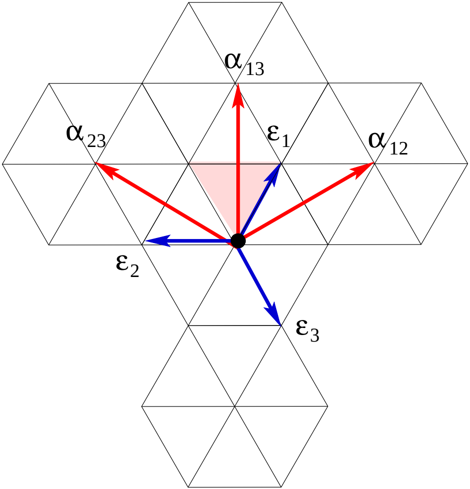
Example. Type A_2.



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Fundamental weights $\{\omega_1, \ldots, \omega_r\} \subset \mathfrak{h}^*_{\mathbb{R}}$ form the dual basis to the simple coroots:

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$$\Lambda = \mathrm{span}_{\mathbb{Z}}(\omega_1, \dots \omega_r) \subset \mathfrak{h}_{\mathbb{R}}^* \,, \quad \Lambda^+ = \mathrm{span}_{\mathbb{Z}_{\geq 0}}(\omega_1, \dots, \omega_r) \,.$$

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$\omega_i = \varepsilon_1 + \ldots + \varepsilon_i = (1, \ldots, 1) = (1^i)$.

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$W = S_n \,,\ \ s_{\alpha_{ij}}$ is the transposition t_{ij} , $s_i = t_{i,i+1}$.

Representations of semisimple Lie algebras

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$$\rho \,:\, \mathfrak{g} \to End(V)\,,$$

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Fact. For each $\lambda \in \Lambda^+$, there is a (finite-dimensional) irreducible representation $V(\lambda)$ of highest weight λ.

Example for $\mathfrak{g} = \mathfrak{sl}_n$

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$$V(\omega_k) = \Lambda^k(V) = \text{span}\left(e_{i_1} \wedge \ldots \wedge e_{i_k} \ : \ i_1 < i_2 < \ldots < i_k\right).$$

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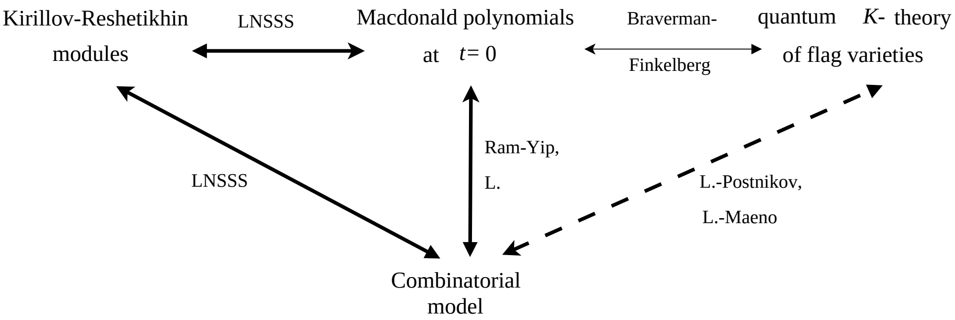
Basis indexed by strictly increasing fillings of $\omega_k = (1^k)$ – Young tableau:

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \cdots \\ \hline \cdots \\ \hline i_k \\ \hline \end{array}\,.$$

Lie algebras, flag varieties, and combinatorics



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Deep connections with:

- affine Lie algebras
- double affine Hecke algebras
- Hilbert schemes
- quantum integrable systems
- conformal field theory
- etc.

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Fact. The highest weight modules $V(\lambda)$ are infinite-dimensional, but can be expressed as infinite tensor products of Kirillov-Reshetikhin modules.

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Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono, 2014)

For all untwisted affine root systems $A_{n-1}^{(1)} - G_2^{(1)}$, *we have*

$$P_\lambda(x;q,0) = X_\lambda(x;q)\,.$$

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Schubert variety $X_w = \overline{BwB/B}$, for $w \in W$.

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A k-point GW invariant (of degree d) counts curves of degree d passing through k given Schubert varieties.

Quantum K-theory

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Theorem (Braverman-Finkelberg, 2011)

In simply-laced types, the Macdonald polynomial specialization $P_\lambda(x; q, 0)$ coincides (up to a scalar factor) with the K-theoretic J-function.

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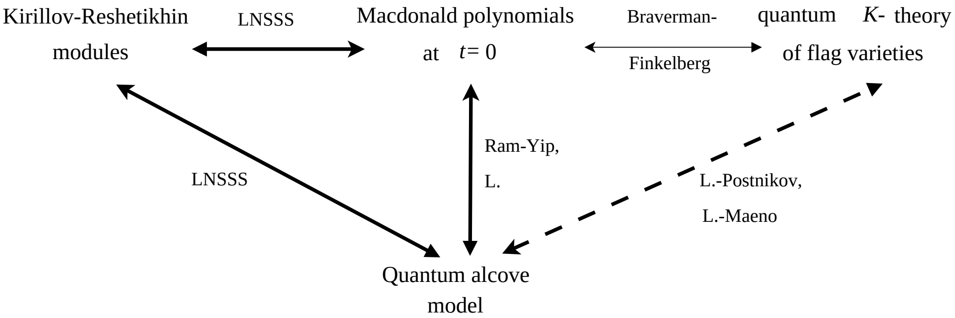
- the quantum K-theory of G/B (conjecture by L.-Postnikov; evidence by L.-Maeno);
- the tensor products of one-column KR modules (LNSSS).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$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of Bruhat order)}\,, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1\,.$$

(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.)

The quantum alcove model

Is uniform for all Lie types, and only depends on the finite root system, of type $A_{n-1} - G_2$.

Main ingredient: the quantum Bruhat graph on W, denoted QBG(W).

This is the directed graph with labeled edges

$$w \stackrel{\alpha}{\longrightarrow} ws_\alpha \,, \qquad \text{where}$$

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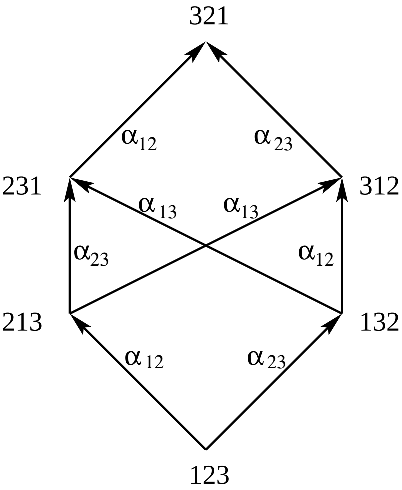
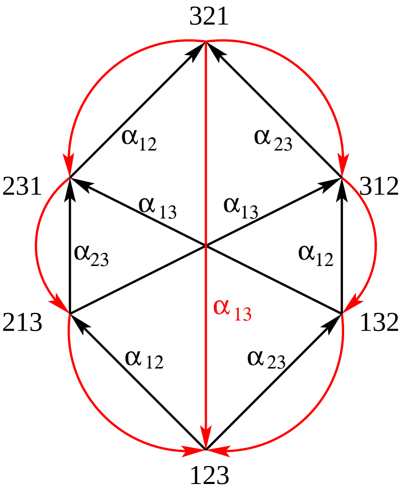
(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.) It originates in the quantum cohomology of flag varieties [Fulton and Woodward].

Hasse diagram of the Bruhat order for S_3:



Quantum Bruhat graph for S_3:

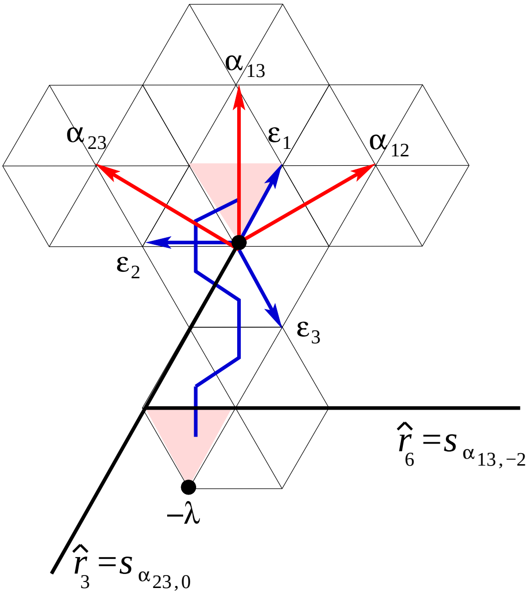


The quantum alcove model

Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \ldots, \beta_m)\,.$$

Example. Type A_2, $\lambda = (3,1,0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\ (1,2),\ (1,3),\ (2,3),\ (1,3),\ (1,2),\ (1,3)\ )$.



The quantum alcove model (cont.)

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For $w \in W$ and J, construct the chain of elements in W:

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$$\mathcal{A}(\Gamma, w) := \{J \;:\; \text{ corresp. chain is a path in QBG}(W)\}\,.$$

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Let $\mathcal{A}(\Gamma) := \mathcal{A}(\Gamma, 1_W)$.

Macdonald polynomials: the Ram-Yip formula

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$$P_\lambda(X; q, 0) = \sum_{J \in \mathcal{A}(\Gamma)} q^{\mathrm{height}(J)} \, x^{\mathrm{weight}(J)} \, .$$

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Remark. For $q = 0$, we recover the LS-gallery/alcove models [Gaussent-Littelmann, L.-Postnikov], which are discrete versions of the Littelmann path model for $\mathrm{ch}(V(\lambda))$.

$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{X_w}] = [\mathcal{O}_w]$, $w \in W$.

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$$[\mathcal{O}_w] * [\mathcal{O}_{s_k}] \stackrel{?}{=} \sum_{J \in \mathcal{A}(\Gamma_{\mathrm{rev}}, w) \setminus \{\emptyset\}} (-1)^{|J|-1} \, q_1^* \ldots q_r^* \, [\mathcal{O}_{\mathrm{end}(w,J)}] \, .$$

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Remark. The above conjecture specializes to:

- the Chevalley formula in $QH^*(G/B)$ [Fulton-Woodward];
- the Chevalley formula for $K(G/B)$ in [L.-Postnikov], which was also generalized to Kac-Moody (infinite) flag varieties [L.-Shimozono].

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- They multiply as in the conjectured Chevalley formula.

Kirillov-Reshetikhin (KR) modules

Recall the KR modules for $\widehat{\mathfrak{g}}$: $W^{r,s}$ and

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The structure of these representations is partially encoded by the corresponding KR crystals.

Kashiwara's crystals

Colored directed graphs encoding certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$ ($\mathfrak{g}$ complex semisimple or affine Lie algebra).

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Fact. V has a crystal basis $B \implies$ in the limit $q \to 0$ we have

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Crystal graph: directed graph on B with arrows colored $i \leftrightarrow \alpha_i$.

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Fact. $B^{\otimes \mathbf{p}}$ is connected (with the 0-arrows).

Models for KR crystals: type $A_{n-1}^{(1)}$ ($\widehat{\mathfrak{sl}}_n$)

Fact. We have as classical crystals (without the 0-arrows):

$$B^{p,1} \simeq B(\omega_p)\,, \qquad \text{where } \omega_p = (1,\ldots,1,0,\ldots,0) = (1^p)\,.$$

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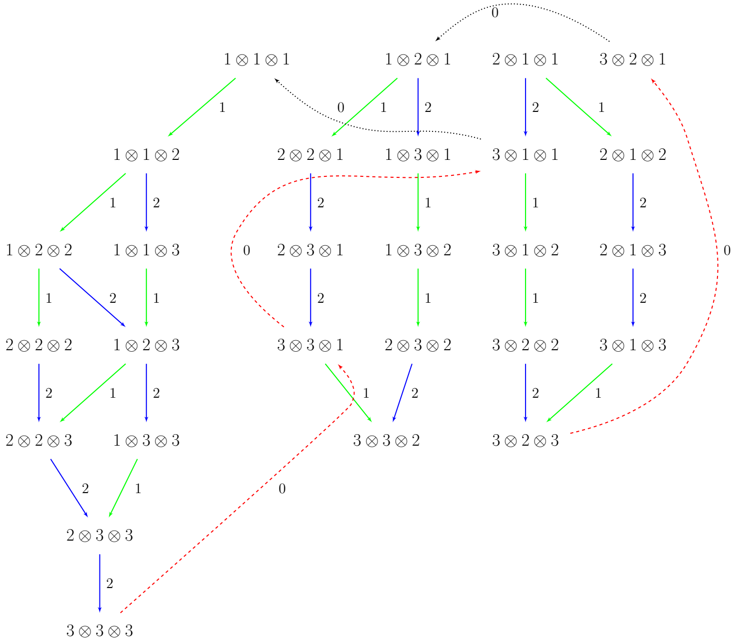
The action of the crystal operators:

$$1 \xrightarrow{\widetilde{f}_1} 2 \xrightarrow{\widetilde{f}_2} \ldots n-1 \xrightarrow{\widetilde{f}_{n-1}} n \xrightarrow{\widetilde{f}_0} 1\,.$$

Example. Type $A_2^{(1)}$, $B^{\otimes(1,1,1)} \simeq B(1) \otimes B(1) \otimes B(1)$.



Models for KR crystals: types $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$

Fact. There are more involved type-specific models for $B^{p,1}$ (based on Kashiwara–Nakashima columns).

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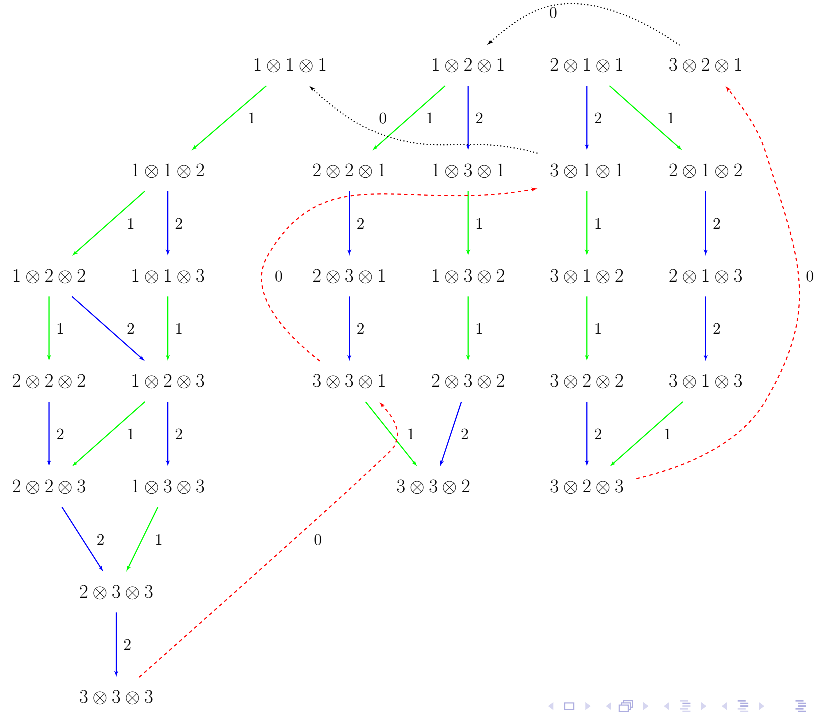
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Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

The (combinatorial) crystal $\mathcal{A}(\Gamma)$ *is isomorphic to the tensor product of KR crystals* $B^{\otimes \mathbf{p}}$.

Applications

In terms of the quantum alcove model, we can do the following computations for $B^{\otimes \mathbf{p}}$, uniformly for all affine types $A_{n-1}^{(1)} - G_2^{(1)}$:

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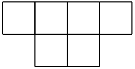
Remark. (2) and (3) were expressed in type A in terms of fillings, but little was known beyond type A.

Example in type A_2

$\mathbf{p} = (1,2,2,1) =$  $; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0).$

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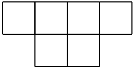
$$\mathbf{p} = (1,2,2,1) =$$  $; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0).$

A λ-chain as a concatenation of ω_1-, ω_2-, ω_2-, and ω_1-chains:

$$\Gamma = (\ \ (1,2),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (1,2),\ \ (1,3)\ \ ).$$

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\left(\ \underline{(1,2)},\ \ \underline{(1,3)}\ \mid\ \underline{(2,3)},\ \ (1,3)\ \mid\ (2,3),\ \ \underline{(1,3)}\ \mid\ \underline{(1,2)},\ \ \underline{(1,3)}\ \right).$$

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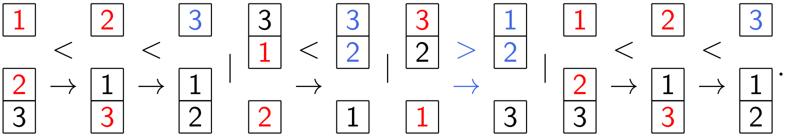
Claim: J is admissible (belongs to $\mathcal{A}(\Gamma)$). Indeed, the corresponding path in the quantum Bruhat graph is



Example. Let $J = \{1,2,3,6,7,8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)}\ \middle|\ \underline{(2,3)},\ (1,3)\ \middle|\ (2,3),\ \underline{(1,3)}\ \middle|\ \underline{(1,2)},\ \underline{(1,3)}\ \right).$$

Claim: J is admissible (belongs to $\mathcal{A}(\Gamma)$). Indeed, the corresponding path in the quantum Bruhat graph is

$$\begin{array}{c}\boxed{1}\\ \\ \boxed{\begin{array}{c}2\\3\end{array}}\end{array} \underset{\to}{<} \begin{array}{c}\boxed{2}\\ \\ \boxed{\begin{array}{c}1\\3\end{array}}\end{array} \underset{\to}{<} \begin{array}{c}\boxed{3}\\ \\ \boxed{\begin{array}{c}1\\2\end{array}}\end{array} \ \Big| \ \begin{array}{c}\boxed{\begin{array}{c}3\\1\end{array}}\\ \boxed{2}\end{array} \underset{\to}{<} \begin{array}{c}\boxed{\begin{array}{c}3\\2\end{array}}\\ \boxed{1}\end{array} \ \Big| \ \begin{array}{c}\boxed{\begin{array}{c}3\\2\end{array}}\\ \boxed{1}\end{array} \underset{\to}{>} \begin{array}{c}\boxed{\begin{array}{c}1\\2\end{array}}\\ \boxed{3}\end{array} \ \Big| \ \begin{array}{c}\boxed{1}\\ \\ \boxed{\begin{array}{c}2\\3\end{array}}\end{array} \underset{\to}{<} \begin{array}{c}\boxed{2}\\ \\ \boxed{\begin{array}{c}1\\3\end{array}}\end{array} \underset{\to}{<} \begin{array}{c}\boxed{3}\\ \\ \boxed{\begin{array}{c}1\\2\end{array}}\end{array}.$$

The corresponding element in $B^{\otimes \mathbf{p}} = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{1,1}$ represented via a column-strict filling:

$$\boxed{3} \otimes \boxed{\begin{array}{c}2\\3\end{array}} \otimes \boxed{\begin{array}{c}1\\2\end{array}} \otimes \boxed{3}.$$