ESSAYS ON MONEY, BANKING, AND FINANCE

by

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A Dissertation
Submitted to the University at Albany, State University of New York
in Partial Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy

College of Arts and Science
Department of Economics
2018
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To my parents: Ngo Quang Lan, Tran Tuyet Lan
And
My sister: Ngo Tran Thuy Tien
ABSTRACT

This doctoral dissertation contains three essays on the topics of money, banking, and finance.

In my first essay, I build a dynamic monetary model with only electronic money to set lights on the effects of unconventional monetary policy since the Great Recession. The model can replicate the sharp rise of reserves in the banking system and the complex inflation dynamics. After quantitative easing, keeping the interest on reserves at zero too long might create deflation in the long run. One good solution to get out of this “low rate-cum-deflation” trap is to “raise rate and money supply simultaneously”.

My second essay extends the first one by including currency into the model. This essay shows that a Taylor rule is not efficient during banking crises, in which banks cut loans. Even with negative interest on reserves or forward guidance, deflation is still likely to be persistent. If the central bank simultaneously targets both the interbank rate and the money supply by a Taylor rule and a Friedman’s k-percent rule, inflation and output are stabilized. When interest on reserves becomes the central bank’s main tool, the traditional view that the central bank cannot target both interest rate and money supply will no longer be true.

My third essay creates a link between financial innovations and income distributions. A general equilibrium model is developed to give a new insight that the introduction of new financial assets causes the income redistribution. Income is transferred from the manufacturing sector to the financial sector as well as from low wealth-holders to top wealth-holders.

Three essays both emphasize the importance of banking and financial sector to understand the modern economy.
ACKNOWLEDGMENT

I would like to express my deepest gratitude to my advisor Professor Adrian Masters for continuous support of my Ph.D study and research. Without his guidance I would have not been able to complete this dissertation.

I would also like to thank the rest of my dissertation committee: Professor Michael Sattinger and Professor Michael Jerison for their insightful advice and thoughtful criticism. I also thank to Prof. John Bailey Jones, Prof. Fang Yang, Prof. Betty Daniel, Prof. Ibrahim Gunay, Prof. Kwan Koo Yun for their helpful comments, time and attention during the semesters.

I also thank my friends Minhee Kim, Gusang Kang, Tu Nguyen, Huy Dang, Garima Siwach, Hai Minh, Huong Tran, Hai Duong to support me.

Finally, I love you, mom, dad, and Thuy Tien. I know you suffer a lot from talking with me, especially during my stressful times. I do not know how to express my love to you. I love you all.
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<th>Description</th>
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<td>GDP</td>
<td>Gross Domestic Product</td>
</tr>
<tr>
<td>IOR</td>
<td>Interest On Reserves</td>
</tr>
<tr>
<td>LSAP</td>
<td>Large Scale Asset Purchase</td>
</tr>
<tr>
<td>MZM</td>
<td>Money Zero Maturity</td>
</tr>
<tr>
<td>QE</td>
<td>Quantitative Easing</td>
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<td>ZMD</td>
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Chapter 1

Introduction

The financial system in general and banking in particular play a vital role to understand the modern economy. Any changes in banking and finance can create a huge impact on output, income distribution, and monetary policy. This dissertation focuses on two main changes of the financial system: (i) the recent popularity of the electronic payment system implies that most money (under the form of saving and checkable deposits) is created by banks, (ii) the recent financial innovations allow the better risk-sharing between different industries. We show that the first point has a huge implication for monetary policy while the second point is related to the income distribution process.

Chapter 2 “Monetary Policy with Electronic Money” introduces a monetary model with the electronic payment system. As deposits are IOUs issued by commercial banks, the model emphasizes the importance of banks in creating money. This model is very different from the common banking and monetary policy literature in two salient features. First, instead of describing banks as intermediaries, who take deposits from households and use that money to lend out, we focus on the function of creating money in banks. Banks, in our model, lend out by issuing their own deposits. As deposits are used for settling transactions in the private sector, households accept these deposits as money. Banks still compete in the deposit market for getting reserves - another type of e-money issued by the central bank and used for settling transactions between bankers. Second, we focus on the connection between the central bank’s balance sheet and commercial banks’ balance sheet in a microfounded way. That builds a foundation to understand the macro effect of monetary policy when banks play the central theme in the economy.
Chapter 2 also uses the above model to understand some aspects of monetary policy during and after the Great Recession. Our model can replicate some key features of the US economy during and after the Great Recession: (i) the debt deleveraging process, (ii) the huge amount of excess reserves in the banking system, (iii) the long duration of the federal funds rate at zero, (iv) the complex inflation path after quantitative easing. Quantitative easing and interest on reserves - two new tools of monetary policy- are also discussed in this chapter. The most important lesson is that monetary policy is much more complex when the central bank uses interest on reserves to manipulate the interbank rate. Interest rate is still the most important indicator; however, it does not provide full information on the monetary policy stance anymore. Committing to keep the short-term rate at zero for a very long time will create inflation in the short-run but cause deflation in the long run. The central bank needs to raise rate and raise money supply to get out of the trap. That kind of monetary policy is very novel for not only academics but also central bankers.

Chapter 3 “Monetary Policy of Targeting both Interest Rate and Money Supply” generalizes the model in the chapter 2 by including currency and discussing new practices of conducting monetary policy. Negative interest on reserves and forward guidance are tested under the banking crisis context. Both tools are effective but their effects are limited. The negative IOR might transmit to negative deposits rate. Hence the public might switch from deposits into currency in this kind of policy, weakening the effect of monetary policy. For forward guidance policy, until all agents are perfect-foresight, it is effective in our model. However, the calibrated model does not generate a huge effect of forward guidance under some reasonable parameters when banks cut loans.

The key message of this chapter is that the money supply is as important as the short term rate. The central bank can target both indicators to stabilize the economy better. The idea of targeting both money supply and interest rate is possible now since the central bank is controlling the interbank rate by adjusting IOR. Simultaneously conducting a Taylor rule and a Friedman’s k-percent rule is very effective by anchoring the public inflation expectation better. The only problem is that the central bank’s stance of monetary policy is not clear anymore. Raising rates is still a loose monetary policy if the central bank simultaneously injects a huge amount of money into the market.

Chapter 4 “Financial Innovation and Income Distribution” proposes a theoretical causal link
between the appearance of many new financial assets and the ever widening income-gap in America. We build a model with two main types of agents: entrepreneurs and financiers. There are idiosyncratic risks in the entrepreneurs’ productions. The market is incomplete; however, the entrepreneurs can partially insure their risks if they pay for financiers - who are experts in the financial market and can trade a wide range of financial assets.

The main theoretical result from our model is that there is an income distribution process when the financial market has a new financial asset. Income is redistributed in two directions. First, income is transferred from the manufacturing sector to the financial sector. Second, income is transferred from the low-wealth holders to top-wealth holders. As the risk-free return on capital is higher when the market is more complete, top-wealth holders enjoy the benefit from their initial wealth.

The dissertation is organized as follows. Chapter 2 discusses monetary policy with electronic money, assessing efficacy of unconventional monetary policy since the Great Recession. Chapter 3 introduces a novel monetary practice of targeting both interest rate and money supply for central banks. Chapter 4 shows the theoretical link between financial innovation and income distribution. Technical derivations and numerical methods for Chapter 2 to Chapter 4 are provided in Appendix.
Chapter 2

Monetary Policy with Electronic Money

2.1 Introduction

Nowadays, money mostly exists in the electronic form. According to data from the Federal Reserve Bank of St. Louis, the total stock of M1 in Jun 2016 is around USD 3274 billion, consisting of USD 1381 billion in currency and USD 1850 billion in checkable deposits. However, as the world currency, most US dollar bills are held outside US. Judson (2012) estimates that 60 percent of US dollar bills are in foreign countries. If we exclude that number from M1 and M2, currency only accounts for 23 percent of M1, 5 percent of M2 and 4.2 percent of MZM\(^1\). In this chapter, we focus on a popular group of e-money issued by commercial banks, including checkable deposits, saving deposits and money market deposit accounts. Together they account for 80 percent of M2. For convenience, we call this group as zero-maturity deposits (ZMDs) thereafter.

ZMDs are different from currency in two salient features. First, in nature, currency is an IOU issued by the central bank while ZMDs are IOUs issued by commercial banks. In the language of economics, currency is outside money while ZMDs are inside money. Second, in the households’ perspective, unlike currency, ZMDs can earn nominal interest. Banks pay interest for saving accounts and money market deposit accounts for a long time, but the tricky parts are checking accounts. In a perfectly competitive banking market, the interest rate on checkable

\(^1\)MZM (Money zero maturity) is equal to M2 less small-denomination time deposits plus institutional money funds.
deposits should be positive and follow the federal funds rate. However, before 2012, under the Regulation Q, banks in US were prohibited from paying interest on checking accounts. During this period, banks still implicitly paid the demand deposit rate under the form of NOW (negotiable order of withdrawal) accounts, giving gifts or reducing the cost of additional services to their customers, see Mitchell (1979), Startz (1979), Dotsey (1983). Becker (1975) estimates that the implicit demand deposit rate in US during 1960-1968 was around 2.64 percent to 3.74 percent.

Since 2012, the Regulation Q has been no longer valid, and most banks are now paying interest rate on checkable deposits. Data in September 2016 of Federal Deposit Insurance Corporation (FDIC) show that the national average interest on checkable account is 0.04 percent, on saving account is 0.06 percent. These rates are low as the federal funds rate is near zero. If the federal funds rate is around 4 percent, these rates are likely from 1 percent to 2 percent. As a result of that, in the era of electronic money, it is more natural to model money as an interest-earning asset that provides liquidity service.

This chapter builds a dynamic general equilibrium model where currency does not exist (a cashless model). There are two forms of money in our model: ZMDs and reserves. ZMDs are inside money issued by commercial banks. They are used for settling transactions between every pair of agents in the private sector, except between bankers-bankers. In these types of transactions, bankers have to use reserves- another type of e-money that is issued by the central bank. The amount of ZMDs that banks can issue is restricted by two constraints: the reserves requirement and the capital requirement. In our model, the central bank only controls the level of reserves while the level of the money supply (amount of ZMDs) depends on the interaction between the central bank, the commercial banks and the public (Mishkin, 2007).

We use our model to discuss unconventional monetary policy during and after the Great Recession. Here are some key results:

i. In normal times, when the central bank controls the federal funds rate by adjusting the level of reserves, the effect of cutting rates in our model is nearly identical to the one founded in the standard New Keynesian model. After the interbank rate goes down, the real rate goes down as price is sticky. Banks lend out more to households by creating more money.

---

2When the interbank rate is negative, the checkable deposits might earn negative nominal rates.
ii. After a shock that makes banks’ capital constraint binding, an interest rate policy following a Taylor rule is not enough to recover the economy quickly. Both output and inflation are lower than their steady state levels for a long time.

iii. A central bank’s large scale asset purchase (LSAP) program, with the aim of directly injecting money into the economy, is very efficient at dealing with the situation when bankers cut loans. Inflation will go up immediately after this program. The byproduct of LSAP is a huge amount of excess reserves in the banking system (Keister and McAndrews, 2009); the reserves requirement is no longer binding; and interest on reserves (IOR) becomes the main tool to control the federal funds rate.

iv. After LSAP, the longer the federal funds rate is committed at the lower bound, the higher is inflation in the short run. As loans have the longer maturity than deposits, commitment to keep the short-term rate near zero for a long time pushes down the loan rate stronger. However, in the long run, it might create a persistent deflation due to the Neo-Fisherian’s effect. The real short-term rate will slowly climb back to the long-run level. The endogenous money supply declines, and deflation realizes. This matches with the US data since the Great Recession (Figure 2.1).

v. It is not easy to safely escape from the “low rate-cum-deflation” trap. If the central bank raises rates (by raising IOR), the amount of banks’ credits declines. The economy will suffer a short recession. Deflation is even more severe in the short run. However, inflation will jump back to the target in the long run. Therefore, the central bank falls into a dilemma between to raise or not to raise rates. Either way the outcome is not bright.

vi. When raising IOR, if the central bank simultaneously commits to target the growth rate of the money supply in response to inflation, the inflation and output path will be stabilized. In the electronic payment system, the central bank somehow can manipulate both interest rate and money supply at the same time. These tools should be utilized simultaneously so that the central bank can hit the inflation target better.

**Related Literature**

On the money supply side, our approach is similar to Bianchi and Bigio (2014) and Afonso
and Lagos (2015) when the central bank can increase the level of the money supply and cut down the federal funds rate by injecting more reserves in the banking system. These papers explicitly model the search and matching process of heterogeneous agents in the interbank market while our model is frictionless with identical bankers. On the other hand, our model can connect the central bank policy to not only banks’ balance sheet but also the production sector, which is missing in both Bianchi and Bigio (2014) and Afonso and Lagos (2015).

On the money demand side, our model follows the cash-in-advance approach in Lucas and Stokey (1987). As our model does not have currency, “cash” here should be interpreted as ZMDs. In Belongia and Ireland (2006, 2014), currency and deposits are bundled together and provide the liquidity service to households. We also extend the Clower constraint to investment (Stockman (1981), Abel (1985), Fuerst (1992), Wang and Wen (2006)). Indeed, most empirical research, for example Friedman (1959) and Mulligan and Sala-I-Martin (1997), usually uses the income, rather than the consumption alone, to estimate the money demand function.

Our model is still in the general New Keynesian framework with the crucial sticky price feature. The important role of financial frictions in the New Keynesian has been emphasized for a long time (see Bernanke, Gertler and Gilchrist (1999), Christiano, Motto and Rostagno (2004)). Recently, many New Keynesian research (Gertler and Kiyotaki (2010), Curdia and Woodford (2011), Gertler and Karadi (2011)) incorporates the banking sector to their models, aiming to answer what happened in the Great Recession and the role of the unconventional monetary policy. There is also a large literature that discusses interest on reserves, see Sargent and Wallace (1985), Goodfriend et al. (2002), Ireland (2014), Cochrane (2014), Keister (2016).
Our model differs mainly from this line of research in the money supply process. We can characterize the micro-foundation link between bank reserves, banks’ balance sheets, money supply, interest rate and output. We emphasize the importance of both money supply and interest rate in monetary policy when the central bank adjusts the interbank rate by IOR.

Our approach also relates to Brunnermeier and Sannikov (2016) where macro shocks can affect strongly to the balance sheets of intermediaries and the amount of inside money. Both papers emphasize the importance of inside money in the deflation episode. However, two papers differ mainly in the role of money and the money supply process. They emphasize the money function as a store of value in a risky environment while our model focuses on the common role of money - medium of exchange - in a deterministic setting.

2.2 The Environment

2.2.1 Notation:

Let \( P_t \) be the price of the final good. We use lowercase letters to represent the real balance of a variable or its relative price to the price of the final good. For example, the real reserves balance \( n_t = N_t/P_t \), or the relative price of the intermediate good to the final good is \( p^m_t = P^m_t/P_t \). The timing notation follows this rule: if a variable is determined or chosen at time \( t \), it will have the subscript \( t \). The gross inflation rate is \( \pi_t = P_t/P_{t-1} \).

2.2.2 Time, Demographics and Preferences

Time is discrete, indexed by \( t \) and continues forever. The model is in the deterministic setting and has five types of agents: bankers, households, wholesale firms, retail firms, and the consolidated government.

There is a measure one of identical infinitely lived bankers in the economy. Bankers discount the future with the discount factor \( \beta \). Each period, they gain utility from consuming the final consumption good \( c_t \). Their utility at the period \( t \) can be written as:

\[
\sum_{i=0}^{\infty} \beta^i \log(c_{t+i})
\]
There is also a measure one of identical infinitely lived households. Households discount the future with the discount factor $\tilde{\beta} < \beta$, so they will borrow from bankers in the steady state. Each period, households gain utility from consuming the final consumption good $\tilde{c}_t$ and lose utility when providing labor $l_t$ to their own production. Household’s utility at the period $t$ can be written as:

$$\sum_{i=0}^{\infty} \tilde{\beta}^i \left( \log(\tilde{c}_{t+i}) - \chi \frac{l_{t+i}^{1+\nu}}{1+\nu} \right)$$

where $\nu$ is the inverse Frisch elasticity of labor supply.

Wholesale firms, retail firms are infinitely lived, owned by households.

The consolidated government includes both the government and the central bank, so for convenience, we assume there is no independence between the government and the central bank.

### 2.2.3 Goods and Production Technology

There are three types of goods in the economy: final good $y_t$ produced by retailers, wholesale goods $y_t(j)$ produced by wholesale firm $j$ and intermediate good $y^m_t$ produced by households.

Each period households self-employ their labor $l_t$ and use the capital $k_{t-1}$ to produce the homogeneous intermediate good $y^m_t$ under the Cobb-Douglas production function:

$$y^m_t = k_{t-1}^\alpha l_t^{1-\alpha}$$

where $\alpha$ is the share of capital in the production function. Capital $k$ depreciates with the rate $\delta_k$. Households also own a technology to convert one unit of final good $y_t$ to one unit of capital type $k$ and vice versa. So each period they also make an investment $i_t = k_t - \delta_k k_{t-1}$. Households sell $y^m_t$ to wholesale firms in the competitive market with price $P^m_t$.

There is a continuum of wholesale firms indexed by $j \in [0, 1]$. Each wholesale firm purchases the homogeneous intermediate good $y^m_t$ from households and differentiates it into a distinctive wholesale goods $y_t(j)$ under the following technology:

$$y_t(j) = y^m_t$$
Then retail firms produce the final good \( y_t \) by aggregating a variety of differentiated wholesale goods \( y_t(j) \):

\[
y_t = \left( \int_0^1 y_t(j) \frac{\varepsilon - 1}{\varepsilon} \, dj \right)^{\frac{\varepsilon}{\varepsilon - 1}}
\]

### 2.2.4 Assets

There are three main types of financial assets (excluding reserves and deposits): bank loans \( B^h_t \), share of wholesale firms \( x_t \) and interbank loans \( B^f_t \).

(a) **Bank loans to households (\( B^h_t \))**: We follow Leland and Toft (1996) and Bianchi and Bigio (2014) to model the loan structure between bankers and households. The market for bank loan is perfectly competitive and the price of loan is \( q^L_t \). When a household wants to borrow 1 dollar at time \( t \), bankers will create an account for her and deposit \( q^L_t \) dollars to her account. In the exchange for that, this household promises to pay \( \delta^b, \delta^b_2, ..., \delta^b_n, \delta^b_{n+1} \) dollars at time \( t+1, t+2, ..., t+n, t+n+1 \)... where \( n \) runs to infinity (Table 3.1). Loans are illiquid and bankers cannot sell loans.

Let \( B^h_t \) be the nominal balance of loan stock in the period \( t \), let \( S_t \) be the nominal flow of new loan issuance, we have:

\[
B^h_t = \delta^b B^h_{t-1} + S_t
\]

(b) **Shares of wholesale firms (\( x_t \))**: are issued by the wholesale firms. Bankers are not allowed to hold this share, so they are only traded between households. Each share has a price \( u_t \) and pays a real dividend \( w_t \). The number of wholesale firms’ shares is 1. In the LSAP, the central bank might purchase these shares to inject money into the market.

(c) **Interbank loan (\( B^f_t \))**: Bankers can borrow reserves from other bankers in the federal funds market. The nominal gross interest rate in the federal funds market is the federal funds rate \( R^f_t \).

### 2.2.5 Money

There are two types of electronic money in our economy: reserves \( n_t \) and zero-maturity deposits \( m_t \).
Table 2.1: Banker issues loans (left) and collects loans (right)

<table>
<thead>
<tr>
<th>Banker</th>
<th>Banker</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loans: (+S_t)</td>
<td>Loans: (- (1 - \delta_b)B_t^{h-1})</td>
</tr>
<tr>
<td>Deposits: (+q_t^L S_t)</td>
<td>Deposits: (-\delta_b B_t^{h-1})</td>
</tr>
<tr>
<td>Net worth: ((1 - q_t^L)S_t)</td>
<td>Net worth: ((2\delta_b - 1)B_t^{h-1})</td>
</tr>
</tbody>
</table>

(a) **Reserve** \((n_t)\): is a type of e-money issued by the central bank. Only government and bankers have an account at the central bank, so only government and bankers have reserves\(^3\). Each period, the central bank pays a gross interest rate \(R_t^n\) on these reserves. The rate \(R_t^n\) is decided solely by the central bank. Reserves are used for settling the transactions between bankers and bankers, bankers and central bank, bankers and government.

(b) **Zero maturity deposit** \((m_t)\): is a type of e-money issued by bankers. Each period, banks pay the interest rate \(R_t^m\) for these ZMDs which is determined by the perfectly competitive market. Money \(m_t\) is used for settling transactions in the non-bank private sector and the ones between households and bankers. These ZMDs are insured by the central bank, so they are totally safe. ZMDs and reserves have the same unit of account.

In the electronic payment system, there is a connection between the flows of reserves and deposits. For example, we assume that wholesale firm A (whose account at bank A) pays 1 dollar for household B (whose account at bank B). Then the flow of payment will follow Table (2.2). For a transaction between the consolidated government and households, money still flows through banks, so we can think that this contains two sub-transactions. One is between the government and banks, which is settled by reserves. One is between banks and households, which is settled by ZMDs.

In the conventional monetary policy, the consolidated government targets the interbank rate by helicopter money or lump-summer tax on households. Each period, the central bank sends \(\tau_t\) dollars in checks to households. It can be thought as a shortcut of the open market operation process when the central bank purchases government bonds from the government (through banks). Then, the government transfers the payoffs to households (Table 3.2). In fractional reserve banking, the amount of \(\tau_t\) needed to change the federal funds rate is extremely small in

---

\(^3\)The amount of US Treasury deposits at the Fed is not considered as reserves in reality. However, for convenience, we also call that money as reserves in our model. In equilibrium, the balance of the government account at the central bank is zero, so it does not matter.
2.2.6 Timing within one period

(i) Production takes place. Households sell goods to wholesalers, who, in turn, sell goods to retailers. All of the payments between households-wholesalers, wholesalers-retailer are delayed until the step (iv).

(ii) The loan market between bankers and households opens.

(iii) The final good market opens. Households need ZMD-in-advance to purchase the final good from retailers. Bankers create ZMD to purchase the final good from retailer.

(iv) Payments in the non-bank private sector are settled.

(v) The banking market opens. Banker can adjust the level of reserves by borrowing in the interbank market, receiving new deposits.
2.3 Agents’ Problems

2.3.1 Bankers

There is a measure one of identical bankers in the economy. These bankers have to maintain a good balance sheet under the regulation of the central bank. There are three types of assets on a banker’s balance sheet: reserves \(n_t\), loans to households \(b^h_t\), loan to other bankers \(b^f_t\). His liability side contains the zero-maturity deposits that households deposit here \(m_t\).

<table>
<thead>
<tr>
<th>Banker</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves:</td>
<td>(n_t)</td>
</tr>
<tr>
<td>Loans to households:</td>
<td>(b^h_t)</td>
</tr>
<tr>
<td>Loans to other bankers:</td>
<td>(b^f_t)</td>
</tr>
<tr>
<td>Zero Maturity Deposits:</td>
<td>(m_t)</td>
</tr>
<tr>
<td>Net worth</td>
<td></td>
</tr>
</tbody>
</table>

**Cost**: We assume that the banker faces a cost of managing loan, which is \(\theta b^h_t\) in terms of final goods.

On the timing of the market, it is worth noting that he can adjust the level of his deposits and reserves after households and firms pay for each other. When the different parties in the economy pay each other, he can witness that the deposits and reserves outflow from or inflow to his bank. Let \(e_t\) be the net inflow of deposits and reserves go into his bank, he will treat \(e_t\) exogenously. When the banking market opens, as the deposit market is perfectly competitive, he can choose any amount \(d_t\) of deposit inflows or outflows to his bank.

In each period, the banker treats all the prices as exogenous and choose \{ \(c_t, n_t, b^h_t, s_t, m_t, b^f_t, d_t\) \} to maximize his utility over a stream of consumptions:

\[
\max \sum_{t=0}^{\infty} \beta^t \log(c_t)
\]

subject to

\[
\frac{R^m_{t-1}}{\pi_t} n_{t-1} + \frac{R^f_{t-1}}{\pi_t} b^f_{t-1} + d_t + e_t + \tau_t = n_t + b^f_t \quad \text{(Reserve Flows)} \tag{2.1}
\]

\[
m_t = \frac{R^m_{t-1}}{\pi_t} m_{t-1} + q^L_t s_t + \delta \frac{b^h_{t-1}}{\pi_t} + c_t + d_t + e_t + \tau_t \quad \text{(Deposit Flows)} \tag{2.2}
\]
\[ b_t^{h} = \delta_{b_t^{h-1}} + s_t \quad \text{(Loan Flows)} \] (2.3)

\[ n_t \geq \varphi m_t \quad \text{(Reserves Requirement)} \] (2.4)

\[ n_t + b_t^{f} + b_t^{h} - m_t \geq \kappa_t b_t^{h} \quad \text{(Capital Requirement)} \] (2.5)

**Reserve Flows:** in each period are shown in the equation (3.1). After receiving the interest on reserves, the previous reserve balance becomes \( R_t^{n} n_{t-1} \pi_t \). He also collects the payment from the interbank loan he lends out to other bankers in the previous period \( R_t^{f} b_t^{f} / \pi_t \). He can also increase his reserves by taking more deposits \( d_t \). When doing that, his reserves and his liability increase by the same amount \( d_t \) (Table 3.3). That is the reason we see \( d_t \) appear on both the equation (3.1) and (3.2). The similar effect can be found on \( \tau_t \) when the central bank drops money. The banker treats \( \tau_t \) exogenously. Then, he can leave reserves \( n_t \) at the central bank’s account to earn interest rate, or lend reserves to another bankers \( b_t^{f} \).

**Deposit Flows:** for the banker are shown in the equation (3.2). He makes loans to households by issuing deposits or creating ZMDs (Table 3.1). So when he makes a loan \( (s_t) \), the balance sheet expands. When he collects the payoffs from loans to households \( (\delta b_t^{h} / \pi_t) \), the balance sheet shrinks\(^4\).

The banker also issues his own ZMDs to purchase the consumption good from retailers \( (c_t) \) and to pay for the cost (in terms of final goods) related to lending activities \( (\theta b_t^{h}) \) (Table 2.4). It is noted that he cannot create infinite amount of money for himself to buy consumption goods as there exists the capital requirement.

**Reserve Requirement:** At the end of each period, he has to hold enough reserves as a fraction of total deposits (3.5)\(^5\). This constraint should be interpreted more broadly than the the real life reserves requirement in the US because the total ZMDs here include not only checkable deposits but also saving deposits and money market deposit account.

**Capital Requirement:** The second constraint plays the key role in our model - the capital requirement constraint. The left hand side of (3.6) is the banker’s net worth (capital), which is

\(^4\)It is assumed that households have to pay loans from the account at the bank they borrow. So if they want to use money from account at bank B to pay for loans from bank A, they need to transfer deposits from bank B to bank A first. In fact, this assumption does not matter in equilibrium.

\(^5\)During one period, his reserve balance can go temporarily negative. But in the end of that period, it must be positive and satisfies the regulation.
equal to total assets minus total liabilities\(^6\). The constraint requires the banker to hold capital greater than a fraction of total loans in his balance sheet. We assume that \(k_t\) is a constant \(k\) in normal times. We later put the unexpected shock on this \(k_t\) to reflect the shock in the Great Recession\(^7\).

Let \(\gamma_t\), \(\mu^{r}_t\) and \(\mu^{c}_t\) be respectively the Lagrangian multipliers attached to the reserves flows, reserves constraint and the capital constraint. The first order conditions of the banker’s problem can be written as:

\[
\gamma_t = \frac{1}{c_t} \quad (2.6)
\]

\[
\gamma_t = \frac{BR_t^r \gamma_{t+1}}{\pi_{t+1}} + \mu^{c}_t \quad (2.7)
\]

\[
\gamma_t = \frac{BR_t^{m} \gamma_{t+1}}{\pi_{t+1}} + \mu^{c}_t + \phi \mu^{r}_t \quad (2.8)
\]

\[
\gamma_t = \frac{BR_t^{n} \gamma_{t+1}}{\pi_{t+1}} + \mu^{c}_t + \mu^{r}_t \quad (2.9)
\]

\[
(q_t^L + \theta) \gamma_t = \frac{B [\delta_b + \delta_q q_{t+1}]}{\pi_{t+1}} \gamma_{t+1} + (1 - k_t) \mu^{c}_t \quad (2.10)
\]

\(^6\)We use the book value \(B^h_t\) rather than the “market value” of loans \(q^L_t B^h_t\) in the capital constraint. The reason is that illiquid bank loans should be treated differently from bonds. In reality, bank loans are often not revalued in the balance sheet when the interest rate changes.

\(^7\)Clearly, it is a simplified way to reflect the increase in the bad loans and the aggregate risk during the Great Recession. Still, we can keep our model simple.
And two complementary slackness conditions:

\[ \mu_i^r \geq 0, \quad n_t - \varphi m_t \geq 0, \quad \mu_i^r (n_t - \varphi m_t) = 0 \]
\[ (2.11) \]

\[ \mu_i^c \geq 0, \quad n_t + b^f_i + (1 - \kappa_i) b^h_i - m_t \geq 0, \quad \mu_i^c (n_t + b^f_i + (1 - \kappa_i) b^h_i - m_t) = 0 \]
\[ (2.12) \]

### 2.3.2 Households

There is a measure one of identical households. These self-employed households produce the homogeneous intermediate good \( y^m \) to sell to the wholesale firms at the price \( P^m_i \), or at the real relative price \( p^m_i \). In each period, a household purchases the final good from the retail firms to consume \( (\tilde{c}_t) \) and make her investment \( (i_t) \).

Let \( \tilde{b}^h_t \) be the nominal debt stock that she borrows from bankers. Recalling from the section 2.2.4, each period she only pays a fraction \( \delta_b \) of the old debts. We impose an exogenous borrowing constraint for households with the debt limit \( \tilde{b}^h_t \leq b^h \).

After the loan market, she brings \( a_t \) amount of ZMDs into the final good market. Basically, she faces the “ZMD-in-advance” constraint when the good market opens. So the amount of loans that she gets from banks will affect her demand for the final goods.

In each period, she chooses \( \{ \tilde{c}_t, \tilde{m}_t, \tilde{b}^h_t, \tilde{s}_t, i_t, k_t, l_t, a_t \} \) to maximize her utility:

\[ \max \sum_{t=0}^{\infty} \beta^t \left( \log(\tilde{c}_t) - \chi \frac{l^1_t}{1+\nu} \right) \]

subject to

**Loan Market:**

\[ a_t + \delta_b \frac{\tilde{b}^h_{t-1}}{\pi_t} = \frac{R^m_{t-1} \tilde{m}_{t-1}}{\pi_t} + q^L_t \tilde{s}_t \]
\[ (2.13) \]

**ZMD-in-advance:**

\[ \tilde{c}_t + i_t \leq a_t \]
\[ (2.14) \]

**Budget:**

\[ \tilde{m}_t + \tilde{c}_t + i_t + \nu_t (\tilde{x}_t - \tilde{x}_{t-1}) = a_t + \tau_t + p^m_i y^m_t + w_t \tilde{x}_{t-1} \]
\[ (2.15) \]

**Investment:**

\[ i_t = k_t - (1 - \delta) k_{t-1} \]
\[ (2.16) \]

**Production:**

\[ y^m_t = k^\alpha_t l^{1-\alpha}_t \]
\[ (2.17) \]

**Loan Flows:**

\[ \tilde{b}^h_t = \delta_b \frac{\tilde{b}^h_{t-1}}{\pi_t} + \tilde{s}_t \]
\[ (2.18) \]

**Borrowing Constraint:**

\[ \tilde{b}^h_t \leq b^h \]
\[ (2.19) \]
We assume that the household faces an exogenous borrowing constraint, rather than a collateral borrowing constraint like Kiyotaki and Moore (1997) and Iacoviello (2005). Our purpose is to emphasize that the mechanism of the shock transmission in our model is not related to the collateral constraint literature. Similar to the capital requirement, we impose the constraint on the face value of the loan.

Let $\eta^z_t$, $\eta^b_t$, $\lambda^a_t$ be the Lagrangian for the cash-in-advance, borrowing constraint and budget constraint. Let $\lambda^b_t$ be defined as the sum of $\eta^z_t$ and $\lambda^a_t$. Let $r^h_t$ be defined as the real short-term borrowing (lending) rate:

$$\frac{1}{c_t} = \eta^z_t + \lambda^a_t = \lambda^b_t \quad (2.20)$$

$$\lambda^a_t = \frac{\tilde{\beta}R^m_t \lambda^b_{t+1}}{\pi_{t+1}} \quad (2.21)$$

$$q^L_t \lambda^b_t = \frac{\tilde{\beta}([\delta_b + \delta_b q^L_{t+1}]\lambda^b_{t+1} + \eta^b_t}{\pi_{t+1}} \quad (2.22)$$

$$\lambda^b_t = \tilde{\beta}(1 - \delta)\lambda^b_{t+1} + \tilde{\beta}\alpha p^m_{t+1} \lambda^a_{t+1} y^m_{t+1} \quad (2.23)$$

$$\chi^v_{t+1} = (1 - \alpha)p^m_{t+1} y^m_{t+1} \lambda^a_t \quad (2.24)$$

$$\lambda^a_t v_t = \tilde{\beta}\lambda^a_{t+1} (v_{t+1} + w_{t+1}) \quad (2.25)$$

$$r^h_t \equiv \frac{\delta_b + \delta_b q^L_{t+1}}{\pi_{t+1} q^L_{t}} \quad (2.26)$$

And two complementary slackness conditions:

$$\eta^z_t \geq 0, \quad a_t - c_t - i_t \geq 0, \quad \eta^z_t (a_t - c_t - i_t) = 0 \quad (2.27)$$

$$\eta^b_t \geq 0, \quad \tilde{b}^h - \tilde{b}^h_t \geq 0, \quad \eta^b_t (\tilde{b}^h - \tilde{b}^h_t) = 0 \quad (2.28)$$

As money plays the role of medium of exchange in our model, it’s value contains the liquidity part. In the steady state, the rate of return on money has to be less than $1/\tilde{\beta}$.

The equations (3.27) and (2.23) give us the marginal cost and the marginal benefit when the household borrows one more unit of loans from bankers and when she makes one more unit of investment. The equation (2.25) is a common asset pricing equation for the wholesalers’ shares.
2.3.3 Retail Firms and Wholesale Firms

Follow Rotemberg pricing, we assume that each wholesale firm $j$ faces a cost of adjusting prices, which is measured in terms of final good and given by:

$$\frac{1}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 y_t$$

where $t$ determines the degree of nominal price rigidity. The wholesale firm $j$ discounts the profit in the future with rate $\tilde{\beta} \lambda_t^a / \lambda_t^a$. Her real marginal cost is $p^m_t$.

In a symmetric equilibrium, all firms will choose the same price and produce the same quantity $P_t(j) = P_t$ and $y_t(j) = y_t = y^m_t$. The optimal pricing rule then implies that:

$$1 - t (\pi_t - 1) \pi_t + t \tilde{\beta} \frac{\lambda_{t+1}^a}{\lambda_t^a} (\pi_{t+1} - 1) \pi_{t+1} \frac{y_{t+1}}{y_t} = (1 - p^m_t) \epsilon$$  \hspace{1cm} (2.29)

2.3.4 The Central Bank and Government

The consolidated government uses the payoffs from tax or their assets to pay for the interest on reserves, then injects (drains) $\hat{\tau}_t$ amount of reserves and deposits by helicopter money (lump-sum tax) to target the interbank rate. All transactions are conducted in the electronic system.

$$\tau_t = -\frac{(R^n_{t-1} - 1)n_{t-1}}{\pi_t} + \hat{\tau}_t$$  \hspace{1cm} (2.30)

In the conventional monetary policy, we assume that the central bank follows the simple Taylor rule, fixing $R^n_t$ at a constant level $\bar{R}$. To connect with the common New Keynesian literature, we assume there is a lower bound for $R^f_t$ that is greater than $\bar{R}$, so there are no excess reserves. Later, we relax the assumption and examine the situation when the banking system is awash of excess reserves and the central bank controls the federal funds rate by adjusting $R^n_t$.

In this chapter, we assume the inflation target in the long-term of the central bank $\pi = 1$.

$$R^n_t = \bar{R}$$ \hspace{1cm} (2.31)

$$R^f_t = \max \left\{ \frac{\pi}{\tilde{\beta}} \frac{(\pi_{t+1})^{\phi}}{\pi}, \bar{R} + \epsilon_f \right\}$$ \hspace{1cm} (2.32)

\[8\] When the reserve requirement is no longer binding, a Taylor rule is not enough for the determinacy as we need a rule governing the motion of reserves.
2.4 Equilibrium

**Definition:** A competitive equilibrium is a sequence of bankers’ decision choice \( \{c_t, n_t, b_t^h, s_t, m_t, b_t^f, d_t\} \), household’s choice \( \{\tilde{c}_t, \tilde{b}_t^h, \tilde{s}_t, \tilde{m}_t, i_t, k_t, l_t, \tilde{y}_t^m, \tilde{x}_t\} \), firms’ choice \( \{y_t\} \), the central bank’s choice \( \{\tau_t, R^n_t\} \), and the market price \( \{q_L^t, R_f^t, \nu_t, \pi_t, p^m_t\} \) such that:

i Given the market price and the central bank’s choices, banker’s choices solve the banker’s problem, household’s choices solve the household’s problem, firm’s choice solves the equation (3.31).

ii All markets clear:

\[
\begin{align*}
\text{Net flows of reserves:} & \quad d_t + e_t = 0 \\
\text{The interbank market:} & \quad b_t^f = 0 \\
\text{Total ZMDs:} & \quad m_t = \tilde{m}_t \\
\text{Loan Market:} & \quad b_t^h = \tilde{b}_t^h \\
\text{Wholesalers’ shares:} & \quad \tilde{x}_t = 1 \\
\text{Good Market:} & \quad y_t = c_t + \tilde{c}_t + i_t + \theta b_t^h + \frac{1}{2} (\pi_t - 1)^2 y_t
\end{align*}
\]

If we consider a model without currency where all banks are identical in the equilibrium, the net flows of reserves to the representative banker will be zero. We also make the following assumption to ensure that in the steady state households will borrow from bankers.

**Assumption 2.1.** The discount factors of bankers and households satisfy:

\[
\frac{\beta \delta_b - \theta \pi}{\pi - \beta \delta_b} > \frac{\tilde{\beta} \delta_b}{\pi - \tilde{\beta} \delta_b}
\]

We also assume that in the long run, the inflation will be at the target level by restricting monetary policy in every regime to satisfy:

**Assumption 2.2.**

\[
\lim_{t \to \infty} \frac{\hat{\tau}_t}{n_t} = \frac{\pi - 1}{\pi}
\]
The relationship between the federal funds rate $R^f_t$, deposit rate $R^m_t$ and interest on reserves $R^n_t$ can be understood under the following theorem:

**Theorem 2.1.** In equilibrium:

1. The lower bound of the federal funds rate and the deposit rate is the interest on reserves. In all cases, $R^n_t \leq R^m_t \leq R^f_t$.
2. When the constraint of reserve requirement is not binding, $R^f_t = R^m_t = R^n_t$.

There are two benefits of holding reserves for bankers. First, bankers can earn the interest on reserves that central bank pays them. Second, it helps bankers satisfy reserve requirement. The cost of holding reserves is the federal funds rate that they give up when they do not lend reserves in the interbank market. When the banking system has a large amount of excess reserves, the second benefit vanishes and the federal funds rate must be equal to the interest on reserves.

In reality, the deposit rate of ZMDs might be lower than the interest on reserves due to the bankers’ cost of providing liquidity services and market power. We ignore these factors in this model to present the main mechanism cleaner.

**Theorem 2.2.** The total level of reserves in equilibrium is decided solely be the central bank:

\[
\frac{n_{t-1}}{π_t} + \hat{τ}_t = n_t
\]

Bankers themselves cannot change the total level of reserves in the banking system. Lending or not lending to households will not change the total level of reserves. The appearance of the huge amount of reserves after the large scale asset purchase is just a byproduct of the central bank’s policy. Later we will examine this kind of policy.

### 2.5 The Steady State

We use $a$ to denote the steady state value of a variable $a_t$. 
**Theorem 2.3.** Under the Assumption (2.1)-(2.2), in every steady state (if exists):

i. The banker’s reserves constraint (3.5), the household’s borrowing constraint (3.21) and the ZMD-in-advance constraint (3.15) are binding.

ii. The banker’s capital constraint (3.6) is not binding.

**Theorem 2.4.** Under the Assumption (2.1) and (2.2), the capital in every steady state (if exists) satisfies the following equation:

\[ \frac{1}{r^{m}\alpha_m k - \delta k + \eta m - \delta_b p^\beta} = \frac{\chi \alpha_l^{\nu + 1} k^\nu}{(1 - \alpha) p^m \alpha_y r^m} \]  

(2.34)

where \( r^m, \alpha_m, \alpha_l, \alpha_y \) are constants independent of \( k \).

We make one more assumption to ensure that there exists a unique steady state. The uniqueness of the steady state will be very important as we mostly examine the global nonlinear dynamic of our model.

**Assumption 2.3.** The parameters satisfy:

\[ \kappa < 1 - \frac{(1 - \varphi)m}{b^\beta} \]

\[ \frac{(\beta \delta_b - \pi \theta)(\pi - \delta)}{\pi - \beta \delta_b} > \delta_b \]

\[ r^m \alpha_m - \delta > 0 \]

where \( m \) is defined in (A.2.21), \( r^m \) and \( \alpha_m \) are defined in (A.2.12) and (A.2.16).

The restriction on the parameter \( \kappa \) is to ensure that the capital constraint is not binding. The last two restrictions are to ensure that the equation (2.34) has a unique positive solution \( k^* \).

**Theorem 2.5.** Under the Assumption (2.1)-(2.3), there is a **unique** steady state.
### Table 2.6: Parameter values

<table>
<thead>
<tr>
<th>Param.</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bankers</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>Banker’s discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>The reserves requirement</td>
<td>0.002</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>The risk weight</td>
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</tr>
<tr>
<td>$\theta$</td>
<td>The monitoring cost</td>
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</tr>
<tr>
<td>$\delta_b$</td>
<td>Loan amortization</td>
<td>0.5</td>
</tr>
<tr>
<td><strong>Households</strong></td>
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<tr>
<td>$\overline{\beta}$</td>
<td>Household’s discount factor</td>
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<tr>
<td>$\chi$</td>
<td>Relative Utility Weight of Labor</td>
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<td>$\nu$</td>
<td>Inverse Frisch Elasticity of Labor Supply</td>
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<tr>
<td>$\overline{b_h}$</td>
<td>The borrowing limit</td>
<td>3.4</td>
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<tr>
<td>$\delta_h$</td>
<td>Capital’s depreciation rate</td>
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<tr>
<td>$\alpha$</td>
<td>Capital share in production function</td>
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<td><strong>Firms</strong></td>
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<td>$\varepsilon$</td>
<td>Elasticity of substitution of wholesale goods</td>
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<tr>
<td>$i$</td>
<td>Cost of changing price</td>
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<td><strong>Central bank</strong></td>
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<td>$\phi_\pi$</td>
<td>Policy respond to inflation</td>
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<tr>
<td>$R^m$</td>
<td>The constant IOR</td>
<td>1+0.25/400</td>
</tr>
<tr>
<td>$R^m + \varepsilon_f$</td>
<td>The lower bound for FFR</td>
<td>1+0.5/400</td>
</tr>
</tbody>
</table>

### 2.6 Quantitative Analysis

#### 2.6.1 Calibration

For the bankers’ parameters, we choose the discount factor $\beta = 0.99$ to match with the federal funds rate of 4% annually before the Great Recession. The reserves requirement is set as the ratio between reserves and the total ZMDs (including checking account, saving account and money market deposit account) before the financial crisis, which is around $\varphi = 0.002$. The monitoring cost $\theta$ and loan amortization $\delta_b$ are set exogenously. The risk weight $\kappa$ is exogenously set so that 10 percent increase of $\kappa$ from steady state will make the capital constraint binding. (Table 4.2)

Most of the households’ parameters are standard in the literature. The only one that needs to be calibrated is the borrowing limit $\overline{b_h}$. We calibrate it to match with the ratio between total...
households’ debts and households’ income before the Great Recession - around 1.3 times. All other parameters are also in the range which is often seen in the macro literature. We tried using as few parameters as possible to illustrate the main mechanism of the model.

2.6.2 Federal funds rate shock

We examine an interest rate shock in the Taylor rule and compare the mechanism of this model to the standard one in the New Keynesian literature.

\[ R_t^f = \max \left\{ \frac{1}{\beta} \left( \frac{\pi_{t+1}}{\pi} \right)^{\phi_{\pi}} \exp(u_t^f), \ R^n + \epsilon_f \right\} \]

\[ R_t^n = \bar{R}^n \]

\[ u_t^f = \rho_f u_{t-1}^f, \ u_0^f \text{ is given} \]  

(P1)

From the steady state, there is an unexpected shock at \( t = 0 \) with \( u_0^f = -2/400 \), then agents know that the shock will die slowly with \( \rho_f = 0.6 \).

Similar to the standard New Keynesian model: As the price is sticky, when the central bank cut the federal funds rate, the real rate goes down and stimulates the economy in the short run. (Figure 2.2)

Difference from the standard New Keynesian model:

i. Banks play an important role in creating money. After the interest rate shock, the real money balance increases by 0.45 percent. Most of that new money is created by banks when they increase loans. The amount of money that the central bank actually “drops” to the economy \( \hat{\tau} \) to change the federal funds rate only accounts for 0.02 percent of this increase. So unlike the standard model in New Keynesian, our model focuses on the money creation process by commercial banks and the pass-through effect from the federal funds rate to the loan rate.

ii. Without any adjustment cost functions, investment still well-behaves after the cut in the real interest rate. The constraint for a huge sudden jump of investment comes naturally from the ZMD-in-advance constraint and the borrowing constraint.

\[ ^9 \text{Except the federal funds rate and the real borrowing rate are converted to the annual level, all other figures show the percentage deviation of a variable from its steady state value.} \]
(a) Federal funds rate and real borrowing rate
(b) Real balance of ZMDs and Reserves

(c) Consumptions
(d) Investment and Capital

(e) Inflation
(f) Output

Figure 2.2: Impulse Response to Interest Rate Shock in (P1)
2.6.3 Financial Crisis - Taylor Rule Response

From the steady state, we illustrate a financial crisis by imposing an unexpected shock at $\kappa_t$ in the capital constraint. This is a simplified way to reflect a sudden increase in the “potential” bad loans in the bankers’ balance sheets. This chapter does not try answering the cause of the Great Recession, so this reduced form is neat to assess different monetary policy rules. In this section, the conventional monetary policy still follows the Taylor rule in (3.33) and (3.34).

$$R^f_t = \max \left\{ \frac{1}{\beta} \left( \frac{\pi_{t+1}}{\bar{\pi}} \right)^{\phi_x}, \bar{R}^m + \epsilon_f \right\}$$

$$R^n_t = \bar{R}^m$$

$$\kappa_t = \rho_k \kappa_{t-1} + (1 - \rho_k) R, \quad \kappa_0 \text{ is given}$$

(P2)

where $\rho_k = 0.95$ is the persistence of the shock and $\kappa_0 = 0.26$, which is 18 percent higher than the one in the steady state level. The capital requirement switches to the binding mode in the short run. The response of the economy is illustrated in the Figure 2.3.

The banking crisis is dangerous as it raises the spread between the prime rate and the federal funds rate. To satisfy the capital requirement (CR), bankers have to cut loans. Loan rate goes up even when the federal funds rate is cut down, as the shadow price of capital requirement $\mu^c_t$ is positive now.

$$\gamma_t = \beta \frac{\delta_b + \delta_b q^L_{t+1}}{\pi_{t+1} (q^L_t + \theta)} \gamma_{t+1} + \frac{(1 - \kappa_t) \mu^c_t}{q^L_t + \theta}$$

Spread due to CR’s binding

Money supply eventually drops due to the consequence of the debt deleveraging process. Deflation will be persistent under the Taylor rule as the conventional monetary policy only focuses on the pass through of federal funds rate to the prime rate, which will not work in this case.

Standard New Keynesian model emphasizes the importance of monetary policy in correcting the deviation of real rate from its natural level due the the price stickiness. Under the framework where the banking sector is modeled clearly, there are two other inefficiencies that monetary policy can intervene to improve the social welfare. The first inefficiency arises from the binding of the capital constraints, which freezes the credit market between bankers and households. The
Figure 2.3: Impulse Response to Capital Constraint Shock (P2)
second inefficiency comes from the households’ borrowing constraint itself. Unconventional monetary policy focuses on the money supply and asset price might be a good remedy for this situation. We only focus on the money supply in this chapter.

2.6.4 Financial Crisis - Large Scale Asset Purchase (LSAP)

Now, assume that central bank injects money directly into the market by purchasing the wholesale firms’ shares. Let $x_t$ be the number of shares that central bank decides to hold at time $t$ and $\Delta x_t = x_t - x_{t-1}$ be the additional number of shares the central bank purchases at time $t$. Recall $v_t$ be the share’s price and $\bar{x}_t$ is the number of shares that households hold.

$$x_t + \bar{x}_t = 1 \quad (2.35)$$

When the central bank makes transactions with households, in the electronic system, the flows of money will follow the Table 2.7.

<table>
<thead>
<tr>
<th>The Fed</th>
<th>Bankers</th>
<th>Households</th>
</tr>
</thead>
<tbody>
<tr>
<td>Securities: $+v_t \Delta x_t$</td>
<td>Reserves: $+v_t \Delta x_t$</td>
<td>Deposits: $+v_t \Delta x_t$</td>
</tr>
<tr>
<td>Reserves: $+v_t \Delta x_t$</td>
<td>Deposits: $+v_t \Delta x_t$</td>
<td>Security: $- v_t \Delta x_t$</td>
</tr>
</tbody>
</table>

Table 2.7: Central Bank’s Asset Purchase

Before time 0, $x_t = 0$. At time 0, there is an unexpected shock of purchasing assets from the central bank in response instantly to the unexpected shock on $\kappa$. Then the central bank will slowly sell these assets back to the market. For the dividends earned from holding securities, we assume that the consolidated government will give them back to households under the form of lump-sum transfers. In equilibrium, the equations for reserve flows and deposit flows become:

$$\frac{n_{t-1}}{\pi_t} + v_t (x_t - x_{t-1}) + \bar{\pi}_t = n_t \quad (2.36)$$

$$m_t = \frac{R^n_{t-1} m_{t-1}}{\pi_t} + q^L_t s_t + \Theta_t b_t^h - \delta_b \frac{b_{t-1}^h}{\pi_t} + c_t + v_t (x_t - x_{t-1}) + \hat{\pi}_t - (R^n_{t-1} - 1) \frac{n_{t-1}}{\pi_t} \quad (2.37)$$
The exogenous shock for $\kappa_t$ and monetary policy rule are:

$$
\kappa_t = \rho_x \kappa_{t-1} + (1 - \rho_x) \bar{\kappa}, \quad \kappa_0 \text{ is given}
$$

$$
x_t = \rho_x x_{t-1}, \quad x_0 \text{ is given}
$$

$$
\hat{\tau}_t = 0, \quad R^n_t = \bar{R}, \quad \forall t \geq 0
$$

where $\rho_x = 0.98$ be the persistence of the asset purchasing shock and $x_0 = 0.0008$. We assume that the central bank does not follow the Taylor rule anymore. They still fix the interest on reserves at the constant level $\bar{R}$ and only use that asset purchase/sale program to adjust the money supply, so $\hat{\tau} = 0$. Figure 2.4 shows the reaction of the economy to this monetary policy.

Here are some important remark for LSAP’s effect:

i  *The excess reserves skyrockets and the long duration of the federal funds rate at the lower bound*: When the central bank purchases assets from the private sector, they inject simultaneously the money supply into the market and banking reserves into the banking system. When the level of reserves increases by 700 percent, the reserve constraint is no longer binding, $\mu^f_r = 0$. As we assume that the central bank fixes IOR at a constant level, it is synonymous that the federal funds rate will be at the lower bound for a long time, around 25 years (100 quarters) in our model. After a long unwinding quantitative easing process, the reserve requirement will be binding again. The federal funds rate climbs back to its long run level. The whole transition process can take around 80 years in our model.

ii  *Positive effect in the short-run*: The combination of new money injection and the long duration of the federal funds rate at the lower bound steer the economy out of recession quickly, unlike the case with the Taylor rule. As loans have the longer maturity than deposits, if the central bank commits to let the federal funds rate at the low level for a long time, the real lending rate will decline sharply. It combines with the relaxation of the liquidity constraint, stimulating the household’s demand and pushing up inflation.

iii  *Negative effect in the long-run*: After inflation jumps up in the short run, it starts declining, below the central bank’s target. This phenomenon can be explained by the Neo-Fisherian’s idea. In the long run, real short-term rate will be back to the long-term level. As $R^f_t = \bar{R}$, the deflation must realize to increase $R^f_t / \pi_{t+1}$.  

28
(a) Federal funds rate and real borrowing rate

(b) Real balance of reserves

(c) Aggregate consumption

(d) Real balance of ZMDs

(e) Inflation

(f) Outputs

Figure 2.4: Response of economy to LSAP (P3) vs Taylor Rule (P2)
iv Intuitive Explanation: When the central bank keeps the interbank rate at 25 basis points, the rate of saving account $R^m$ will be at 25 basis points as deposits and interbank loans have the same short-term maturity. However, the real return on capital in the long-run recovers to the pre-crisis level. In equilibrium, the real return on money (plus the liquidity premium) must follow the real return on capital. The endogenous money supply declines gradually. Deflation must realize to ensure this condition.

2.6.5 Interest on Reserves (IOR) as Monetary Policy Tool

IOR: To raise or not to raise?

In the previous section, we know that after the LSAP program without adjusting $R^n_t$, the inflation - the central bank’s main target - is high in the short run but below the target in the long-run. How long should the central bank keep the federal funds rate at the zero lower bound? And if the central bank decides to raise rate, what is the best strategy for the central bank?

In this section, we still conduct the experiment similar to the previous section with one twist. We assume that after $T_u$ periods, the central bank will raise IOR and after $T_d$ periods, IOR will be brought back to the initial level. We choose the different level for $T_u$ at 20, 40 and 80 quarters to see the effect of the prolonged low interest rate environment on output and inflation in the short run and long run. $T_d$ is chosen at 200 quarters.

$$
\kappa_t = \rho \kappa_{t-1} + (1 - \rho) \bar{R}, \quad \kappa_0 \text{ is given}
$$

$$
x_t = \rho x_{t-1}, \quad x_0 \text{ is given}
$$

$$
\hat{\tau} = 0, \quad \forall t \geq 0
$$

$$
R_t = \begin{cases} 
\bar{R} & \text{if } t < T_u \\
1/\beta & \text{if } T_u \leq t \leq T_d \\
\bar{R} & \text{if } t > T_d
\end{cases}
$$

Here are some remarks from our experiments: (Figure 2.5)

i. The longer is the duration of the federal funds rate at the lower bound, the higher is inflation in the short run. This forward guidance effect is well-documented in the New Keynesian
(a) Federal funds rate

(b) Real rate of borrowing

(c) Real balance of money supply

(d) Outputs

(e) Inflation - Short run

(f) Inflation - Long run

Figure 2.5: Raise IOR at different time horizons (P4)
literature when the central bank commits to set the short-term at the zero lower bound for a long time (Eggertsson and Woodford (2003)). However, the hyperinflation never happens in our model even with 20 years that rate is set at the lower bound. Due to the household’s borrowing constraint and banker’s capital constraint, the amount of the money supply is restricted even with the huge amount of excess reserves in the banking system.

ii. The longer is the duration of the federal funds rate at the lower bound, the bigger is the negative effect on output and deflation in the long run. It emphasizes that our model is Keynesian in the short run, but Neo-Fisherian in the long run.

iii. The endogenous money supply drops sharply when the central bank raises rates. As price is sticky, the real fed funds rate and real lending rate must go up after this rate hike. Hence, the total of amount of bank credits declines, also implying a huge fall in the money supply. However, after some periods, the neo-Fisherian effect dominates the Keynesian effect, stabilizing inflation at the target level. After all, the central bank still needs to pay a big cost for a rate hike in the short run.

The last point implies an important hint for monetary policy when the central bank decides to raise rate. The central bank can still stabilize inflation and the aggregate demand if it commits to a rule of targeting the money supply at the time of raising rates. The appearance of interest on reserves and electronic payment system allow the central bank to manipulate both the money supply and interest rate at the short run, which is very different from Keynesian theory with only paper money. In this sense, our research is very near to the Monetarism when the growth rate of the money supply always decides the inflation path in the long run.

**Raise rate and raise money supply - Money Supply Rule**

We do an experiment similar to (P4) but at the time of raising IOR, the central bank also commits to a money supply rule (massive helicopter money if necessary) to target the inflation rate. The money supply rule simply responds to the deviation of the inflation rate from its target:

\[
\frac{M_t}{M_{t-1}} = \left(\frac{\bar{\pi}}{\pi_t}\right)^{\rho_m}
\]

(2.38)
where $\rho_m = 0.5$ is the coefficient showing how much the central bank will change the growth rate of the money supply in response to inflation.

To create the same interest path like the previous section, we assume this money supply rule only applies since the time the central bank decides to raise rates. The complete list of exogenous shocks and monetary policy for this experiment can be written as follows:

$$\kappa_t = \rho_x \kappa_{t-1} + (1 - \rho_x) \bar{\kappa}, \quad \kappa_0 \text{ is given}$$

$$x_t = \rho_x x_{t-1}, \quad x_0 \text{ is given}$$

$$\begin{cases} \hat{\kappa}_t = 0 & \text{if } t < T_u \\ \log(m_t) - \log(m_{t-1}) = -(1 + \rho_m) \log(\pi_t) & \text{if } t \geq T_u \end{cases} \quad (P5)$$

$$R^n_t = \begin{cases} \bar{R} & \text{if } t < T_u \\ \frac{1}{\beta} & \text{if } T_u \leq t \leq T_d \\ \bar{R} & \text{if } t > T_d \end{cases}$$

Figure 2.6, by comparing (P5) to (P4), shows the effectiveness of combining raise rate with the rule of targeting money supply:

i. Even though the federal funds rate paths are nearly identical in the first 200 periods in our experiments, the dynamics of output and inflation are very different. It implies that interest rate path does not give enough information for the stance of monetary policy when central bank use IOR as the main tool. When there is no excess reserves, federal funds rate path conveys all information about monetary policy. It is not this case with the current situation, when the central bank can manipulate both money supply and interest rate.

ii. Money supply targeting is extremely efficient in stabilizing inflation and output. The inflation is anchored at the target rate since the time the central bank target the growth rate of the money supply in our model.

iii. At the time of raising rate (period 20), to stabilize the inflation and avoid a severe short recession, money supply targeting implies that the central bank should conduct a massive helicopter money. With this commitment, the central bank can anchor the household’s expectation about inflation path and get out of the dilemma to raise or not to raise rate.
Figure 2.6: Raise IOR with (P5) and without (P4) the money supply rule after 20 periods
2.7 Conclusion

Our research shows that, when the central bank controls the federal funds rate by adjusting interest on reserves, the interest path does not provide full information on the stance of monetary policy. The endogenous money supply can complete go down when the federal funds rate is near zero for a long time. However, if the central bank simply raises rate, the economy will fall into a short recession and deflation is worse in the short run. Basically, the central bank falls into a dilemma to raise or not to raise rate, where outcome is not bright in either way.

One feasible solution for the central bank is to target the growth rate of the money supply in response to inflation when they raise rates. With that, they can completely avoid the negative short term effect and do a better job in hitting the inflation target.
Chapter 3

Monetary Policy of Targeting both Interest Rate and Money Supply

3.1 Introduction

In the Great Recession, the federal funds rate was near zero; however, the deflation pressure was still high as banks cut loans. That phenomenon raised many concerns among academics and policy makers on how the central bank’s policy should be designed when the interest rate channel was weak. Possible solutions for pushing up inflation in this circumstance are forward guidance, helicopter money and quantitative easing (Bernanke, 2016c,b). The last two tools can change the money supply directly (rather than indirectly through a money creation process by banks); however they need to be integrated with standard monetary policy frameworks and cannot be a daily tool like an interest rate targeting policy. This chapter argues that the central bank can utilize both instruments, interest rate and money supply, to do a better job at hitting the inflation target in a banking crisis.

The common traditional consensus among economists is that the central bank cannot target both interest rate and money supply at the same time. The central bank chooses either the monetary base as its main instrument (Meltzer, 1987; McCallum, 1988; Friedman, 1960) or the common interbank rate (Taylor, 1993). However, the introduction of a new monetary policy tool - interest on reserves (IOR) - and the transformation of the economy from making transactions with currency to with demand deposits will allow the central bank to use both above instruments.
With IOR, the price of reserves might disconnect from the quantity of reserves in the banking system. With demand deposits, money can earn a positive nominal interest rate. Therefore, it is a possible scenario that the central bank can increase the money supply and raise IOR at the same time.

This chapter builds a dynamic model with bank reserves, currency, and demand deposits. The monetary base in our model is controlled by the central bank, while the money supply is determined by the interactions between the central bank, banks and the public. The interbank rate is controlled either by open market operations or by adjusting IOR, so a wide range of conventional and unconventional monetary policy can be assessed in this model.

We find that in normal times, an interest rate policy following a Taylor rule is a transparent and effective means of controlling the economy. When the central bank cuts rates, the amount of the money supply increases because banks create more loans. As the price is sticky in our model, the economy is stimulated due to the rise in the aggregate demand. The effect is identical to the standard New Keynesian model.

However, in banking crises, when banks cannot make loans due to capital constraints, a policy following a Taylor rule is insufficient for pushing the real interest rate down. Even with the negative IOR or forward guidance, the outcome is only slightly better. The main reason is that inflationary expectations also depend on the path of the money supply. In the case of banking crisis, the endogenous money supply declines. If the central bank does not inject liquidity directly into the market, the deflation pressure will be huge. Because of deflation and the wedge created by capital constraints, the real prime rate will be high even though the interbank rate touches the zero lower bound.

Targeting both the money supply and the interest rate is very efficient in this situation. We find that the central bank only needs to follow a simple Taylor rule and a Friedman’s k-percent rule so that both output and inflation will be stabilized. After the crisis time, the central bank can always come back to a simple Taylor rule.
Related Literature

Our model shares many similarities with the standard New Keynesian framework with the existence of the banking sector\(^1\). Banks play a role of intermediaries channeling funds from savers to borrower in these models; while ours focuses on the function of creating money in the banking sector (McLeay, Radia and Thomas, 2014). Banking crises create liquidity problem for agents in the private sector as the money supply declines. In this sense, our model is identical to Benigno and Nisticò (2017), where a shock worsening the liquidity of pseudo-safe assets can create a crisis with a persistent deflation.

The money supply in our model is endogenous. The interest rate channel and the credit channel are interdependent. In this aspect, our model is related to Bianchi and Bigio (2014) and Afonso and Lagos (2015), where these authors apply the search and matching theory to study the banking sector. On the other hand, our banking sector is perfectly competitive and frictionless, so we can focus more on the impact of monetary policy on output and inflation. On the money demand side, we follow the cash-in-advance literature in (Lucas and Stokey, 1987; Belongia and Ireland, 2006, 2014), so households hold currency and bank deposits as they provide the liquidity. Cash goods and deposit goods are bundled together in a constant elasticity of substitution utility function.

To target both the money supply and interest rate, the central bank has to use IOR. This tool is already mentioned in the literature (Sargent and Wallace, 1985; Goodfriend et al., 2002; Kashyap and Stein, 2012; Ireland, 2014; Cochrane, 2014; Keister, 2016). Our model, different from this line of research, can connect IOR with banking reserves, money supply and the inter-bank rate in a micro-founded dynamic setup.

We also discuss two unconventional monetary policies: negative IOR and forward guidance. Analysis of monetary policy with a negative interest rate can be found in Rognlie (2015). Our model emphasizes the transmission through a negative IOR to the interbank rate and the deposit rate while the framework in Rognlie (2015) assumes directly that the central bank can impose a negative short term rate. In both papers, the negative rate might be an important tool when the interbank rate is near zero. We also examine the effect of the central bank’s forward guidance.

policy\(^2\) in a banking crisis context.

The rest of the chapter is divided into six parts. Section 3.2 and 3.3 describe the model. Section 3.4 and 3.5 study the equilibrium conditions as well as some theoretical results of this model. Section 3.6 performs some experiments to assess the efficacy of different policies when banking crises happen. Section 3.7 gives the conclusion.

### 3.2 The Environment

#### 3.2.1 Goods and Production Technology

Our model extends the previous model in the chapter 2 to the environment where currency and demand deposits coexist. There are four types of goods in the economy: cash-goods \(y_{1,t}\) produced by \(c\)-retailers who only accept currency as the mean of payment, deposit-goods \(y_{2,t}\) produced by \(d\)-retailers who only accept payment through banks, wholesale goods \(y_{t}(j)\) produced by wholesale firm \(j\) and intermediate good \(y_{t}^{m}\) produced by households.

Each period households self-employ their labor \(l_{t}\) to produce the homogeneous intermediate good \(y_{t}^{m}\) under the production function:

\[
y_{t}^{m} = l_{t}
\]

Households sell \(y_{t}^{m}\) to wholesale firms in the competitive market with the price \(P_{t}^{m}\).

There is a continuum of wholesale firms indexed by \(j \in [0, 1]\). Each wholesale firm purchases the homogeneous intermediate good \(y_{t}^{m}\) from households and differentiates it into a distinctive wholesale goods \(y_{t}(j)\) under the following technology:

\[
y_{t}(j) = y_{t}^{m}
\]

Wholesale firms faces the Rotemberg adjustment cost when they change their prices.

Two types of retail firms both produce the final good \(y_{i,t}\) \((i = 1, 2)\) by aggregating a variety

\(^{2}\text{See (Eggertsson et al., 2003; Levin et al., 2009; Del Negro, Giannoni and Patterson, 2012; Campbell et al., 2012; Keen, Richter and Throckmorton, 2017a)}\)
of differentiated wholesale goods $y_i(t)$:

$$y_{i,t} = \left( \int_0^1 y_t(j) \frac{e^{-1}}{e} d j \right)^{\epsilon-1}, \ i = 1, 2$$

As the markets for cash-goods and deposit-goods are perfectly competitive and they have the same constant return to scale production function, they have the same price $P_t$.

### 3.2.2 Time, Demographics and Preferences

Time is discrete, indexed by $t$ and continues forever. The model is in the deterministic setting and has six types of agents: bankers, households, wholesale firms, two types of retail firms, and the consolidated government.

There is a measure one of identical infinitely lived bankers in the economy. Bankers discount the future with the discount factor $\beta$. Each period, they gain utility from consuming a basket $c_t$ that contains cash-goods $c_{1,t}$ and deposit-goods $c_{2,t}$. Their utility at the period $t$ can be written as:

$$\sum_{s=0}^{\infty} \beta^s \log(c_{t+s}), \text{ with } c_t = \left[ \sum_{i=1}^{2} \alpha_i \frac{1}{\sigma_i} c_{i,t}^{\sigma_i-1} \right]^{\frac{1}{\sigma_i-1}}$$

where $\alpha_i$ is the share of cash-goods or deposit-goods in the basket and $\sigma$ is the elasticity of substitution between two goods in the basket.

There is also a measure one of identical infinitely lived households. Households discount the future with the discount factor $\tilde{\beta} < \beta$, so they will borrow from bankers in the steady state. Similar to bankers, each period households gain utility from consuming the basket $\tilde{c}_t$ and lose utility when providing labor $l_t$ to their own production. Household’s utility at the period $t$ can be written as:

$$\sum_{s=0}^{\infty} \tilde{\beta}^s \left[ \log(\tilde{c}_{t+s}) - \chi l_{t+s} \right], \text{ with } \tilde{c}_t = \left[ \sum_{i=1}^{2} \alpha_i \frac{1}{\sigma_i} c_{i,t}^{\sigma_i-1} \right]^{\frac{1}{\sigma_i-1}}$$

where $\chi$ is the weight of labor in the utility function.

Wholesale firms, retail firms are infinitely lived, owned by households.

The consolidated government includes both the government and the central bank, so for convenience, we assume there is no independence between the government and the central bank.
3.2.3 Assets

There are two types of financial assets: bank loans to households $B^h_t$ and interbank loans $B^f_t$.

(a) **Bank loans to households ($B^h_t$)**: We follow Leland and Toft (1996) and Bianchi and Bigio (2014) to model the loan structure between bankers and households. The market for bank loan is perfectly competitive and the price of loan is $q^l_t$. When a household wants to borrow 1 dollar at time $t$, bankers will create an account for her and deposit $q^l_t$ dollars to her account. In the exchange for that, this household promises to pay $\delta^b_1$, $\delta^b_1(1 - \delta^b)$, $\delta^b_1(1 - \delta^b)^n$, ... dollars at time $t+1$, $t+2$, ..., $t+n$, $t+n+1$... where $n$ runs to infinity and $0 < \delta^b \leq 1$ (Table 3.1). Loans are illiquid and bankers cannot sell loans.

Let $B^h_t$ be the nominal balance of loan stock in the period $t$, let $S_t$ be the nominal flow of new loan issuance, we have:

$$B^h_t = (1 - \delta^b)B^h_{t-1} + S_t$$

(b) **Interbank loan ($B^f_t$)**: Bankers can borrow reserves from other bankers in the interbank market. The nominal gross interest rate in the interbank market is the interbank rate $R^f_t$.

3.2.4 Money and Payment System

There are three types of money in our economy: currency $x_t$, zero-maturity deposits $m_t$ and reserves $n_t$.

(a) **Currency ($x_t$)**: is issued by the central bank and held by households. If currency is held by bankers, it is automatically converted to reserve. Currency is used for transactions between households/bankers and c-retailers who sell cash-goods. Currency does not pay nominal interest. The amount of currency in circulation is endogenous in the equilibrium.
(b) **Zero maturity deposit (ZMD) \((m_i)\):** is a type of e-money issued by bankers. ZMDs have the same unit of account as currency. When holding these deposits, households earn the gross nominal interest rate \(R^m_t\) which is determined by the perfectly competitive banking market. ZMDs are used for settling all transactions in the private sector, except for transactions between households/bankers and c-retailers. When the market between bankers and households open, household can convert ZMDs to currency or currency to ZMDs. They are insured by the central bank, so they are totally safe.

(c) **Reserve \((n_t)\):** is a type of e-money issued by the central bank for only bankers. It has the same unit of account with currency. The central bank pays the gross interest rate \(R^n_t\) for these reserves. Interest on reserves \(R^n_t\) is a monetary policy tool of the central bank. At any moment, bankers can convert these reserves to currency and pay households. Reserves are used for settling transactions between bankers and bankers, bankers and the consolidated government.

Transactions with currency are simple. However, transactions with zero maturity deposits relates to many parties that we can see in the chapter 2.

### 3.2.5 Central Bank and Monetary Policy

The central bank always uses the electronic payment system to conduct monetary policy. Each period, the central bank transfers \(\tau_t\) dollars in checks to households\(^3\). For any transactions between the central bank and households, as the payments are conducted through the banking system, we should think that they contain two sub-transactions: one between the central bank-bankers is settled by reserves, one between bankers-households is settled by ZMDs.

<table>
<thead>
<tr>
<th>The Central Bank</th>
<th>Banks</th>
<th>Public</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves:+(\tau_t)</td>
<td>Reserves:+(\tau_t)</td>
<td>Deposits:+(\tau_t)</td>
</tr>
<tr>
<td>Net worth:-(\tau_t)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: Helicopter Money / Lump-sum tax

---

\(^3\)This can be seen as a shortcut of an open market operation process when the central bank purchases government bonds from the government. Then the government transfers the payoffs to households. When \(\tau_t\) is negative, it is equivalent to a lump-sum tax.
3.2.6 Timing within one period

(i) Production takes place. Households sell goods to wholesalers, who, in turn, sell goods to retailers. All of the payments between households-wholesalers, wholesalers-retailer are delayed until the step (v).

(ii) The cash-good market opens. Households need cash-in-advance to purchase from c-retailers. Bankers can convert reserves to cash to purchase from c-retailers.

(iii) The loan market between households and bankers opens. All the debt payments and loan issuance will be conducted electronically. The government transfers money to households. Households cannot exchange cash and deposits in this step.

(iv) The deposit-good market opens. Households need ZMD-in-advance to purchase goods from d-retailers. Bankers can create ZMDs to purchase d-goods.

(v) Payments in the non-bank private sector are settled. Profits from firms are transferred back to households under either form of cash or ZMDs. Households can go to banks and readjust their portfolio between cash and deposits.

(vi) The banking market opens. Bankers can adjust the level of reserves by borrowing in the interbank market, receiving new deposits.

3.3 Agents’ Problems

3.3.1 Bankers

There is a measure one of identical bankers in the economy. These bankers have to follow the central bank’s regulations. There are three types of assets on a banker’s balance sheet: reserves ($n_t$), loans to households ($b^h_t$), loans to other bankers ($b^f_t$). His liability side contains the zero-maturity deposits that households deposit here ($m_t$).

Cost: The banker faces a cost of managing loan, which is $\theta b^h_t$ in terms of deposit-goods.

The banker can adjust the level of his deposits and reserves after households and firms pay each other or when households withdraw currency from bank account. When these happen, the banker can witness that the deposits and reserves outflow from or inflow to his bank. Let $e_t$ be
the net inflow of deposits and reserves go into his bank, he will treat \( e_t \) as exogenous. When the 
banking market opens, as the deposit market is perfectly competitive, he can choose any amount 
\( d_t \) of deposit inflows or outflows to his bank.

In each period, the banker treats all the prices as exogenous and chooses \( \{c_t, c_{1,t}, c_{2,t}, n_t, b^h_t, s_t, m_t, b^f_t, d_t\} \) to maximize his utility over a stream of consumption:

\[
\max \sum_{t=0}^{\infty} \beta^t \log(c_t)
\]

subject to

\[
\begin{align*}
\frac{R^n_{t-1}}{\pi_t}n_{t-1} + \frac{R^f_{t-1}}{\pi_t}b^f_{t-1} + d_t + e_t + \tau_t &= n_t + b^f_t + c_{1,t} \quad \text{(Reserve Flows)} \quad (3.1) \\
\frac{m_t}{{R}_t} &= \frac{q_t}{\pi} + q_t + \theta b^h_t - \delta b^h_t - \delta b^f_t + c_{2,t} + d_t + e_t + \tau_t \quad \text{(Deposit Flows)} \quad (3.2) \\
b^h_t &= (1 - \delta b^h_t) - b^h_t + s_t \quad \text{(Loan Flows)} \quad (3.3) \\
c_t &= \left[ \sum_{i=1}^{2} \alpha^i \right]^{\frac{1}{\sigma}} \quad \text{(Consumption)} \quad (3.4) \\
n_t &\geq \varphi m_t \quad \text{(Reserves Requirement)} \quad (3.5) \\
n_t + b^f + b^h_t - m_t &\geq \kappa b^h_t \quad \text{(Capital Requirement)} \quad (3.6)
\end{align*}
\]

The equation (3.1) shows the change in reserves in the banker’s balance sheet. After receiving 
the IOR, the previous balance of reserves becomes \( \frac{R^n_{t-1}}{\pi_t}n_{t-1} \). He also collects the payment from the interbank loans he lends out to other bankers in the previous period \( \frac{R^f_{t-1}}{\pi_t}b^f_{t-1} \). He can also increase his reserves by taking more deposits \( d_t \). When doing that, his reserves and 
his liability increase by the same amount \( d_t \) (Table 3.3). That is the reason we see \( d_t \) appear on 
both the equation (3.1) and (3.2). The similar effect can be found on \( \tau_t \)- helicopter money and
Table 3.3: The banker takes more deposits (left) and makes interbank loan (right)

<table>
<thead>
<tr>
<th>Banker</th>
<th>Banker</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves: $+d_t$</td>
<td>Deposits: $+d_t$</td>
</tr>
<tr>
<td>Interbank loan: $+b_t^f$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4: Banker buys goods from c-retailers

<table>
<thead>
<tr>
<th>Central Bank</th>
<th>Banker</th>
<th>c-Retailers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves: $-c_{1,t}$</td>
<td>Reserves: $-c_{1,t}$</td>
<td>Net worth: $-c_{1,t}$</td>
</tr>
<tr>
<td>Currency: $+c_{1,t}$ (Vault Cash)</td>
<td></td>
<td>Currency: $+c_{1,t}$</td>
</tr>
<tr>
<td>Inventory: $-c_{1,t}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.5: Banker buys goods from d-retailers

<table>
<thead>
<tr>
<th>Banker</th>
<th>d-Retailers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deposits: $(c_{2,t} + \theta b_t^h)$</td>
<td>Deposits: $(c_{2,t} + \theta b_t^h)$</td>
</tr>
<tr>
<td>Net worth: $-(c_{2,t} + \theta b_t^h)$</td>
<td>Inventory: $-(c_{2,t} + \theta b_t^h)$</td>
</tr>
</tbody>
</table>

$e_t$. The banker treats $\tau_t$ and $e_t$ exogenously. Then, he can leave reserves $n_t$ at the central bank’s account to earn interest rate, or lend reserves to another bankers $b_t^f$. To purchase the cash-goods $c_{1,t}$ from c-retailers, he converts reserves into currency (Table 3.4).

The equation (3.2) shows the change in the banker’s deposits. He makes loans to households by issuing deposits or creating ZMDs (Table 3.1). The banker also issues his own ZMDs to purchase the consumption good from d-retailers ($c_{2,t}$) and to pay for the cost (in terms of deposit-goods) related to lending activities ($\theta b_t^h$) (Table 3.5). It is noted that he cannot create infinite amount of money for himself to buy consumption goods as there exists the capital requirement. Even without the capital requirement, because deposits are bankers’ debts, the No-Ponzi condition is enough to prevent that from happening.

The banker faces two constraints in every period. At the end of each period, he has to hold enough reserves as a fraction of total deposits, which is showed in the inequation (3.5).

---

4. During one period, his reserves balance can go temporary negative. But in the end of every period, it must be positive and satisfies the regulation. Hence, the constraint in purchasing cash-goods implicitly lies in the reserve requirement.

5. It is assumed that households have to pay loans from the account at the bank they borrow. So if they want to use money from account at bank B to pay for loans from bank A, they need to transfer deposits from bank B to bank A first. In fact, this assumption does not matter in equilibrium.
The second constraint is the capital requirement constraint. The left hand side of (3.6) is the banker’s net worth (capital), which is equal to total assets minus total liabilities. The constraint requires the banker to hold capital greater than a fraction of total loans in his balance sheet. We assume that \( \kappa_t \) is a constant \( \kappa \) in normal times. We later put the unexpected shock on this \( \kappa_t \) to reflect the shock in a banking crisis.

Let \( \gamma_t, \mu^r_t \) and \( \mu^c_t \) be respectively the Lagrangian multipliers attached to the reserves flows, reserves constraint and the capital constraint. Let \( r^h_t \) be defined as the real short-term lending rate. The first order conditions of the banker’s problem can be written as:

\[
\gamma_t = \left( \frac{\alpha_t c_t}{c_{i,t}} \right)^{1/\sigma} \frac{1}{c_{i,t}}, \quad i = 1, 2 \tag{3.7}
\]

\[
\gamma_t = \frac{\beta R^f_t \gamma_{t+1}}{\pi_{t+1}} + \mu^c_t \tag{3.8}
\]

\[
\gamma_t = \frac{\beta R^m_t \gamma_{t+1}}{\pi_{t+1}} + \mu^c_t + \varphi \mu^{r'}_t \tag{3.9}
\]

\[
\gamma_t = \frac{\beta R^n_t \gamma_{t+1}}{\pi_{t+1}} + \mu^c_t + \mu^{r'}_t \tag{3.10}
\]

\[
(q^l_t + \theta) \gamma_t = \frac{\beta [\delta_b + (1 - \delta_b)q^l_{t+1}] \gamma_{t+1}}{\pi_{t+1}} + (1 - \kappa) \mu^c_t \tag{3.11}
\]

\[
r^h_t = \frac{\delta_b + (1 - \delta_b)q^l_{t+1}}{(q^l_t + \theta)(\pi_{t+1})} \tag{3.12}
\]

And two complimentary slackness conditions:

\[
\mu^r_t \geq 0, \quad n_t - \varphi m_t \geq 0, \quad \mu^c_t (n_t - \varphi m_t) = 0 \tag{3.13}
\]

\[
\mu^c_t \geq 0, \quad n_t + b^f_t + (1 - \kappa_t)b^h_t - m_t \geq 0, \quad \mu^c_t (n_t + b^f_t + (1 - \kappa_t)b^h_t - m_t) = 0 \tag{3.14}
\]

### 3.3.2 Households

There is a measure one of identical households. These self-employed households produce the homogeneous intermediate good \( y^m_t \) to sell to the wholesale firms at the price \( P^m_t \). In each period, a household consumes the cash-goods \( (\tilde{c}_1, t) \) from c-retailers and the deposit-goods \( (\tilde{c}_2, t) \) from d-retailers.

---

\(^6\)We use the book value \( B^h_t \) rather than the “market value” of loans \( q^l_t B^h_t \) in the capital constraint. The reason is that illiquid bank loans should be treated differently from bonds. In reality, bank loans are often not revalued in the balance sheet when the interest rate changes.
Let \( \tilde{B}_h \) be the nominal debt stock that she borrows from bankers. The loan structure follows the description in Table (3.1). There is an exogenous borrowing constraint for households with the debt limit \( \tilde{b}_h \leq b^h \).

In each period, households choose \( \{\tilde{c}_t, l_t, a_t, x_t, \tilde{b}_h, \tilde{m}_t, \tilde{s}_t, \tilde{c}_{1,t}, \tilde{c}_{2,t}\} \) to maximize their expected utility:

\[
\max \sum_{t=0}^{\infty} \tilde{\beta}^t \left( \log(\tilde{c}_t) - \chi l_t \right)
\]

subject to

**CIA:**

\[
\tilde{c}_{1,t} \leq \frac{x_{t-1}}{\pi_t}
\]  

(3.15)

**Loan Market:**

\[
a_t + \delta_b \frac{\tilde{b}_h^{t-1}}{\pi_t} = \frac{R^{m}_{t-1} m_{t-1}}{\pi_t} + q_l s_t + \tau_t
\]

(3.16)

**DIA:**

\[
\tilde{c}_{2,t} \leq a_t
\]

(3.17)

**Budget:**

\[
m_t + x_t + \tilde{c}_{1,t} + \tilde{c}_{2,t} = a_t + \frac{x_{t-1}}{\pi_t} + p^m y^m_t + \frac{\Pi_t}{P_t}
\]

(3.18)

**Production:**

\[
y^m_t = l_t
\]

(3.19)

**Loan flows:**

\[
\tilde{b}_h^t = (1 - \delta_b) \frac{\tilde{b}_h^{t-1}}{\pi_t} + \tilde{s}_t
\]

(3.20)

**Constraint:**

\[
\tilde{b}_h^t \leq \tilde{b}^h
\]

(3.21)

**Consumption:**

\[
\tilde{c}_t = \left[ \sum_{i=1}^{2} \alpha^\frac{1}{\sigma} \tilde{c}_{i,t}^{\frac{\sigma-1}{\sigma}} \right]^\frac{\sigma}{\sigma-1}
\]

(3.22)

When the cash-good market opens, the household brings \( (x_{t-1}/\pi_t) \) in cash to make transactions there. She faces the cash-in-advance (3.15) constraint when purchasing goods from c-retailers.

The loan market between bankers and households (3.16) only opens after that. Here the household pays a fraction of her old debts \( (\delta_b b_h^{t-1}/\pi_t) \) and borrows new loan \( (q_l s_t) \). All of the transactions are conducted electronically. We have assumed that she cannot readjust her portfolio between cash and deposits in this step. In the end, she brings \( a_t \) amount of ZMDs to purchase goods from d-retailers.

The equation (3.18) is the household’s general budget constraint. After receiving the profits \( (\Pi_t/P_t) \) from wholesalers and revenue \( (p^m_t, y^m_t) \) from selling the intermediate good, she can go to
banks and readjust her portfolio between deposits \((m_t)\) and currency \((x_t)\).

Let \(\eta_{1,t}, \eta_{2,t}, \eta_t^b\) be the Lagrangian for the cash-in-advance, the deposit-in-advance and the borrowing constraint. Let \(\lambda_t\) be the Lagrangian for the budget constraint.

\[
\lambda_t + \eta_{i,t} = \left( \frac{\alpha_i \bar{c}_t}{\bar{c}_t} \right)^{1/\sigma} \frac{1}{\bar{c}_t}, \quad i = 1, 2
\]  
(3.23)

\[
p_t^m \lambda_t = \chi
\]  
(3.24)

\[
\lambda_t = \frac{\bar{\beta}(\lambda_{t+1} + \eta_{1,t+1})}{\pi_t + 1}
\]  
(3.25)

\[
\lambda_t = \frac{\bar{\beta} R_t^m (\lambda_{t+1} + \eta_{2,t+1})}{\pi_t + 1}
\]  
(3.26)

\[
q_t^l(\lambda_t + \eta_{2,t}) = \frac{\bar{\beta} \delta_t + (1 - \delta_t) q_{t+1}^l (\lambda_{t+1} + \eta_{2,t+1})}{\pi_t + 1} + \eta_t^b
\]  
(3.27)

And three complimentary slackness conditions:

\[
\eta_{1,t} \geq 0, \quad \frac{x_t-1}{\pi_t} - \bar{c}_{1,t} \geq 0, \quad \eta_{1,t} \left( \frac{x_t-1}{\pi_t} - \bar{c}_{1,t} \right) = 0
\]  
(3.28)

\[
\eta_{2,t} \geq 0, \quad a_t - \bar{c}_{2,t} \geq 0, \quad \eta_{2,t} (a_t - \bar{c}_{2,t}) = 0
\]  
(3.29)

\[
\eta_t^b \geq 0, \quad \bar{b}^h - b^h_t \geq 0, \quad \eta_t^b (\bar{b}^h - b^h_t) = 0
\]  
(3.30)

### 3.3.3 Retail Firms and Wholesale Firms

Following Rotemberg pricing, each wholesale firm \(j\) faces a cost of adjusting prices, which is measured in terms of final good and given by:

\[
\frac{\lambda_t}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 y_t
\]

where \(\lambda_t\) determines the degree of nominal price rigidity and \(\bar{\pi}\) is the long-run inflation target. The wholesale firm \(j\) discounts the profit in the future with rate \(\bar{\beta} \lambda_t / \lambda_t\). Her real marginal cost is \(p_t^m\).

In a symmetric equilibrium, all firms will choose the same price and produce the same
quantity \( P_t(j) = P_t \) and \( y_t(j) = y_t \). The optimal pricing rule then implies that:

\[
1 - \bar{\iota} \left( \pi_t - \bar{\pi} \right) \pi_t + t \bar{\beta} \left[ \frac{\lambda_{t+1}}{\lambda_t} \left( \pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right] \frac{y_{t+1}}{y_t} = \left( 1 - p^n_t \right) \varepsilon
\]  

(3.31)

### 3.3.4 The Central Bank and Government

The consolidated government uses the payoffs from tax or their assets to pay for the IOR, then injects (withdraws) \( \hat{\tau}_t \) amount of money to (from) households to target the interbank rate. All transactions are conducted in the electronic system.

\[
\tau_t = -\frac{(R^n_t - 1)n_{t-1}}{\pi_t} + \hat{\tau}_t
\]  

(3.32)

In the conventional monetary policy, we assume that the IOR \( R^n_t \) is fixed at a constant level \( \bar{R}^n \). The interbank rate follows a common Taylor rule. To connect with the common New Keynesian literature, we assume that the central bank do not want to have excess reserves in the banking system so they never set \( R_f^t \) lower than \( \bar{R}^n + \delta_f \) where \( \delta_f > 0 \). Later, we relax the assumption and examine the situation when the banking system is awash of excess reserves and the central bank controls the interbank rate by adjusting \( R^n_t \).

The conventional monetary policy rule can be described as:

\[
R^n_t = \bar{R}^n
\]  

(3.33)

\[
R_f^t = \max \left\{ \frac{\pi_t}{\bar{\beta}} \left( \frac{\pi_{t+1}}{\bar{\pi}} \right)^{\phi_{\pi}}, \bar{R}^n + \delta_f \right\}
\]  

(3.34)

### 3.4 Equilibrium

**Definition:** A perfect foresight equilibrium is a sequence of bankers’ decision choice \( \{ c_t, c_{i,t}, n_t, b^h_t, s_t, m_t, b^f_t, d_t \} \), household’s choice \( \{ \tilde{c}_t, \tilde{c}_{i,t}, \tilde{b}^h_t, \tilde{s}_t, \tilde{m}_t, x_t, l_t, y^m_t \} \), the firms’ choice \( \{ y_t \} \), the central bank’ choice \( \{ \tau_t, R^n_t \} \), and the market price \( \{ q_t, R^f_t, \pi_t, p^n_t \} \) such that:

1. Given the market price, the initial conditions and the central bank’s choices, banker’s

---

\(^7\)When the reserve requirement is no longer binding, there are infinite levels of reserves that can satisfy the interbank rate at its lower bound. In this case, we need a rule governing the motion of reserves and change the standard Taylor Rule.
choices solve the banker’s problem, household’s choices solve the household’s problem, firm’s choice solves the firm’s problem.

ii All markets clear:

Net inflows of deposits: \( d_t + e_t = -(x_t - \frac{x_{t-1}}{\pi_t} - c_{1,t}) \)  \hspace{1cm} (3.35)

The interbank market: \( b_t^f = 0 \)  \hspace{1cm} (3.36)

Total ZMDs: \( m_t = \tilde{m}_t \)

Loan Market: \( b_t^h = \tilde{b}_t^h \)

Good Market: \( y_t = \sum_{i=1}^{2}(c_{i,t} + \tilde{c}_{i,t}) + \theta b_t^h + \frac{1}{2}(\pi_t - \bar{\pi})^2 y_t \)  \hspace{1cm} (3.37)

Later, we set some different central bank’s monetary policies subject to a set of equations in this perfect foresight equilibrium. In each case, we might also change the set of the central bank’s monetary policy tools. For convenience, we define \((AD)\) as the set of equations in the perfect foresight equilibrium, excluding the monetary policy and exogenous shocks.

**Definition:** Let \((AD)\) contain the set of equations and conditions in \((B.1.1)-(B.1.24)\).

### 3.5 Theoretical Results

We make the following assumption to ensure that in the steady state, households borrow from bankers.

**Assumption 3.1.** The discount factors of bankers and households satisfy:

\[
\frac{\beta \delta_b - \theta \pi}{\pi - \beta (1 - \delta_b)} > \frac{\tilde{\beta} \delta_b}{\pi - \tilde{\beta} (1 - \delta_b)}
\]

The next assumption ensures that in the steady state, inflation is equal to the central bank’s inflation target.

**Assumption 3.2.** The monetary policy tools satisfy:

\[
\lim_{t \to \infty} \frac{\hat{c}_t}{n_t + x_t} = \frac{\pi - 1}{\pi}
\]
\[ R^n + \delta_f < \frac{\pi}{\beta} \]

We start with the first result showing the relationship between the interest on reserves, the deposit rate and the interbank rate.

**Theorem 3.1.** In equilibrium:

i. The lower bound of the interbank rate and the deposit rate is the interest on reserves. In all cases, \( R^n_t \leq R^m_t \leq R^f_t \)

ii. When the constraint of reserve requirement is not binding, \( R^f_t = R^m_t = R^n_t \).

The benefits of holding reserves come from two sources. First, bankers earn the interest on reserves. Second, bankers satisfy the reserve requirement, showing in the shadow price of reserve constraint \( \mu^r_t \geq 0 \). When the banking system has a huge amount of excess reserves, second benefit is no longer there \( \mu^r_t = 0 \), and the interbank rate is equal to the interest on reserves.

**Theorem 3.2.** In equilibrium, the level of the monetary base, as the sum of reserves and currency in circulation, is decided solely by the central bank:

\[
\frac{n_{t-1} + x_{t-1}}{\pi_t} + \hat{\tau}_t = n_t + x_t \quad (3.38)
\]

When households withdraw currency from their bank accounts, it only changes the level of reserves but does not affect the level of monetary base. We assume that the central bank will target the interbank rate, so it implies that the central bank will never leave the banking system with the negative amount of reserves.

**Theorem 3.3.** Under the Assumption (3.1)-(3.2) and if \( \kappa \) satisfies:

\[
\kappa < 1 - \frac{(1 - \phi)\bar{m}}{\bar{b}^h}
\]

where \( \bar{m} \) is the steady state value of \( m \), then there exists a unique steady state. Moreover, in this steady state, the reserve requirement is binding while the capital constraint is not binding.
Table 3.6: Parameter values

<table>
<thead>
<tr>
<th>Param.</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bankers</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>Banker’s discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>The reserves requirement</td>
<td>0.002</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>The risk weight</td>
<td>0.18</td>
</tr>
<tr>
<td>$\theta$</td>
<td>The monitoring cost</td>
<td>0.0005</td>
</tr>
<tr>
<td>$\delta_b$</td>
<td>Loan amortization</td>
<td>0.5</td>
</tr>
<tr>
<td><strong>Households</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{\beta}$</td>
<td>Household’s discount factor</td>
<td>0.985</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Relative Utility Weight of Labor</td>
<td>0.586</td>
</tr>
<tr>
<td>$\bar{b}$</td>
<td>The borrowing limit</td>
<td>1.5</td>
</tr>
<tr>
<td><strong>Consumption Basket</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>Share of cash goods in the basket</td>
<td>0.06</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>Share of deposit goods</td>
<td>0.94</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Elasticity of substitution between two goods</td>
<td>10</td>
</tr>
<tr>
<td><strong>Firms</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Elasticity of substitution of wholesale goods</td>
<td>4</td>
</tr>
<tr>
<td>$t$</td>
<td>Cost of changing price</td>
<td>80</td>
</tr>
<tr>
<td><strong>Central bank</strong></td>
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<tr>
<td>$\pi$</td>
<td>Inflation long-run target</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_\pi$</td>
<td>Policy responds to inflation</td>
<td>1.25</td>
</tr>
<tr>
<td>$\bar{R}^n$</td>
<td>The constant IOR</td>
<td>1+0.25/400</td>
</tr>
<tr>
<td>$\bar{R}^n + \delta_f$</td>
<td>The lower bound for FFR</td>
<td>1+0.251/400</td>
</tr>
</tbody>
</table>

This unique steady state reflects well the banking system in the US before the Great Recession. There were no excess reserves and the federal funds rate was around 4 percent. The central bank’s main tool was open market operations at that time, rather than the IOR. After the Great Recession, due to many rounds of quantitative easing (unconventional monetary policy), the excess reserves skyrocketed. However, in the long term, the central bank has a plan to scale down its balance sheet’s size to the level before the Great Recession, so this unique steady state might reflect well the long-term position of the central bank.
3.6 Numerical Experiments

3.6.1 Calibration

The time period is one quarter. Data are calibrated to match the US economy before the Great Recession. The bankers’ discount factor is set to the standard value 0.99. The reserves requirement is calibrated to reflect the ratio between the total level of reserves and the total ZMDs. In Dec 2007, before the financial crisis, the total level of reserves was around 9 billion dollars. The total MZM (Money Zero Maturity) was approximately 8130 billion, 75-80 percent of which are checkable deposits, saving deposits and money market deposit accounts. The level of ZMDs was therefore 6000 billion, and the ratio between reserves and ZMD is 0.0015, which we round up to 0.002. The monitoring cost and the loan amortization factor are set exogenously. The risk weight is calibrated so that it satisfies the condition in the Theorem 3.3 to ensure the unique steady state. For $\kappa \geq 0.2$, the capital constraint is binding in the steady state. Therefore, we set $\kappa$ at 0.18.

The consumption basket is calibrated to match the ratio between currency and ZMDs in the economy. In Dec 2007, the total level of currency was 760 billion. Judson (2012) estimates that more than half of US dollar bills are held overseas, so we end up with around 330 billion in currency. At the same time, ZMDs was 6000 billion, so currency accounts for approximately 6 percent of the total money supply (MZM). We calibrate $\alpha_1 = 0.06$ and $\alpha_2 = 0.94$ to match with this fact. The elasticity of substitution between cash goods and deposit goods is set exogenously 10.

For the central bank’s parameters, the only unusual parameter is the interest on reserves. We set it at 25 basis points and consider it as the lower bound for IOR at most cases in our quantitative exercises. All other parameters are in the standard range in the New Keynesian literature.
### Table 3.7: Parameters in Numerical Experiments

<table>
<thead>
<tr>
<th>Param.</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Experiment (M1)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_0^f$</td>
<td>Initial interest shock</td>
<td>$-2/400$</td>
</tr>
<tr>
<td>$\rho_f$</td>
<td>The persistence of the interest shock</td>
<td>0.7</td>
</tr>
<tr>
<td><strong>Experiment (M2)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_0$</td>
<td>Initial shock on the risk weight</td>
<td>0.224</td>
</tr>
<tr>
<td>$\rho_\kappa$</td>
<td>The persistence of shock</td>
<td>0.95</td>
</tr>
<tr>
<td><strong>Experiment (M3)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^n$</td>
<td>The negative lower bound for IOR</td>
<td>1-0.35/400</td>
</tr>
<tr>
<td>$T_e$</td>
<td>Number of periods $R^n_t = R^n$</td>
<td>50</td>
</tr>
<tr>
<td><strong>Experiment (M4)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>The forward guidance signal</td>
<td>$-1/400$</td>
</tr>
<tr>
<td>$T_{FG}$</td>
<td>Number of periods in forward guidance</td>
<td>16</td>
</tr>
<tr>
<td><strong>Experiment (M5)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi_m$</td>
<td>Coefficient in mixed rule</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Experiment (M6)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_n$</td>
<td>Coefficient in Taylor Rule</td>
<td>0.8</td>
</tr>
<tr>
<td>$T_e$</td>
<td>Number of periods targeting both MS and IR</td>
<td>50</td>
</tr>
</tbody>
</table>

### 3.6.2 Shock on the interbank rate

The first numerical experiment is to examine the response of the economy when the central bank cuts the interbank rate. The list of equations for monetary policy and exogenous shock is:

$$
R_t^f = \max \left\{ \frac{\pi}{\beta} \left( \frac{n_{t+1}}{n_t} \right)^{\phi_n} \exp(u_t^f), \quad \overline{R}^n + \delta_f \right\}
$$

$$
R_t^n = \overline{R}^n
$$

$$
u_t^f = \rho_f u_{t-1}^f, \quad u_0^f \text{ is given}
$$

From the steady state, there is an unexpected shock on the interbank rate $u_0^f$, then agents know the shock will die slowly with the persistence $\rho_f$. These two parameters’ values are in Table 3.7. We can summarize this problem as (P1) containing the set of conditions in (AD) and (M1). Figure 3.1 shows the response of the economy under this experiment.

When the central bank cuts the interbank rate by increasing the level of reserves, $q^l$ increases

54
Figure 3.1: Impulse Response to Interest Rate Shock (P1)
and the real lending rate $r^h$ is lower for households. As bankers lend out by creating money under the form of ZMDs, the money supply increases. It is noting that both currency and ZMD go up after this shock. The aggregate demand is stimulated and inflation goes up.

Basically, this is identical to the reaction in the standard New Keynesian model. The only key difference here is the role of commercial banks in creating money. Money supply is totally endogenous and depends on the interaction between the central bank, banks and the public. Another crucial point is that the effect of monetary policy, in this conventional setting, depends on the transmission from the interbank rate to the lending rate in the loan contract between bankers and households.

### 3.6.3 Financial Crisis - Taylor Rule

We examine a simple form of banking crisis by imposing an exogenous shock on $\kappa_t$, reflecting the increase in the bad loans that causes the capital constraint to bind. The central bank is assumed to respond to this crisis using a Taylor Rule.

\[
R_f^t = \max \left\{ \frac{\pi^t}{\beta} \left( \frac{\pi^t+1}{\pi} \right)^{\phi_\pi} \left( R^m + \delta_f \right) \right\}
\]

\[
R_n^t = R^m
\]

\[
\kappa_t = \rho_\kappa \kappa_{t-1} + (1 - \rho_\kappa) \bar{\kappa}, \quad \kappa_0 \text{ is given}
\]

From the steady state, there is an unexpected shock $\kappa_0$. After that, the shock dies with the persistence $\rho_\kappa$. These two parameters’ values are reported in the Table 3.7. We can summarize this problem as (P2) containing the set of conditions in (AD) and (M2).

As the lower bound on the interbank rate is the IOR, the Taylor rule is constrained by the IOR. To see whether the negative IOR helps the central bank, we conduct another similar exper-
ment but allow the interest on reserves is negative during $T_e$ periods.

$$R_i^f = \max \left\{ \frac{\pi}{\beta} \left( \frac{\pi_{t+1}}{\pi} \right)^{\phi_{\pi}} R^n_i + \delta f \right\}$$

$$R_i^n = \begin{cases} R^n_i, & \text{if } t \leq T_e \\ \bar{R}^n_i, & \text{if } t > T_e \end{cases} \quad (M3)$$

$$\kappa_t = \rho_{\kappa} \kappa_{t-1} + (1 - \rho_{\kappa}) \bar{\kappa}, \quad \kappa_0 \text{ is given}$$

According to Bernanke (2016a), the Fed estimates that the interest rate paid on bank reserves in the U.S. could not practically be brought lower than about -0.35 percent to avoid the bank withdrawal. Hence, we set $R^n = 1 - 0.35/400$ and $T_e = 50$. We can set the problem $(P3)$ contains the equations in $(AD)$ and $(M3)$. Figure 3.2 shows the response of the economy under these two experiments. Here are some important remarks:

i. A Taylor rule with a negative IOR is more efficient than the one with zero lower bound. Rognlie (2015) gets the same result from a standard New Keynesian framework. However, the positive effect is very small at dealing with this type of financial crisis.

ii. The banking crisis is dangerous as the central bank cannot rely on the pass-through from the interbank rate to the prime rate any more. In our simulation, the interbank rate is at its lower bound for 12 quarters with the IOR at 25 basis points and 4 quarters with the IOR at -35 basis points; however, the real lending rate still goes up. When the capital constraint is binding $\mu_t^c > 0$, the wedge between the interbank rate and the prime rate must reflect this shadow price.

iii. The banking crisis is often accompanied by the deflation episode and the insufficient demand. Bankers cut loan; therefore, the total money supply plummets even though the monetary base increases. In our model, the level of currency goes up a little bit during the crisis. The deposit rate is near zero or even negative in our two experiments, making currency more favorable in households’ eyes. However, this change does not affect much the total money supply because currency only accounts 6 percent of the total money supply in the steady state.

iv. Lacking liquidity, households cut their own consumption and output declines. A Taylor
Figure 3.2: Financial Crisis - Taylor Rule (M2 and M3)
rule, even with a negative IOR, is not enough to stimulate the aggregate demand in this case. When the link between the interbank rate and the prime rate breaks, the conventional monetary policy is generally not effective.

### 3.6.4 Forward Guidance

The recent literature in monetary economics focuses on the forward guidance policy when interest policy is restricted by the zero lower bound. In this section, we do a simple experiment to see whether the forward guidance policy is useful in the banking crisis. There are two common ways to model how the central bank informs the public about the interest rate path in the future: (i) interest rate peg and (ii) news shock on the Taylor rule. We follow the latter in Keen, Richter and Throckmorton (2017b) to characterize the forward guidance as follows:

\[
R_t^f = \max \left\{ \frac{\pi_t}{\beta} \left( \frac{\pi_{t+1}}{\pi_t} \right)^{\phi_\pi} \exp(\epsilon_t), \quad R^n + \delta_f \right\}
\]

\[
\epsilon_t = \begin{cases} 
\bar{\epsilon}, & \text{if } t \leq T_{FG} \\
0, & \text{if } t > T_{FG}
\end{cases}
\]

\[
R_t^n = \bar{R^n}
\]

\[
\kappa_t = \rho \kappa_{t-1} + (1 - \rho) \bar{R}, \quad \kappa_0 \text{ is given}
\]

The central bank still follows the common Taylor rule. However, during the forward guidance period \(0 \leq t \leq T_{FG}\), the central bank commits to lower the intercept of the Taylor rule by \(\exp(\bar{\epsilon})\).

We set \(\bar{\epsilon} = -1/400\). In the previous experiment, when following the Taylor rule, the interbank rate is bounded by the IOR during the first 12 periods. Hence, we set the horizon for forward guidance as 4 years \((T_{FG} = 16)\) to evaluate its efficacy in pushing up inflation.

The key channel that forward guidance affects the real economy is through increasing the expected inflation. Hence, it lowers the real short-term interbank rate, which in turn passes through to the real lending rate. In comparison to a common Taylor rule, forward guidance is much more effective at pushing up the expected inflation. Like all monetary models with the forward looking feature, path of inflation affects the current activities.

The effectiveness of forward guidance policy depends mostly on how far households look forward in the future. In our model, if the horizon of forward guidance is around 5 years, this
Figure 3.3: Financial Crisis - Forward Guidance
policy cannot push up inflation. As banks cut loans due to the capital constraint, inflation should be also consistent with the decline in the money supply path.

### 3.6.5 Financial Crisis - Mixed Rule

The previous section shows that a monetary policy targeting only the interbank rate is not efficient to deal with the banking crisis. What can the central bank do to improve the situation? If the problem is a lack of liquidity in the private sector when banks cut loans, a natural guess should be a policy of targeting the money supply directly. In this section, we examine a modification of a Taylor rule. In normal times, the central bank still targets the interbank rate by a Taylor rule. However, in crises, when the interbank rate is pushed down to the level of IOR and the deflation is still severe, the central bank will switch to target the growth of money supply $m_t^s = m_t + x_t$. All the exogenous shocks and monetary policy can be described by the following system of equations:

\[
\begin{align*}
R_f^t & = \frac{\pi}{\beta} \left( \frac{\pi_{t+1}}{\pi} \right) \phi_\pi \left( \frac{m_t^s}{m_{t-1}^s} \right) \\
R_n^t & = \bar{R}^n \\
\kappa_t & = \rho_\kappa \kappa_{t-1} + (1 - \rho_\kappa) \bar{\kappa}, \quad \kappa_0 \text{ is given}
\end{align*}
\]

(\text{M5})

where $\phi_m = 0.2$ measures the reaction of the interbank rate to the growth of the money supply. The problem (\text{P5}) is defined to contain the equations in (\text{AD}) and (\text{M5}).

We set $\phi_m$ small relative to $\phi_\pi$ for two reasons. First, it means that in normal times, the interbank rate is not influenced much by the growth rate of the money supply. Second, it implies that, in crises, when the interbank rate is not enough for raising inflation, the central bank will respond aggressively by raising the money supply through helicopter money. Why? We can rewrite the modified Taylor rule as:

\[
\frac{m_t^s}{m_{t-1}^s} = \left( \frac{R_f^t \beta}{\pi} \right)^{1/\phi_m} \left( \frac{\pi}{\pi_{t+1}} \right) \phi_\pi / \phi_m
\]

When the interbank rate is at its lower bound $R_f^t = R_n^t = \bar{R}^m$, it means that $\phi_\pi / \phi_m$ shows how aggressively the central bank will raise the money supply to deal with inflation. Figure 3.4 shows the economy’s response under this modified Taylor rule. Here are some important remarks:
Figure 3.4: Financial Crisis - Mixed Rule
i. In this banking crisis, when targeting the interbank rate is not efficient anymore, switching to target the growth rate of the money supply is very effective. Both inflation and output paths in this experiment are smoother and less volatile than a Taylor rule with a negative IOR. A modified Taylor rule can anchor inflationary expectations better; thereby restricting the increase in the real lending rate. This result is similar to Christiano and Rostagno (2001). Their research also shows that when inflation is out of a bounded region, switching from a Taylor rule to target the growth rate of the money supply can reduce the volatility of the economy.

ii. The interest rate path alone does not reflect the stance of the monetary policy in a banking crisis. If we only look at the paths of the interbank rate, a Taylor rule with a negative IOR keeps the interbank rate not only longer at the lower bound but also lower in every period in comparison to a mixed rule in this section (Figure 3.4a). If we follow the common New Keynesian logic, inflation should have been higher in the previous experiment. However, this is not the case here. When we model explicitly the microfoundation in the banking sector, the link between the money supply and the interest rate is not as tight as the one in the New Keynesian literature. Money supply is not determined solely by the central bank. Moreover, the central bank do not control the interest rate by directly changing the money supply here.

iii. inflationary expectation is anchored by both the money supply and the interest rate. Seemly, money supply is a more credible signal for the inflation path in banking crises.

### 3.6.6 Taylor Rule and Friedman’s k-percent Rule

The new tool IOR allows the central bank to target both the interest rate and money supply at the same time. In this section, the central bank is assumed to follow the Friedman’s k-percent rule during our crisis. The Friedman’s k-percent rule indicates that the growth of the money supply is fixed at a constant level:

\[
\frac{M_t}{M_{t-1}} = \pi
\]
At the same time, IOR is adjusted following a Taylor rule. The set of monetary policies and the exogenous shock can be written as:

\[
\begin{align*}
\frac{m_t}{m_{t-1}} &= \frac{\pi_t}{\pi_t}, \quad \text{if } t \leq T_e \\
R_t^f &= \max \left[ \frac{\pi_t}{\beta} \left( \frac{\pi_{t+1}}{\pi_t} \right)^{\phi} , \ R^n + \delta_f \right], \quad \text{if } t > T_e \\
R_t^n &= \begin{cases} 
(1 - \alpha_n)\overline{R^n} + \alpha_n \frac{\pi_t}{\beta} \left( \frac{\pi_{t+1}}{\pi_t} \right)^{\phi} & \text{if } t \leq T_e \\
\overline{R^n} & \text{if } t > T_e
\end{cases}
\end{align*}
\]

where \( \alpha_n = 0.8 \) and \( T_e = 50 \). Together with the equations in (AD), (M6) sets up the problem (P6). This experiment can be summarized as followings. Before time \( T_e \), the central bank targets both the IOR and the money supply by, respectively, a Taylor rule and the Friedman’s k-percent rule. After time \( T_e \), the central bank comes back to its Taylor rule and only targets the interbank rate. Figure 3.5 compares the effect of monetary polices in (P6) with the previous experiment. Here are some important remarks:

i. Targeting both the money supply and the interest rate is extremely efficient. The inflation rate is nearly anchored at the target level for the whole time. As our model does not have any real rigidities, it implies the output is also at the steady state level.

ii. The obvious byproduct of targeting the growth of money supply directly is clearly the sharp increase of reserves and excess reserves. Reserves increase by 25 times in our model and the reserves requirement is no longer binding for 25 periods. Many economists worry that a huge amount of excess reserves might prevent the effectiveness of the monetary policy or create hyperinflation. Our model shows these concerns have no foundations. By using the IOR, the central bank can control the interbank rate. The effect is very similar to the one when the central bank adjusts by using open market operations. There are also no reasons to believe the huge amount of reserves will create a huge amount of money supply. When the reserves requirement is no longer binding, we cannot use the logic in the money multiplier model to create the link between the monetary base and money supply anymore. Until there is still a borrowing constraint and capital requirement, the endogenous money
Figure 3.5: Financial Crisis - a Taylor Rule and the Friedman’s k-percent Rule
supply is always bounded. Furthermore, the inflation, in the long run, is always determined by the central bank.

iii. Once again, we emphasize that the stance of monetary policy can only be judged when we observe both the nominal interbank rate, the money supply and the real short-term rate. The money supply and the interest rate can move in any directions. It can be the case that the central bank increases the money supply and the IOR at the same time. It might be a serious mistake to infer that it would cause a deflation.

iv. The above point raises an important issue about the central bank’s communication. In reality, the stance of the monetary policy, when sending to the public, is often summarized by only one indicator: the interbank rate. Indeed, this is a common and good practice as the interbank rate is a unique short-term target that the central bank controls completely. In normal times, it is a good predicator of inflation path. However, the current situation is very tricky. The interbank rate in most developed countries has been near zero for a long time and the inflation is persistently lower than its target. The growth rate of the money supply should be included as part of the central bank’s communication with the public.

3.7 Conclusion

With a huge amount of excess reserves in the banking systems, IOR is now the most crucial tool for the Fed. This tool opens a new opportunity for the conduct of monetary policy as the central bank can target both the interest rate and the growth rate of the money supply at the same time. Of course, in normal times, adjusting the interbank rate alone is always timely and much more transparent than targeting the money supply, which is not entirely controlled by the central bank. However, in banking crises, our research shows that the link between the interbank rate and the lending rate is very weak. If the central bank simultaneously targets both the interbank rate and the money supply, they can hit the inflation target.
Chapter 4

Financial Innovation and Income Distribution

4.1 Introduction

This chapter is motivated by three stylized features of the US economy since 1980:

1. *The rise of the financial sector, especially of subsectors trading in the asset market*: From 1977 to 1997, the share of finance industry in GDP increased from 4.7 percent to 7 percent, in which the sector “Security, commodity contract, and investment” ¹ alone contributed 47 percent of this growth (Figure 4.1). The relative wage between the finance industry and the non-financial private sector, which was very stable at 1 before 1980, also skyrocketed after 1980 (Figure 4.2).²

2. *Income inequality*: According to Piketty and Saez (2003), the top 1%’s income share nearly tripled during the period 1977-2007. The representation of Wall Street in the top 1% and top 0.1% earners also dominated all other sectors in the economy (Table 4.1).

¹According to the definition of North American Industry Classification System (NAICS): “Industries in the Securities, Commodity Contracts, and Other Financial Investments and Related Activities subsector group establishments that are primarily engaged in one of the following: (1) underwriting securities issues and/or making markets for securities and commodities; (2) acting as agents (i.e., brokers) between buyers and sellers of securities and commodities; (3) providing securities and commodity exchange services; and (4) providing other services, such as managing portfolios of assets; providing investment advice; and trust, fiduciary, and custody services.”

²The average wage of the non-financial private sector is normalize to 1 in Figure 4.2. Both Figure 4.1 and 4.2 use data from U.S. Bureau of Economic Analysis.
3. **Financial Innovation**: The most remarkable change in the financial market during 1980s was the increase in trading volume of a wide range of new financial derivatives. Many financial products, which existed long before 1980 like options or futures contract, have only started becoming popular and useful tools for firms in managing risks since 1980. Besides options and futures, many types of asset backed securities (ABS) also made their debut during 1980s. Allen and Gale (1994) emphasize the biggest motivation for the introduction of many new financial instruments is risk sharing.

This chapter gives a novel theoretical insight that the appearance of many new financial assets (fact 3) can spur the growth of the financial sector (fact 1) and deepen the income inequality trend in US (fact 2).

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<table>
<thead>
<tr>
<th>Year</th>
<th>Top 1 percent</th>
<th>Top 0.1 percent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Executives, managers, supervisors (non-finance)</td>
<td>Financial professions, including management</td>
</tr>
<tr>
<td>1979</td>
<td>35.3</td>
<td>7.7</td>
</tr>
<tr>
<td>1993</td>
<td>33.3</td>
<td>10.8</td>
</tr>
<tr>
<td>1997</td>
<td>33.2</td>
<td>11.9</td>
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<tr>
<td>1999</td>
<td>32.7</td>
<td>12.8</td>
</tr>
<tr>
<td>2001</td>
<td>31.0</td>
<td>13.1</td>
</tr>
<tr>
<td>2002</td>
<td>30.8</td>
<td>13.0</td>
</tr>
<tr>
<td>2003</td>
<td>30.2</td>
<td>12.9</td>
</tr>
<tr>
<td>2004</td>
<td>30.0</td>
<td>13.4</td>
</tr>
<tr>
<td>2005</td>
<td>30.0</td>
<td>13.2</td>
</tr>
</tbody>
</table>

Table 4.1: Primary taxpayers in top 1% and top 0.1%. Source:Bakija, Cole and Heim (2008)
To this end, we build a model with two agents: entrepreneurs and financiers. Entrepreneurs, due to in different industries, face the different risk structures in their production. They can hedge their production risks by paying financiers to purchase a wide range of financial assets in the economy. New assets allow the better risk-sharing between entrepreneurs in the different industries, making the access to the financial market become more valuable. Financiers themselves, due to the increase in the number of assets, can also earn the higher income from actively trading in the asset market. As the agents are risk averse, the more complete market also implies the higher rate of return on capital, which in turn transmits and amplifies the cycle between the wealth inequality and the income inequality.

The contribution of our paper is threefold. First, it gives a new theoretical explanation for the income inequality trend and the rise of the financial sector in US by putting the asset market in the main theme. There is a literature about the relationship between the development of finance and the income inequality (see Galor and Zeira (1993) and Levine (1997)), but the theme rotates around the credit market where intermediaries (banks) play the main role. This literature does not match with the data from 1980 when nearly 50 percent of the growth of finance lies in the subsector dealing with the asset market. Most closely related to our paper is the work of Greenwood and Jovanovic (1990), in which intermediaries pool and diversify the risk, leading to the higher rate of return on capital and therefore deepening the income inequality. In our model, risk-sharing is conducted in the decentralized asset market by allowing agents to trade a range of assets. The income inequality in our model is widened when the number of assets increases while the model in Greenwood and Jovanovic (1990) predicts the income inequality eventually declines with the development of finance.

Second, our model can explain the fact that more financiers go to the top income level. To my limited knowledge, this is the first paper investigating this phenomenon theoretically. Some research, like Cagetti and Nardi (2006), Jones and Kim (2014), give insights why the income share of top entrepreneurs increases over years. However, there is a big gap in the literature in explaining the rise of the financiers into the top income, which is itself the main feature of the US inequality trend since 1980. New assets in our model generate two inequality trends simultaneously: income is redistributed from entrepreneurs to financiers and from low-wealth holders to top-wealth holders.

Third, we extend Simsek (2013)’s endowment economy to characterize the existence and uniqueness of an equilibrium for an economy with production where agents can trade a range of
financial assets. Although we cannot get the closed-form solution for the vector price of assets like Simsek (2013), we can still get some theoretical results about the effect of new financial assets on the income distribution and financiers’ transaction income. In Simsek (2013), due to the heterogeneity in agents’ beliefs on the fundamental risks, new assets can increase the portfolio risk. We, however, assume that agents have the same belief about the risk structure, so new assets allow the better risk sharing between agents, which is emphasized in Van Horne (1985), Allen and Gale (1994), Duffie and Rahi (1995), Gabrielle and Guy (1995).

The chapter is structured as follows. Section 4.2 describes our model. Section 4.3 gives some analytical results about the effect of new assets on the income distribution. Section 4.4 does a numerical example to illustrate the impact of new financial assets on the income distribution.

4.2 The model

4.2.1 The Environment

Consider an economy with two dates, \( t = 0, 1 \), and one single good which can be used as either capital or consumption. Except by using this good in the production technology, it cannot be stored to the next period. There are two types of agents in the economy: entrepreneurs and financiers.

**Entrepreneurs:** There is a continuum measure of 1 of entrepreneurs in each sector \( i \in I = \{1, 2, \ldots, |I|\} \). Entrepreneurs cannot move across sectors. Sectors differ from each other by the risk structure in production. The entrepreneurs in the same sector \( i \) have different initial endowments (initial wealth) \( e \) at date 0, where \( e \) follows a continuous distribution function with the density \( f(e) \) on the support \([\underline{e}, \overline{e}]\). So an entrepreneur can be denoted by a pair \((i, e)\), where \( i \) shows his sector and \( e \) shows his initial endowment level. Let \( \bar{e} = \frac{1}{\bar{e}} \int_{\underline{e}}^{\overline{e}} ef(e) de \) be the average initial endowment of each sector at date 0, then the total endowment of economy is \( I\bar{e} \). Entrepreneurs receive no additional endowments at date 1.

At date 0, the entrepreneur, before making his decision about production, is matched to a financier who can help him to access the financial market\(^3\). If the entrepreneur agrees to pay the

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\(^3\)All results of this paper still hold if we assume that entrepreneur can always access to a set of financial assets as the subset of the one that financiers can access
bargained fees for the financier at date 1, he can purchase or short sell the financial assets before going into the production stage.

An entrepreneur has a CARA utility function over his net worth \( n_{i,e} \) at the end of date 1:

\[
U_E = E[-\exp(-\theta n_{i,e})]
\]

where \( \theta \) is the coefficient of risk aversion, \( \theta > 0 \).

**Financiers:** There is a mass \( I \) of financiers in the economy. Financiers cannot produce; but they can make transactions in the asset market or help entrepreneurs to make transactions there. We assume that financiers do not have any endowments at date 0 and date 1. A financier also has a CARA utility function over his net worth at the end of date 1 like entrepreneurs:

\[
U_F = E[-\exp(-\theta n_F)]
\]

**Production Technology and Risk Structure:** We model the risk and asset structure similar to Simsek (2013). A fundamental risk (or uncertainty) in the economy is captured by the \( m \)-dimensional random vector \( \mathbf{v} = (v_1, ..., v_m)' \), which follow the multivariate normal distribution \( \mathbf{v} \sim N(\mu, \Omega) \), where \( \mu \in \mathbb{R}_m^+ \). The risk when producing in the sector \( i \) is characterized by a random variable \( \mathbf{W}_i'\mathbf{v} \), where \( \mathbf{W}_i \in \mathbb{R}^m \). As \( \mathbf{v} \) follows a multivariate normal distribution, \( \mathbf{W}_i'\mathbf{v} \) follows a normal distribution. An entrepreneur, if investing \( k > 0 \) amount of capital in the production at date 0, can produce the amount of output \( y \) at date 1 as:

\[
y = (\mathbf{W}_i'\mathbf{v} + z)k^\eta, \quad \frac{1}{2} \leq \eta < 1
\]

where \( z > 0 \) is the parameter indicating the general productivity level of the economy, \( \eta \) is the coefficient showing the level of control. We assume that capital is totally depreciated.

The production function exhibits decreasing returns to scale. The normal distribution of the sector shock has a lot of advantages when we introduce the asset market. The only problem is the output can be negative. However, if we let the the value of the general productivity level \( z \) be big enough, for example more than 10 times the standard deviation of the shock \( \mathbf{W}_i'\mathbf{v} \), the probability of \( y < 0 \) is almost 0. This setup of the risk structure makes the model simple and tractable.
Matching between financiers and entrepreneurs: At the beginning of date 0, financiers and entrepreneurs are matched randomly 1-to-1. Here we assume the mass of financiers is equal to the mass of entrepreneurs, so everyone is matched. If an entrepreneur meets a financier, his type \((i, e)\) is observed. To access the financial market, the entrepreneur must pay fees for the financier. They go into the Nash bargaining process with \(\alpha\) is the bargaining power of the entrepreneur. The bargained fees \(w_{i,e}\) will depend on the type of entrepreneur and only be paid at date 1. The fees here should be interpreted as the sum of consulting fees, servicing fees and transaction fees. In equilibrium, every entrepreneur will participate in the asset market.

Asset Market: There are bonds and \(H\) assets in the market, \(H + 1 < m\). A bond promises to pay 1 unit of goods at date 1. The inverse price of a bond is \(s\), which is endogenous. It means that 1 unit of goods at date 0 can exchange for \(s\) units of bonds.

Unlike bonds, the payoffs of assets are uncertain. Each asset \(h\) promises to give a random payoff \(a_h = A_h'v\) at date 1, where \(A_h \in \mathbb{R}^m\). So the payoff of an asset depends on the realization of the fundamental risk \(v\). The vectors, \(\{A_h\}_{h=1,...,H}\), are linearly independent which ensures no assets are redundant. We can say the asset matrix \(A = [A_1 \ A_2 \ ... \ A_H]_{m \times H}\) is full rank. The price vector of assets is \(p \in \mathbb{R}^H\), which is endogenous. The total net supply of assets and bonds are both 0 in equilibrium (we allow agents to short sell in the financial market).

Timing: The timing of events can be summarized as followings:

\(t = 0:\)
- Entrepreneurs and financiers are matched and they go into the bargaining process.
- Financiers and entrepreneurs make transactions in the asset market.
- Production stage: entrepreneurs make decisions about production.

\(t = 1:\)
- Fundamental risk \(v\) is realized.
- Outputs are realized; agents receive the payoffs from the financial assets and bonds; entrepreneurs pay the committed fees to financiers.

---

\(^4\)This assumption can be relaxed by letting the number of matches between financiers and entrepreneurs follows a standard matching function in search theory. If entrepreneurs are not matched with a financier, he cannot buy any financial instruments. The main insight of this chapter is still valid under this set up.
Like Simsek (2013), we assume there is no default in the model. That assumption is justified if the fluctuation of fundamental risk is small enough. It is also standard in the literature of general equilibrium model with a set of financial assets.

### 4.2.2 The entrepreneur’s problem:

When bargaining fees with the financier, the entrepreneur treats the price in the financial market as given and compares the change in his utility in two cases: with and without access to the financial market.

**No access to the financial market**

Let $k_{i,e}^U$ be the capital the entrepreneur of type $(i,e)$ will put into the production if he cannot get access to the financial market. The entrepreneur problem is:

\[
V_{i,e}^U = \max_{k_{i,e}^U} \quad E[-\exp(-\theta n_{i,e}^U)]
\]

subject to

\[
n_{i,e}^U = (W'_i v + z)(k_{i,e}^U)^\eta \tag{4.1}
\]
\[
0 \leq k_{i,e}^U \leq e \tag{4.2}
\]

As $W'_i v$ follows the normal distribution, $[\exp(-\theta n_{i,e}^U)]$ follows the log-normal distribution. Under the CARA preference, we can rewrite the problem as:

\[
\max_{k_{i,e}^U} \quad E(n_{i,e}^U) - \frac{\theta}{2} \text{Var}(n_{i,e}^U)
\]

subject to the constraints (4.1) and (4.2).

Let $\sigma_i$ be the variance of the shock in the sector $i$, then $\sigma_i = W'_i \Omega W_i$. The mean and variance of entrepreneur’s net worth in this case are:

\[
E(n_{i,e}^U) = (W'_i \mu + z)(k_{i,e}^U)^\eta
\]
\[
\text{Var}(n_{i,e}^U) = (k_{i,e}^U)^{2\eta} W'_i \Omega W_i = (k_{i,e}^U)^{2\eta} \sigma_i
\]
As the entrepreneur is risk averse, he must consider the trade-off between the expected return from the production and his sector risk $\sigma_i$—the variance of the shock in the sector $i$. Without access to the financial market, he must bear the risk himself. Lemma 4.1 shows the unique solution for the entrepreneur.

**Lemma 4.1.** The solution for the entrepreneur $(i,e)$ in case of no access to the financial market is:

$$k_{i,e}^U = \min\{\tilde{k}_{i,e}^U, e\}$$

where

$$\tilde{k}_{i,e}^U = \left(\frac{W_i'\mu + z}{\theta \sigma_i}\right)^{1/\eta}$$

In equilibrium, every entrepreneur will pay financiers to get access to the financial market so $V_{i,e}^U$ only plays the role as the reference point to identify the bargaining fees.

**With access to the financial market**

If the entrepreneur participates in the asset market, he can buy bonds and assets to hedge the risk when producing in the sector $i$. Let $x_{i,e}^h$ be the position of the entrepreneur $(i,e)$ on the asset $h$. Let $x_{i,e}$ be the vector showing his position for all $H$ assets and $k_{i,e}$ be the capital invested in the production. The entrepreneur problem can be written as:

$$V_{i,e} = \max_{k_{i,e} \geq 0, x_{i,e}} E[-\exp(-\theta n_{i,e})]$$

subject to

$$n_{i,e} = (e - k_{i,e} - x_{i,e}'p)s + (W_i'v + z)(k_{i,e})^\eta + x_{i,e}'A'v - w_{i,e}$$

(4.3)

Like the case of no access to the financial market, the problem can be rewritten in the mean-variance form:

$$\max_{k_{i,e} \geq 0, x_{i,e}} E(n_{i,e}) - \frac{\theta}{2} \text{Var}(n_{i,e})$$
And the mean and variance of net worth are:

\[
E(n_{i,e}) = (e - k_{i,e} - x_{i,e}'p)s + (W_{i}'\mu + z)(k_{i,e})^\eta + x_{i,e}'A'\mu - w_{i,e}
\]

\[
Var(n_{i,e}) = \sigma_i(k_{i,e})^2 + x_{i,e}'\Omega A x_{i,e} + 2x_{i,e}'\lambda_i(k_{i,e})^\eta
\]

where \( \lambda_i = A'\Omega W_i \) is the covariance between the risk in the sector \( i \) and the asset payoffs, \( \Omega_A = A'\Omega A \) is the variance-covariance of assets’ payoffs matrix.

The capability of making transactions in the asset market adds two layers to the entrepreneur’s decision. First, he can borrow and lend through the bond market or short-sell through the asset market, so the resource restriction is no longer binding. Second, he can hedge the production risk through the asset market. His capital investment still depends on the sector risk \( \sigma_i \); but it also depends on the asset market. He wants to buy more asset \( h \) if it offers him the high expected return \( (A'\mu - p_s) \) or its return is negative correlated to his sector risk \( \lambda_i \). His decision also depends on the variance-covariance of all asset payoffs \( \Omega_A \). Gaining access to the asset market helps the entrepreneurs to hedge the risk better and therefore leveraging and investing more into the risky production process.

Assumption 4.1 ensures that market is incomplete, so that entrepreneurs could not share all the idiosyncratic risks through trading \( H \) assets.

**Assumption 4.1.** \( \sigma_i \neq \lambda_i^\prime \Omega_A^{-1} \lambda_i, \ \forall i = 1, ..., I \)

**Lemma 4.2.** Under Assumption 4.1 and \( s > 0 \), the solution for the entrepreneur \( (i, e) \) if he can access to the financial market is unique. Let \( \beta = 1/\eta \), his capital choice \( k_{i,e} \) and portfolio choice \( x_{i,e} \) are the solution of the following system:

\[
(W_{i}'\mu + z) - s\beta(k_{i,e})^\eta(\beta - 1) - \theta \sigma_i(k_{i,e})^\eta - \theta x_{i,e}'\lambda_i = 0 \quad (4.4)
\]

\[
(A'\mu - p_s) - \theta \left( \Omega A x_{i,e} + (k_{i,e})^\eta \lambda_i \right) = 0 \quad (4.5)
\]

It is also easy to see that the value of gaining access to the financial market (less the fees paid to the financier) is always greater than (or at least equal to) the autarky case. Moreover, the decision rules for invested capital \( k_{i,e} \) and portfolio \( x_{i,e} \) do not depend on the initial endowment level \( e \).
4.2.3 The financier:

Financiers can always access and make transactions in the financial market. The financier’s problem can also be written in the mean-variance trade-off like the entrepreneurs. Financiers solve:

$$\max_{x_F} E(n_F) - \frac{\theta}{2} \text{Var}(n_F)$$

subject to

$$n_F = (-x'_F p)s + x'_F A'v + w_{i,e}$$

(4.6)

Lemma 4.3. The mean and variance of financiers’ net worth are:

$$E(n_F) = (-x'_F p)s + x'_F A'\mu + w_{i,e}$$

$$\text{Var}(n_F) = x'_F \Omega_A x_F$$

The solution for financiers is:

$$x_F = \frac{\Omega_A^{-1}(A'\mu - ps)}{\theta}$$

(4.7)

Unlike the entrepreneurs, the financiers only care about the expected return of asset payoffs ($A'\mu - ps$) and the variance-covariance of asset’s payoff ($\Omega_A$). The purpose of the financiers when holding risky assets is fundamentally different from the entrepreneurs’ choice. There is no hedging motivation there, the financiers hold the risky assets to earn the positive income from making transactions in the financial market.

4.2.4 Fee bargaining:

When an entrepreneur ($i,e$) meets a financier, his type is observed by the financier. They treat the bond price $s$ and asset prices $p$ in the financial market as given and go into Nash bargaining. Recall $V_{i,e}$, $V^U_{i,e}$ are the indirect utilities of the entrepreneur in case he can and cannot access the financial market. Let $V_F$, $V^N_F$ be respectively the indirect utility of the financier in case of bargaining is successful and not. We shorten the notation as all indirect utility functions depend on $p$ and $s$.

Let $\hat{V}_{i,e}$ be the indirect utility of the entrepreneur if he can access the asset market without
paying fees to the financier.

\[ \hat{V}_{i,e} = -\exp(-\theta \hat{n}_{i,e}) \]

where \( \hat{n}_{i,e} = (e - k_{i,e} - x'_{i,e}p)s + (W_i'\mu + z)(k_{i,e})^\eta + x'_{i,e}A'\mu \)

Then \( V_{i,e} = \exp(\theta w_{i,e})\hat{V}_{i,e} \). We also have \( V_F = \exp(-\theta w_{i,e})V_F^N \). With the entrepreneur’s bargaining power as \( \alpha \) (0 < \( \alpha \) < 1), the bargained fees will solve this problem:

\[
\max_{w_{i,e}} \left( \exp(\theta w_{i,e})\hat{V}_{i,e} - V_{i,e}^U \right)^\alpha \left( \exp(-\theta w_{i,e})V_F^N - V_F^N \right)^{1-\alpha}
\]

**Lemma 4.4.** The bargain fee \( w_{i,e}^* \) is:

\[
\exp(\theta w_{i,e}^*) = \frac{2\alpha - 1}{2\alpha} + \sqrt{\left( \frac{2\alpha - 1}{2\alpha} \right)^2 + \frac{(1-\alpha)}{\alpha} \left( \frac{V_{i,e}^U}{\hat{V}_{i,e}} \right)}
\]

**In the special case** \( \alpha = 1/2 \), then:

\[
w_{i,e}^* = \frac{1}{2} \left[ \left( E(\hat{n}_{i,e}) - \frac{\theta}{2} Var(\hat{n}_{i,e}) \right) - \left( E(n_{i,e}^U) - \frac{\theta}{2} Var(n_{i,e}^U) \right) \right]
\]

Like all the other Nash bargaining problems, the bargained fee is set up to divide the total surplus from the successful match. As \( \hat{V}_{i,e} \geq V_{i,e}^U \), all matches are successful in equilibrium.

### 4.3 Equilibrium:

**Definition:** Market equilibrium consists of asset price vector \( p \) and bond price \( s > 0 \), an allocation \( \{k_{i,e}\} \), a vector of asset holdings \( \{\{x_{i,e}\}, x_F\} \), and the fees \( \{w_{i,e}\} \) such that:

(i) Given \( p \) and \( s \), \( \{k_{i,e}, x_{i,e}\} \) solves the problem of the entrepreneur of type \( (i,e) \); \( \{x_F\} \) solves the problem of financiers; \( \{w_{i,e}\} \) solves the problem of fee bargaining between entrepreneurs and financiers.

(iii) All markets clear:

\[
\text{Asset Market:} \quad \sum_{i=1}^{I} \left( \int_{L}^{\bar{e}} x_{i,e} f(e) de \right) + Ix_F = 0 \quad (4.8)
\]
Goods Market:  \[ \sum_{i=1}^{I} \left( \int_{\mathcal{E}} k_{i,e} f(e) de \right) = I \bar{e} \]  

(4.9)

This chapter only considers the market equilibrium when the inverse bond price is positive (bonds will exist in equilibrium). When the amount of endowment is bigger than the level economy can absorb, the risk averse entrepreneurs might want to get rid of excess resources rather than put it into the production function (see Lemma 4.1). In this case, \( s \leq 0 \). However, the Assumption 4.2 ensures that the case with negative \( s \) will not happen.

**Assumption 4.2.** The total endowment of the economy satisfies:

\[
\sum_{i=1}^{I} (W_i \mu + z) - \theta \sigma_i (\bar{\sigma}_i I \bar{e})^\eta > 0; \quad \text{where} \quad \bar{\sigma}_i = \frac{\sigma_i^{1/(1-\eta)}}{\sum_{j=1}^{I} \sigma_j^{1/(1-\eta)}}
\]

**4.3.1 The existence and uniqueness of the equilibrium**

We prove the existence and the uniqueness of the equilibrium in two steps. First, we prove that the set of allocation of market equilibria (if exist) is identical to the set of solutions of a problem faced by a social planner. Second, we prove the solution for the social planner’s problem is unique; and, therefore, the market equilibrium exists and must be unique. We can also recover all the prices in the market equilibrium from the social planner’s problem.

Assume there is a social planner who wants to maximize the aggregate certainty equivalent net worth of the whole economy \( Y \), which is defined by:

\[
Y = \left[ \sum_{i=1}^{I} \int_{\mathcal{E}} \left( E(n_{i,e}) - \frac{\theta}{2} \text{Var}(n_{i,e}) \right) f(e) de \right] + I \left( E(n_F) - \frac{\theta}{2} \text{Var}(n_F) \right)
\]

(4.10)

The social planner takes the structure of assets as given and chooses the portfolios as well as the capital invested in the production for each agent. However, she faces four constraints. First, she must allocate the capital invested in the production and the portfolios for all agents in the same sector identically. It means that \( k_{i,e} = k_i \) and \( x_{i,e} = x_i \). Second, the total allocated capital must be less than or equal to the available endowment in date 0. Third, the portfolios for all agents must satisfy the net supply of assets equal to 0. Fourth, she could not interfere and reallocate the agents’ income at date 1.
Lemma 4.5. A social planner’s problem can be defined as

$$\max_{k_i, x_i} \ Y = \left[ \sum_{i=1}^{l} (W_i \mu + z)(k_i)^{\eta} - \frac{\theta}{2} \left( \sigma_i(k_i)^{2\eta} + x_i' \Omega_A x_i + 2(k_i)^{\eta}x_i' \lambda_i \right) \right] - \frac{\theta l}{2} x_F' \Omega_A x_F$$

subject to

$$\sum_{i=1}^{l} k_i \leq l \hat{e}$$

$$\sum_{i=1}^{l} x_i + l x_F = 0$$

Then under the Assumptions (4.1), the social planner’s solution is unique.

Theorem 4.1. Under the Assumptions (4.1)-(4.2), the market equilibrium exists and is unique.

We can also recover the inverse price of bond in the market equilibrium from solving the social planner’s problem. From the proof of Theorem 4.1, the Lagrangian multiplier for the resource constraint in the solution of social planner problem is $s$. In the following sections, we analyze the impact of new assets on the income distribution in the economy.

4.3.2 New assets and the rise of the financial sector

We characterize the change in the economy and in the financial sector if the list of assets expands. An economy can be denoted by $\xi = (\{W_i\}_{i=1}^{l}, \{A_h\}_{h=1}^{H})$. Assume there is a new additional asset $A_{H+1}$ for trading, the new economy can be denoted by $\xi' = (\{W_i\}_{i=1}^{l}, \{A_h\}_{h=1}^{H+1})$. We make the following assumption for this new asset.

Assumption 4.3. $A_{H+1} \in \mathbb{R}^m$, $A_{H+1} \notin \text{span}\{A_1, A_2, ..., A_H\}$.

This assumption makes sure the new asset is not redundant and the asset matrix is still full rank. Now we characterize the rise of the financial sector when a new asset is added. Let $Y_F$ be the “adjusted” certainty equivalent (CE) net worth of all financiers ($Y_F$ contains all the consulting fees but only half of financiers’ transaction income) and $\varphi$ be the share of $Y_F$ in total.
economy:

\[
Y_F = \left( \sum_{i=1}^{I} \int_{\xi}^e \bar{w}_{i,e} f(e) \, de \right) + \frac{1}{2} \left( -x_F^p s + x_F^p \mu - \frac{\theta}{2} x_F^p \Omega_A x_F \right)
\]

(4.11)

\[
\phi = \frac{Y_F}{Y}
\]

(4.12)

**Theorem 4.2.** If both economies \(\xi\) and \(\xi'\) satisfy the Assumption (4.1)-(4.2), the new asset satisfies the Assumption (4.3) and the bargaining power of entrepreneur \(\alpha = 0.5\), then:

(i) The certainty equivalent net worth of the whole economy \(Y'\) in \((\xi')\) is greater than or equal to \(Y\) in \((\xi)\).

(ii) The certainty equivalent net worth of all financier \(Y_F'\) in \((\xi')\) is greater than or equal to \(Y_F\) in \((\xi)\).

(iii) The financial sector’s share \(\phi\) is (weakly) increasing when the economy transforms from \((\xi)\) to \((\xi')\).

Theorem 4.2 shows the rise of the financial sector when a new asset is added in the economy. The appearance of the new financial asset can increase the risk-sharing efficiency; therefore, it raises the value of being accessed to the financial market. Entrepreneurs are ready to pay more for the financiers to hedge their risks in production. So the fees \(w_{i,e}\) increase to reflect the benefit from accessing the financial market to entrepreneurs. Intuitively, if the CE net worth of the whole economy \(Y\) increases due to the innovation in the financial sector, financiers will capture half of this gain in the Nash bargaining process with entrepreneurs.

What do we know about the financiers’ expected transaction income \(x_F'(A'\mu - sp)\) when new assets appear? First, it is surely not negative, otherwise they would be better by not making transaction \(x_F = 0\). While entrepreneurs join the financial market to hedge the production risk, financiers join the market to earn the transaction income.

Second, the change in the financiers’ transaction income depends on the risk correlation between the production sectors. Let’s assume that the economy has two production sector \(I = 2\). If the risk of two production sectors are perfectly negative correlated, or there is no aggregate risk in the economy, then transaction income is always zero; and fees income are increasing with the appearance of new asset (Figure 4.3a). In the opposite case, when risk of two sectors are
extremely positive correlated, the expected financier’s transaction income is generally increasing with the appearance of new assets as there are more financial tools for financiers to share risks with the production sectors (Figure 4.3b). We formalize two above ideas for some specific economies.

**Proposition 4.1.** If the risks of two production sectors are perfectly negative correlated I = 2, $W_1 + W_2 = 0$ and $W_1\mu = W_2\mu = 0$, then financiers’ expected transaction income $x_F'(A'\mu - sp) = 0$.

To illustrate for the change in the financiers’ transaction income when the risks of two production sectors are perfectly positive correlated, we consider the case when two sectors are identical $W_1 = W_2$, or $I = 1$.

**Proposition 4.2.** If both economies $\xi$ and $\xi'$ satisfy the Assumption (4.1)-(4.2), the new asset satisfies the Assumption (4.3) and $I = 1$, then financiers’ expected transaction income $x_F'(A'\mu - sp)$ is weakly increasing when the economy transforms from ($\xi$) to ($\xi'$).

### 4.3.3 New assets and the rise of income inequality

New assets also deepen the income inequality in the economy. From the result of Theorem 4.2, new assets can transfer the income from entrepreneurs to financiers. This creates the income inequality between the finance industry and non-finance industries. However, new assets also widen the income gap within entrepreneurs. They allow the better chance to hedge the risk in the production, pushing up the capital demand and therefore the rate of return on capital $s$ and
the income share of top wealth holders.

To see the mechanism clearly, we consider the economy with two production sectors with perfectly negative risk correlation

**Assumption 4.4.** \( I = 2, W_1 + W_2 = 0, W'_1 \mu = W'_2 \mu = 0 \)

From the Assumption 4.4, we have \( \lambda_1 = -\lambda_2 \) and \( k^*_1 = k^*_2 \) (result in the proof of Proposition 4.1). Risk sharing in this economy is conducted between two sectors. Solving the social planner problem with bond price \( s \) as the Lagrangian multiplier with the resource constraint, we get: (equation C.0.15 in the Appendix)

\[
(W'_i \mu + z) - s \beta (k^*_i)^{1-\eta} = \theta \sigma_i (k^*_i)^\eta - \frac{\theta}{2} (k^*_i)^\eta \lambda_i' \Omega_A^{-1} \lambda_i, \quad i = 1, 2
\]

Recall that \( \beta = 1/\eta \), multiply both sides by \( \eta (k^*_i)^{\eta-1} \):

\[
(W'_i \mu + z) \eta (k^*_i)^{\eta-1} = \underbrace{s}_{\text{Marginal Benefit}} + \underbrace{\theta \sigma_i (k^*_i)^{2\eta-1}}_{\text{Production Risk}} - \underbrace{\frac{\theta}{2} \eta (k^*_i)^{2\eta-1} \lambda_i' \Omega_A^{-1} \lambda_i}_{\text{Risk Sharing}} \quad (4.13)
\]

The above equation shows the capital demand in each sector in the economy. The left hand side of (4.13) is the marginal benefit for entrepreneurs from investing one more unit of capital in the production. The right hand side could be considered as the marginal cost of putting capital into the production. The first component of this cost is the opportunity cost from giving up the chance to buy bonds. This second component is associated with the risk-averse attitude of entrepreneurs, as more investment means the higher risk. However, this cost can be reduced if the list of financial assets allows the entrepreneurs to hedge the production risk. The effect of new asset on the capital demand comes from the increase of the risk-sharing effect \((\theta/2) \eta (k^*_i)^{2\eta-1} \lambda_i' \Omega_A^{-1} \lambda_i\), when entrepreneurs have more financial instrument tools to hedge their production risk.

First, we prove formally the important result about the impact of new asset on the right hand side of (4.13). From now, we use the notation \( \hat{a} \) for variable (or parameter) \( a \) in the economy \((\xi)\) and \( \tilde{a} \) for variable (or parameter) \( a \) in \((\xi')\).

---

5The detail derivation is conducted in the proof of the Theorem 4.1.
As the risk structures in two sectors are symmetric, we have $\sigma_1 = \sigma_2$ and $\lambda_1 \Omega_A^{-1} \lambda_1 = \lambda_2 \Omega_A^{-1} \lambda_2$. We drop the subscript $i$ to shorten the notation.

**Lemma 4.6.** Let $\tilde{\Omega}_A, \hat{\Omega}_A$ be respectively the variance-covariance payoffs of assets in the economy ($\xi'$) and ($\xi$). Then, under the Assumption (4.3) and $H + 1 < m$, we have: $\hat{\lambda}' \hat{\Omega}_A^{-1} \hat{\lambda} \leq \tilde{\lambda} \tilde{\Omega}_A^{-1} \tilde{\lambda}$

For the case two symmetric sectors, under the Assumption (4.2), the total capital demand will equate the total capital supply $I \bar{e}$ in equilibrium, or $k^*_1 = k^*_2 = \bar{e}$. With the appearance of new asset, the only way to balance the equation (4.13) is the inverse bond price $s$ must go up. Intuitively, because new financial securities allow the better risk sharing between two sectors, the capital demand increases. While the total supply of capital in inelastic at $I \bar{e}$, the rate of return on capital $s$ will go up.

Figure 4.4 shows the effect of the new assets on the inverse bond price. The introduction of new financial assets shift the capital demand to the right, raising up the inverse of bond price. This implies that the people in the top of the wealth distribution enjoy the bigger chunk in GDP as their capital income increases. Recall that the difference between the net worth at date 1 of two entrepreneurs in the same sector with initial wealth $e_1$ and $e_N$ will be $(e_N - e_1)s$ (as they will
chose the same portfolio and capital invested); therefore, the new financial assets, by increasing $s$, also deepen the income inequality from the initial wealth inequality. The entrepreneurs with the low initial wealth must borrow the capital with the higher interest rate while the one with high initial wealth enjoys the higher income from lending capital. The higher rate of return on capital creates the inequality cycle when transmitting the wealth inequality to income inequality and reverse.

To see it clearly, let $y(e)$ be the expected income of all entrepreneurs with the initial endowment $e$ and the financiers who are matched to them. We know that transaction income and the fees from the financial market only transfer from members in this group, so it does not affect the expected income of the whole group. The capital invested in equilibrium will be $k_{i,e} = \bar{e}$. We have:

$$y(e) = (e - \bar{e})s + \sum_{i=1}^{2} (W_i \mu + z)\bar{e}^\eta = (e - \bar{e})s + 2\varepsilon \bar{e}^\eta$$

Let $y$ be the expected GDP of the economy, then:

$$y = 2\varepsilon \bar{e}^\eta$$

Figure 4.5: The effect of new assets on the income distribution
Let $H(.)$ be the cumulative income distribution of $y(e)$, then Gini coefficient measuring the income inequality of the economy is:

$$ G = \frac{1}{2e\eta} \int_{y(e)}^{\bar{y}(e)} H(y(e))(1 - H(y(e)))dy(e) $$

**Theorem 4.3.** If both economies $\xi$ and $\xi'$ satisfy the Assumption (4.1)-(4.4), then the Gini coefficient is weakly increasing when the economy transforms from ($\xi$) to ($\xi'$).

For the group in the bottom of wealth distribution, $\xi < \bar{\xi}$. The increase in $s$ reduces their income while the group at the top of wealth distribution $\bar{\xi} > \bar{\xi}$ enjoys the bigger share in GDP. The effect of new assets on the distribution of income can be summarized by the Figure 4.5.

The introduction of new financial assets increases the income share of financiers in GDP. It also pushes up the share of top wealth people in the country. Although the model is very simple, it matches two income inequality trends we observe from the 1980: the rapid rise of top wealth people as well as many top financial executives in the top 0.1% income in the economy.

### 4.4 Numerical Example

#### 4.4.1 Calibration

In our numerical example, we set up the number of sector $I = 2$, and the dimension of fundamental risk $m = 56$. We can think $m$ as the number of industries in the economy. The vector $W_1=[1 \ 0 \ 1 \ 0 \ldots \ 1 \ 0]'$ showing the risk for the sector 1 is set up to have zero in the even positions and 1 in the odd position. It means that sector 1 ($W_1v = v_1 + v_3 + \ldots + v_{55}$) only contains the risks of industries in the odd positions of $v$. Similarly, the vector $W_2=[0 \ 1 \ldots \ 0 \ 1]'$ showing the risk for sector 2 means that the sector 2 only contains the risks of industries in the even positions of $v$.

We calibrate the variance-covariance matrix $\Omega$ by two parameters. We estimate the covariance matrix of output growth between 56 industries in US between 1970-1982, the period before new financial instruments appear. Then we calculate the average covariance between the growth of two industries, which is $1.05 \times 10^{-4}$. We set up all entries $\Omega(i, j) = 1.05 \times 10^{-4}$, where $i \neq j$.

---

6Here we use the concept of Gini-coefficient being the Gini’s mean difference divided by twice the mean. See Yitzhaki and Schechtman (2012) for more details.
For all the diagonal entries, we set $\Omega(i, i) = 39 \times 10^{-4}$, which is the average variance of growth of 56 industries. We set $\mu = 0$.\(^7\)

The asset matrix $A$ in this experiment is set up so that asset $h$ can fully insure the risk in the industry $h$. In this way $A_h$ is the unit vector with the entry in the position $h$ equal to 1, other positions equal to 0.

For the production function, the level of span of control $\eta$ is set to equal 0.9 and the general productivity $z$ is set at 1.38 to match with the target that return on capital in the economy with only bond is around 7 percent in 1980.

The wealth distribution of entrepreneurs is assumed to follow a Pareto distribution with density:

$$f(e) = \frac{\zeta e^{\zeta}}{e^{\zeta+1}}$$

We calibrate $\zeta$ and $\bar{e}$ to satisfy two conditions. First, the aggregate wealth $\bar{e}$ of each sector is 1. Second, the initial wealth share of the top 1% entrepreneurs is 24% to match the data in 1980 from Saez and Zucman (2014). The parameters are summarized in the Table 4.2.

### Table 4.2: Parameter values

<table>
<thead>
<tr>
<th>Param.</th>
<th>Value</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>56</td>
<td>Dimension of fundamental risk</td>
</tr>
<tr>
<td>$l$</td>
<td>2</td>
<td>Number of sectors</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1</td>
<td>Risk aversion coefficient</td>
</tr>
<tr>
<td>$z$</td>
<td>1.38</td>
<td>General technology level</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.9</td>
<td>Span of control level</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.5</td>
<td>Bargaining power of entrepreneurs</td>
</tr>
<tr>
<td>$\bar{e}$</td>
<td>0.29</td>
<td>Scale of initial wealth distribution</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>1.42</td>
<td>Shape of initial wealth distribution</td>
</tr>
</tbody>
</table>

\(^7\)We experimented with the model when $\Omega(i, j)$ matches the real variance-covariance matrix of 56 industries in data and the asset matrix is random, all the trends of results are identical to our simplified version.
4.4.2 Results

We start from the economy with only bonds (zero assets) and keep adding the new asset into the economy and calculate the new equilibrium. As the dimension of fundamental risk is \( m = 56 \), the market is complete when we have 56 assets. All results are calculated in the expected value as the actual income of agents in the economy at date 1 depends on the realization of shock \( v \). Table 4.3 shows the main statistics of the economy when the financial innovations happen.

Table 4.3: The change in economy with new financial assets

<table>
<thead>
<tr>
<th>No of assets ((H))</th>
<th>Rate of return on capital ((s))</th>
<th>GDP</th>
<th>Share of finance in GDP</th>
<th>Share of top 0.1% entrepreneurs</th>
<th>Share of top 0.1% financiers</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.072</td>
<td>2.76</td>
<td>11.06</td>
<td>11.88</td>
<td>8.24</td>
</tr>
<tr>
<td>1</td>
<td>1.075</td>
<td>2.76</td>
<td>11.21</td>
<td>11.91</td>
<td>8.27</td>
</tr>
<tr>
<td>2</td>
<td>1.078</td>
<td>2.76</td>
<td>11.35</td>
<td>11.93</td>
<td>8.30</td>
</tr>
<tr>
<td>3</td>
<td>1.081</td>
<td>2.76</td>
<td>11.49</td>
<td>11.96</td>
<td>8.33</td>
</tr>
<tr>
<td>4</td>
<td>1.084</td>
<td>2.76</td>
<td>11.63</td>
<td>11.98</td>
<td>8.35</td>
</tr>
<tr>
<td>5</td>
<td>1.087</td>
<td>2.76</td>
<td>11.75</td>
<td>12.00</td>
<td>8.38</td>
</tr>
<tr>
<td>6</td>
<td>1.089</td>
<td>2.76</td>
<td>11.88</td>
<td>12.02</td>
<td>8.40</td>
</tr>
<tr>
<td>7</td>
<td>1.091</td>
<td>2.76</td>
<td>12.00</td>
<td>12.04</td>
<td>8.42</td>
</tr>
<tr>
<td>8</td>
<td>1.094</td>
<td>2.76</td>
<td>12.11</td>
<td>12.06</td>
<td>8.45</td>
</tr>
<tr>
<td>9</td>
<td>1.096</td>
<td>2.76</td>
<td>12.22</td>
<td>12.08</td>
<td>8.47</td>
</tr>
<tr>
<td>10</td>
<td>1.098</td>
<td>2.76</td>
<td>12.32</td>
<td>12.10</td>
<td>8.49</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>54</td>
<td>1.155</td>
<td>2.76</td>
<td>15.01</td>
<td>12.58</td>
<td>9.02</td>
</tr>
<tr>
<td>55</td>
<td>1.156</td>
<td>2.76</td>
<td>15.05</td>
<td>12.59</td>
<td>9.03</td>
</tr>
<tr>
<td>56</td>
<td>1.157</td>
<td>2.76</td>
<td>15.09</td>
<td>12.60</td>
<td>9.04</td>
</tr>
</tbody>
</table>

There are three important observations. First, the expected level of GDP does not change with the introduction of new assets. In our model, the technology level remains constant; therefore, the introduction of new assets mainly affects the welfare and the distribution of income rather than the level of output. Second, new assets increase the income share of the financial sector in the economy. In our simulation, ten new assets can increase the share of the finance industry by 1.25%. Third, the top-wealth entrepreneur and the top financiers enjoy the bigger chunk of GDP when the financial innovation happens. The income share of entrepreneurs at bottom in fact shrinks. That replicates the two important income inequality trends we observe since 1980: the rise of financiers and top 0.1% in wealth.
To see the income inequality trend more clearly, we take the economy with only bonds as the benchmark and calculate the income growth of top 0.1% entrepreneurs by wealth, top 0.1% financiers (who are matched with top 0.1% entrepreneurs) and the entrepreneurs at the bottom 10% when the number of assets increases. Table 4.3 displays the income growth of three groups of agents with the financial innovations. The introduction of new assets benefits the top financiers most. Compared to the economy with only bonds, the income of top financiers grows by 28 percent if the market becomes complete. This explains the dominance of Wall Street against Main Street observed in the empirical research by Kaplan and Rauh (2010) since 1980. The top wealth entrepreneurs also enjoys the higher growth of income due to the higher rate of return on capital; however, their income growth is less than the top financiers. The agents at the bottom of wealth ladder suffer the decline in income share.

We break down the financiers’ income into two parts to understand more clearly the force behind the rise of the financial sector. In our model, financiers have two sources of income: the consulting fees is paid from entrepreneurs and the income they earn from making transaction in the financial market. Both income sources go up with the financial innovations, but the transaction income grows with the faster rate.

The transaction income grows due to the fact that new asset open more opportunities for financiers earns money from the asset market. There is no uncertainty in the fees financiers are paid, so they only hold assets if the expected income outweighs the variance of assets’ payoffs. Financiers always earn the positive expected transaction income. This fundamentally differs from the purpose of hedging the production risks when entrepreneurs hold assets.

The divergent trend of income growth between the different types of agents is the most crucial insight in our simple model. This numerical result confirms the two income inequality trends we characterized in the theoretical results. Financial innovations push up the income of financiers and the people in the top distribution of wealth.

4.5 Conclusion

This chapter builds the simple model to understand the link between the appearance of new financial assets and the income inequality trend we observed in US since 1980. The model predicts that new assets lead to two trends in the income distribution. First, the income share of
the financial sector will go up, due to both the consulting fees and the profits they earn from the transactions in asset market. Second, the cycle between wealth inequality to income inequality will become more severe, as new assets allow the better risk-sharing and therefore raise the rate of return on capital. Both predictions of the model are very consistent with the data trend from 1980.
Chapter 5

Conclusion and Further Research

Direction

Finance, especially banking, becomes more and more important in macroeconomics. The connection between the central bank and the commercial banks implies that monetary policy should pay more attention on the money supply. The connection between the finance industry and the manufacturing factor implies that financial innovations have a huge impact on the income distribution. Money, banking and finance should be a key theme in macroeconomics in the twentieth century.

Even though our dissertation supports the view that the central bank should observe the money supply more closely, finding a best practice to manipulate the money supply is not an easy task. If the central bank uses the helicopter money, the independence between the central bank and the government is threatened. Purchasing the government bonds will not be efficient if the government bond themselves are good and liquid collaterals. We definitely need more research on how to change the money supply optimally.
Bibliography


Keister, Todd, and James McAndrews. 2009. “Why are banks holding so many excess reserves?”


Appendix A

Mathematical Appendix for Chapter 2

A.1 System of Equations in Equilibrium

A.1.1 Conventional Monetary Policy: Taylor Rule

We have a system of 29 equations for 29 variables.

Bankers:

\[
g_t = \frac{1}{c_t} \tag{A.1.1}
\]

\[
y_t = \frac{\beta R_f y_{t+1}}{\pi_{t+1}} + \mu_t^c \tag{A.1.2}
\]

\[
y_t = \frac{\beta R_m y_{t+1}}{\pi_{t+1}} + \mu_t^c + \varphi \mu_t^r \tag{A.1.3}
\]

\[
y_t = \frac{\beta R_n y_{t+1}}{\pi_{t+1}} + \mu_t^c + \mu_t^r \tag{A.1.4}
\]

\[
(q_t^r + \theta) y_t = \frac{\beta \left[ \delta_b + \delta_b q_{t+1}^r \right] y_{t+1}}{\pi_{t+1}} + (1 - \kappa) \mu_t^c \tag{A.1.5}
\]

\[
n_{t-1} \frac{1}{\pi_t} + \hat{\tau}_t = n_t \tag{A.1.6}
\]

\[
m_t = \frac{R_{t-1}^m m_{t-1}}{\pi_t} + q_t^r s_t + \theta_t b_t^h - \delta_b \frac{b_{t-1}^h}{\pi_t} + c_t + \hat{\tau}_t - (R_t^m - 1) \frac{n_{t-1}}{\pi_t} \tag{A.1.7}
\]

\[
\mu_t^c \geq 0, \quad n_t - \varphi m_t \geq 0, \quad \mu_t^r (n_t - \varphi m_t) = 0 \tag{A.1.8}
\]

\[
\mu_t^c \geq 0, \quad n_t + (1 - \kappa) b_t^h - m_t \geq 0, \quad \mu_t^c \left( n_t + (1 - \kappa) b_t^h - m_t \right) = 0 \tag{A.1.9}
\]

\[
b_t^h = \delta_b \frac{b_{t-1}^h}{\pi_t} + s_t \tag{A.1.10}
\]

Households:

\[
\frac{1}{c_t} = \eta_t^r + \lambda_t^a \tag{A.1.11}
\]

\[
\frac{1}{c_t} = \lambda_t^b \tag{A.1.12}
\]
\[ \lambda_t^a = \frac{\bar{\beta} R_t^m \lambda_{t+1}^b}{\pi_{t+1}} \]  
(A.1.13)

\[ d_t^l \lambda_t^b = \frac{\bar{\beta} \delta b + \delta b q_{t+1}^L \lambda_{t+1}^b}{\pi_{t+1}} + \eta_t^b \]  
(A.1.14)

\[ \lambda_t^b = \bar{\beta} (1 - \delta) \lambda_{t+1}^b + \bar{\beta} \alpha \frac{p_{t+1}^m \lambda_{t+1}^a y_{t+1}}{k_t} \]  
(A.1.15)

\[ \chi l_t^{\nu+1} = (1 - \alpha) p_t^m y_t \lambda_t^a \]  
(A.1.16)

\[ \lambda_t^a v_t = \bar{\beta} \lambda_{t+1}^a (v_{t+1} + w_{t+1}) \]  
(A.1.17)

\[ a_t + \delta b^{b_t-1} = \frac{R_{t-1}^m m_t-1}{\pi_t} + d_t^s \]  
(A.1.18)

\[ \eta_t^\tau \geq 0, \quad a_t - c_t - i_t \geq 0, \quad \eta_t^\tau (a_t - c_t - i_t) = 0 \]  
(A.1.19)

\[ \eta_t^b \geq 0, \quad \bar{b}^h - b_t^h \geq 0, \quad \eta_t^b (\bar{b}^h - b_t^h) = 0 \]  
(A.1.20)

**Firms:**

\[ 1 - t (\pi_t - 1) \pi_t + t \bar{\beta} \frac{\lambda_{t+1}^a}{\lambda_t^a} (\pi_{t+1} - 1) \pi_{t+1} \frac{y_{t+1}}{y_t} = (1 - p_t^m) \varepsilon \]  
(A.1.21)

\[ y_t = k_{t-1}^{\alpha} l_{t-1}^{1-\alpha} \]  
(A.1.22)

\[ w_t = (1 - p_t^m) y_t - \frac{1}{2} (\pi_t - 1)^2 y_t \]  
(A.1.23)

**Central bank:**

\[ R_t^l = \max \left\{ R_t^l \left( \frac{\pi_{t+1}}{\pi_t} \right)^{\phi e}, \quad R_t^m + \varepsilon_f \right\} \]  
(A.1.24)

\[ R_t^m = R_t^m \]  
(A.1.25)

**Markets Clear:**

\[ y_t = c_t + \bar{c}_t + i_t + \theta b_t^h + \frac{1}{2} (\pi_t - 1)^2 y_t \]  
(A.1.26)

\[ k_t = (1 - \delta) k_{t-1} + i_t \]  
(A.1.27)

\[ \bar{x}_t = 1 \]  
(A.1.28)

**Shock:**

\[ \kappa_t = \rho_{\kappa} \kappa_{t-1} + (1 - \rho_{\kappa}) \bar{\kappa} \]  
(A.1.29)

### A.1.2 Unconventional Monetary Policy: LSAP

In comparison to the conventional monetary policy, we have one more variable \( x_t \) - the number of wholesale firms’ shares held by the central bank. In timing within one period, it is assumed that asset purchases happen after the credit market between bankers and households. Therefore, only banker’s reserves flows
(A.1.6) and deposits flows (A.1.7) are modified:

\[ \frac{n_{t-1}}{\pi_t} + \nu_t(x_t - x_{t-1}) + \hat{\tau}_t = n_t \]

\[ m_t = \frac{R^m_{t-1}m_{t-1}}{\pi_t} + q^t \nu_t + \theta_t b^h_t - \delta_t b^h_{t-1} - c_t + \nu_t(x_t - x_{t-1}) + \hat{\tau}_t - (R^m_{t-1} - 1) \frac{n_{t-1}}{\pi_t} \]

Equation (A.1.28) is replaced by:

\[ x_t + \bar{x}_t = 1 \]

The Taylor Rule (A.1.24) is replaced by:

\[ \hat{\tau} = 0 \]

One more equation for the evolution of \( x \):

\[ x_t = \rho x_{t-1} \]

**A.2 Mathematical Proof**

**Proof for Theorem 2.1:**

From the first order condition of bankers’ problem, we have:

\[ \gamma_t = \frac{\beta R^f_t \gamma_{t+1}}{\pi_t} + \mu^c_t \]  \hspace{1cm} (A.2.1)

\[ \gamma_t = \frac{\beta R^m_t \gamma_{t+1}}{\pi_t} + \mu^c_t + \varphi \mu^f_t \]  \hspace{1cm} (A.2.2)

\[ \gamma_t = \frac{\beta R^r_t \gamma_{t+1}}{\pi_t} + \mu^c_t + \mu^r_t \]  \hspace{1cm} (A.2.3)

As \( \mu^c_t \) and \( \mu^f_t \) are non-negative shadow price of capital constraint and reserve constraint, \( \gamma_t > 0 \) as \( \nu_t > 0 \), we have \( R^r_t \leq R^m_t \leq R^f_t \).

The "" = "" happens when \( \mu^r_t = 0 \), or when the reserve requirement is no longer binding.

**Proof for Theorem 2.2:**

The equation for reserves flows is:

\[ \frac{R^n_{t-1}n_{t-1}}{\pi_t} + \frac{R^f_{t-1}b^f_{t-1}}{\pi_t} + d_t + e_t + \tau_t = n_t + b^f_t \]

In equilibrium, \( b^f_t = 0 \), \( d_t + e_t = 0 \) and:

\[ \tau_t = -\frac{(R^n_{t-1} - 1)n_{t-1}}{\pi_t} + \hat{\tau}_t \]

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Substitute that into the reserves flow:
\[ \frac{n_{t-1}}{\pi_t} + \dot{r}_t = n_t \]
So the total level of reserves only depend on \( \hat{\tau} \), which is decided solely be the central bank.

**Proof for Theorem 2.3:**
We use \( a \) to denote the steady state value of a variable \( a_t \). From the Theorem 2.2, in every steady state:
\[ \pi = \frac{1}{1 - \tau/n} \]  
(A.2.4)

Under the Assumption (2.2):
\[ \frac{\tau}{n} = \frac{\pi - 1}{\pi} \]  
(A.2.5)

(A.2.4) and (A.2.5) \( \rightarrow \pi = \bar{\pi} \). Money supply rule ensures that inflation reaches to its target in the steady state.

From (A.1.24), we have:
\[ R^f = \max \{ \bar{\pi}/\beta, \bar{R}^f + \epsilon_f \} \]  
(A.2.6)

Under the assumption (2.2): \( \bar{R}^f + \epsilon_f < \bar{\pi}/\beta \), we get \( R^f = \bar{\pi}/\beta \). The equation (A.1.2) can be rewritten in the steady state as:
\[ \gamma = \frac{\beta R^f}{\pi} \gamma + \mu^c \]
When \( R^f = \bar{\pi}/\beta \), we get \( \mu^c = 0 \), the capital constraint is not binding (if steady state exists). As \( R^f > \bar{R}^f \), from the Theorem 2.1, \( \mu^c > 0 \), or the reserve requirement is binding.

When \( \mu^c = 0 \), from (A.1.5), at the steady state:
\[ q^l = \frac{\beta \delta_b - \theta \bar{\pi}}{\pi - \beta \delta_b} \]  
(A.2.7)

Under the Assumption (2.1) and (A.1.14), at the steady state, \( \eta_b > 0 \), so the borrowing constraint is binding.

As \( \mu^c > 0 \), we get \( R^m < R^f = \pi/\beta \). From (A.1.12) and (A.1.13), at the steady state, \( \eta^c > 0 \), so the ZMD-in-advance constraint is binding.

**Proof for Theorem 2.4:**
Let \( r \) denote the gross real rate such that \( r = R/\pi \). In the steady state, we have:
\[ 1 = \beta r^f + \frac{\mu^c}{\gamma} \]  
(A.2.8)
\[ 1 = \beta r^m + \frac{\mu^c}{\gamma} + \frac{\phi \mu^f}{\gamma} \]  
(A.2.9)
\[ 1 = \beta r^n + \frac{\mu^c}{\gamma} + \frac{\mu^f}{\gamma} \]  
(A.2.10)
As \( r^f = 1/\beta \) and \( \mu^e = 0 \), we have:

\[
\frac{\mu^f}{\gamma} = 1 - \beta r^n
\]  

(A.2.11)

\[
r^m = \frac{1 - \varphi(1 - \beta r^n)}{\beta}
\]  

(A.2.12)

Besides that, it is easy to see that:

\[
q^L = \frac{\beta \delta b - \pi \theta}{\pi - \beta \delta b}, \quad p^m = \frac{\epsilon - 1}{\epsilon}; \quad b^h = \beta^h; \quad s = (1 - \frac{\delta b}{\pi})b^h
\]

Substitute \( r^m \) into the equation (A.1.13) showing the liquidity premium of ZMDs:

\[
\frac{\lambda^a}{\lambda^b} = \tilde{\beta} r^m
\]  

(A.2.13)

Use (A.2.13) to substitute into (A.1.15), then define \( \alpha_y \) as:

\[
\frac{y}{k} = \frac{1 - \tilde{\beta}(1 - \delta)}{\tilde{\beta} \alpha p^m} \left( \frac{\lambda^a}{\lambda^b} \right) = \frac{1 - \tilde{\beta}(1 - \delta)}{\tilde{\beta} \alpha p^m \beta r^m} = \alpha_y
\]  

(A.2.14)

Use (A.2.14) to substitute into the production function, then define \( \alpha_l \) as:

\[
\frac{l}{k} = \left( \frac{y}{k} \right)^{1/(1 - \alpha)} = \left( \frac{1 - \tilde{\beta}(1 - \delta)}{\tilde{\beta} \alpha p^m \beta r^m} \right)^{1/(1 - \alpha)} = \frac{1}{\alpha_l}
\]  

(A.2.15)

From the banker’s deposit flows:

\[
m = r^n m + q^f s - \frac{\delta b^h}{\pi} + \theta b^h + c + \tilde{\tau} - \frac{(R^n - 1)}{\pi} \frac{n}{\pi}
\]

\[
m = c + \tilde{c} + i + \theta b^h + \varphi m \left( 1 - \frac{1}{\pi} \right) - \frac{(R^n - 1)}{\pi} \frac{\varphi m}{\pi} \quad \text{(Use ZMD in advance)}
\]

\[
\left[ 1 - \varphi \left( 1 - \frac{1}{\pi} \right) + \frac{(R^n - 1)}{\pi} \varphi \right] m = y = \alpha_y k
\]

So we can write:

\[
m = \alpha_m k \quad \text{where} \quad \alpha_m = \frac{\alpha_y}{1 - \varphi \left( 1 - \frac{1}{\pi} \right) + \frac{(R^n - 1)}{\pi} \varphi}
\]  

(A.2.16)

From the ZMD-in-advance constraint:

\[
\tilde{c} = r^n \alpha_m k - \delta k + q^f s - \frac{\delta b^h}{\pi}
\]  

(A.2.17)
From the household’s foc w.r.t labor:

\[
\lambda^a = \frac{\chi l^{y+1}}{(1 - \alpha) p^m y} = \frac{\chi (\alpha l k)^{y+1}}{(1 - \alpha) p^m \alpha y k} = \frac{\chi \alpha_l^{y+1} k^y}{(1 - \alpha) p^m \alpha y}
\]  \hspace{1cm} (A.2.18)

So we have:

\[
\lambda^b = \frac{\lambda^a}{\beta r^m} = \frac{\chi \alpha_l^{y+1} k^y}{(1 - \alpha) p^m \alpha y r^m}
\]  \hspace{1cm} (A.2.19)

So we have an equation with a single variable \(k\):

\[
\frac{1}{r^m \alpha m k - \delta k + q^L s - \delta b \frac{\pi}{\pi}} = \frac{\chi \alpha_l^{y+1} k^y}{(1 - \alpha) p^m \alpha y r^m}
\]  \hspace{1cm} (A.2.20)

**Proof for Theorem 2.5**

Consider the following function:

\[
f(k) = \frac{1}{r^m \alpha m k - \delta k + q^L s - \delta b \frac{\pi}{\pi}} - \frac{\chi \alpha_l^{y+1} k^y}{(1 - \alpha) p^m \alpha y r^m}
\]

Under the Assumption (2.3), it is clear that \(f(k)\) is decreasing with \(k\) when \(k > 0\). Moreover under this assumption, we have:

\[
f(0) = \frac{1}{q^L s - \delta b \frac{\pi}{\pi}} > 0
\]

So \(f(k) = 0\) has a unique positive root \(k^* > 0\). It is equivalent that (A.2.20) has a unique solution \(k^* > 0\).

The steady state value of \(m\) is:

\[
m = \alpha_m k^*
\]  \hspace{1cm} (A.2.21)

We still need to ensure that the capital constraint at this steady state is not binding. That’s why we need the restriction on \(\kappa\) in the Assumption (2.3).
Appendix B

Mathematical Appendix for Chapter 3

B.1 System of Equations in Equilibrium

Bankers:

\[ \gamma_i = \left( \frac{\alpha_i c_i}{c_{i,t}} \right)^{1/\sigma} \frac{1}{c_t}, \quad i = 1, 2 \]  \hspace{1cm} (B.1.1)

\[ \gamma_i = \frac{\beta R_i^t \gamma_{i+1}}{\pi_{i+1}} + \mu_i^c \]  \hspace{1cm} (B.1.2)

\[ \gamma_i = \frac{\beta R_i^m \gamma_{i+1}}{\pi_{i+1}} + \mu_i^c + \varphi \mu_i^r \]  \hspace{1cm} (B.1.3)

\[ \gamma_i = \frac{\beta R_i^n \gamma_{i+1}}{\pi_{i+1}} + \mu_i^c + \mu_i^r \]  \hspace{1cm} (B.1.4)

\[ (q_i^t + \theta) \gamma_i = \frac{\beta [\delta_b + (1 - \delta_b) q_{i+1}^t] \gamma_{i+1}}{\pi_{i+1}} + (1 - \kappa) \mu_i^c \]  \hspace{1cm} (B.1.5)

\[ \frac{n_t - 1}{\pi_t} + \frac{x_t - 1}{\pi_t} + \hat{\tau}_t = n_t + x_t \]  \hspace{1cm} (B.1.6)

\[ m_t = \frac{R_t^n m_{t-1}}{\pi_t} + q_t^t s_t + \theta b_t^h - \delta_b \frac{b_{t-1}^h}{\pi_t} + c_{2,t} + c_{1,t} - x_t + \frac{x_{t-1}}{\pi_t} - \frac{(R_{t-1}^n - 1)n_{t-1}}{\pi_t} + \hat{\tau}_t \]  \hspace{1cm} (B.1.7)

\[ \mu_i^r \geq 0, \quad n_t - \varphi m_t \geq 0, \quad \mu_i^c (n_t - \varphi m_t) = 0 \]  \hspace{1cm} (B.1.8)

\[ \mu_i^c \geq 0, \quad n_t + (1 - \kappa) b_t^h - m_t \geq 0, \quad \mu_i^c \left( n_t + (1 - \kappa) b_t^h - m_t \right) = 0 \]  \hspace{1cm} (B.1.9)

\[ b_t^h = (1 - \delta_b) \frac{b_{t-1}^h}{\pi_t} + s_t \]  \hspace{1cm} (B.1.10)

Households:

\[ \lambda_i + \eta_{i,t} = \left( \frac{\alpha_i \tilde{c}_i}{\tilde{c}_{i,t}} \right)^{1/\sigma} \frac{1}{\tilde{c}_t}, \quad i = 1, 2 \]  \hspace{1cm} (B.1.11)

\[ p_t^n \lambda_t = \chi \]  \hspace{1cm} (B.1.12)

\[ \lambda_t = \frac{\tilde{\beta} (\lambda_{t+1} + \eta_{1,t+1})}{\pi_{t+1}} \]  \hspace{1cm} (B.1.13)
\[ \lambda_t = \frac{\beta R^m_t (\lambda_{t+1} + \eta_{2,t+1})}{\pi_{t+1}} \]  
\[ q'_t(\lambda_t + \eta_{2,t}) = \frac{\beta [\delta_b + (1 - \delta_b)q'_t(\lambda_{t+1} + \eta_{2,t+1})]}{\pi_{t+1}} + \eta'_b \]  
\[ a_t + \delta_b \frac{\tilde{b}_{t-1}^h}{\pi_t} = \frac{R^m_{t-1} m_{t-1}}{\pi_t} + q'_t s_t + z_t \]  
\[ \eta_{1,t} \geq 0, \quad \frac{x_{t-1}}{\pi_t} - \tilde{c}_{1,t} \geq 0, \quad \eta_{1,t} \left( \frac{x_{t-1}}{\pi_t} - \tilde{c}_{1,t} \right) = 0 \]  
\[ \eta_{2,t} \geq 0, \quad a_t - \tilde{c}_{2,t} \geq 0, \quad \eta_{2,t} (a_t - \tilde{c}_{2,t}) = 0 \]  
\[ \eta_t^b \geq 0, \quad \widetilde{b}_t^b - b_t^b \geq 0, \quad \eta_t^b \left( \widetilde{b}_t^b - b_t^b \right) = 0 \]  

Firms:

\[ 1 - t(\pi_t - \bar{\pi}) \pi_t + t \frac{\lambda_{t+1}}{\lambda_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \frac{y_{t+1}}{y_t} = (1 - \rho^m) \varepsilon \]  
\[ y_t = l_t \]  

Market Clearing:

\[ y_t = \sum_{i=1}^{2} (c_{i,t} + \tilde{c}_{i,t}) + \theta \tilde{b}_t^h + \frac{1}{2} (\pi_t - \bar{\pi})^2 y_t \]  
\[ c_t = \left[ \sum_{i=1}^{2} \alpha_{i}^{\frac{1}{2}} \tilde{c}_{i,t} \right]^{\frac{\sigma}{\sigma - 1}} \]  
\[ \tilde{c}_t = \left[ \sum_{i=1}^{2} \alpha_{i}^{\frac{1}{2}} \tilde{c}_{i,t} \right]^{\frac{\sigma}{\sigma - 1}} \]  

Central bank:

\[ R^f_t = \max \left\{ \tilde{R}^f \left( \frac{\pi_{t+1}}{\bar{\pi}} \right)^{\theta_{\pi}}, \bar{R}^m + \delta f \right\} \]  
\[ R^m_t = \bar{R}^m \]  

Shock:

\[ \kappa_t = (1 - \rho_{\kappa}) \bar{\kappa} + \rho_{\kappa} \kappa_{t-1} \]  

### B.2 Mathematical Proof

Proof for Theorem 3.1:

We rewrite the equation (B.1.2), (B.1.3) and (B.1.4):

\[ \gamma_t = \frac{\beta R^f_t \gamma_{t+1}}{\pi_{t+1}} + \mu_{t}^\gamma \]
\[
\gamma_t = \frac{\beta R^n_{t+1}}{\pi_{t+1}} + \mu^c_t + \varphi \mu^r_t
\]
\[
\gamma_t = \frac{\beta R^m_{t+1}}{\pi_{t+1}} + \mu^c_t + \mu^r_t
\]

As \(\mu^c_t\) and \(\mu^r_t\) are non-negative and \(\gamma_t > 0\), we have \(R^n_t \leq R^m_t \leq R^f_t\).

The \(\gamma_t = \gamma_t\) happens when \(\mu^c_t = 0\), or when the reservor requirement is no longer binding.

**Proof for Theorem 3.2:**

Substitute (3.32), (3.36) and (3.35) into the reserve flows equation (3.1), we have:
\[
\frac{n_{t-1} + x_{t-1}}{\pi_t} + \hat{\tau}_t = n_t + x_t
\]

**Proof for Theorem 3.3:**

We omit the subscript “t” to denote the steady state value of a variable. Under the Assumption (3.2) and the result of the Theorem (3.2) \(\pi = \bar{\pi}\). Under the Assumption (3.2) and the Taylor rule (3.34) \(R^f = \bar{\pi}/\beta\). Replace this value of \(R^f\) into (3.8) \(\Rightarrow \mu^c = 0\) (the capital requirement is not binding). From (3.10) and (3.9) \(\Rightarrow R^m\); from (3.11) \(\Rightarrow q^f\).

\[
\frac{\mu^r_t}{\gamma} = 1 - \frac{\beta R^m}{\pi}
\]
\[
R^m = \left(1 - \varphi \frac{\mu^c}{\gamma}\right) \frac{\pi}{\beta}
\]
\[
q^f = \frac{\beta \delta_b - \theta \pi}{\pi - \beta (1 - \delta_b)}
\]
\[
\hat{\tau} = \frac{\pi - 1}{\pi}
\]

Under the steady state, (3.31) \(\Rightarrow p^m = (e - 1)/e\). Next we move on the household’s equation and can find the steady state of:

(3.24) \(\Rightarrow \lambda = \frac{\lambda}{p^m}\)

(3.25) \(\Rightarrow \eta_1 = \left(\lambda - \frac{\beta \lambda}{\pi}\right) \frac{\pi}{\beta}\)

(3.26) \(\Rightarrow \eta_2 = \left(\lambda - \frac{\beta R^m \lambda}{\pi}\right) \frac{\pi}{\beta R^m}\)

(3.27) \(\Rightarrow \eta^b = q^f (\lambda + \eta_2) - \frac{\beta [\delta_b + (1 - \delta_b)q^f] (\lambda + \eta_2)}{\pi} > 0\)

(3.21) \(\Rightarrow b^h = \bar{b}^h\)

(3.3) \(\Rightarrow s = \left(1 - \frac{1 - \delta_b}{\pi}\right) b^h\)
From (3.23), we can perform $\tilde{c}_i$ as a function of $\tilde{c}$:

$$\tilde{c}_i = \frac{\alpha_i (\tilde{c})^{1-\sigma}}{(\lambda + \eta_i)^\sigma}$$  \hspace{2cm} (B.2.1)

Substituting (B.2.1) into the aggregate consumptions (3.22), we can find the steady state value of $\tilde{c}$, then $\tilde{c}_1$ and $\tilde{c}_2$:

$$\tilde{c} = \left( \sum_{i=1}^{2} \frac{\alpha_i}{(\lambda + \eta_i)^{\sigma-1}} \right)^{\frac{1}{\sigma-1}}$$

All the constraints (3.15), (3.17) are binding:

$$x = \pi \tilde{c}_1$$
$$a = \tilde{c}_2$$

From (3.5), (3.32) and (3.16), we can find $m$ and $n$ from the following equations:

$$\frac{[R^m - (R^n - 1) \varphi] m}{\pi} = a + \frac{\delta b^h}{\pi} - \hat{c} - q' s$$

$$n = \varphi m$$

Replacing that into the deposit flows:

$$\sum_{i=1}^{2} c_i = m - \left( \frac{R^m}{\pi} + q' s + \theta b^h - \frac{\delta b^h}{\pi} - x + \frac{x}{\pi} - \frac{(R^n - 1) n}{\pi} + \tau \right)$$ \hspace{2cm} (B.2.2)

From(7), (4) and use (B.2.2):

$$c_i = \frac{\alpha_i c^{1-\sigma}}{\gamma^\sigma}$$

$$\left( \alpha_1 + \alpha_2 \right) = c_1 + c_2$$

$$c_i = \frac{\alpha_i (c_1 + c_2)}{\alpha_1 + \alpha_2}$$

$$c = \left[ \sum_{i=1}^{2} {\alpha_i}^{1/\sigma} \left( \frac{c_i}{c^{1/\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \right]^{\frac{\sigma-1}{\sigma}}$$

We know that this steady state only exists if $\kappa$ satisfy the condition such that capital constraint is not binding, so we need the condition:

$$\kappa < 1 - \frac{(1 - \varphi)m}{b^h}$$
Appendix C

Mathematical Appendix for Chapter 4

Lemma 4.1

In case of no access to the financial market, the entrepreneur problem can be rewritten as:

$$\max_{k_{i,e}^U} (W_i'u + z)(k_{i,e}^U)\eta - \frac{\theta}{2}\sigma_i(k_{i,e}^U)^{2\eta}$$  \hspace{1cm} (C.0.1)

subject to

$$k_{i,e}^U \leq e$$

With $1/2 \leq \eta < 1$, the objective function is strictly concave. Let $\gamma$ be the Lagrangian multiplier with the resource constraint, take the first order condition of (C.0.1) with respect to $k_{i,e}^U$:

$$\eta(W_i'u + z)(k_{i,e}^U)^{\eta-1} - \theta\eta\sigma_i(k_{i,e}^U)^{2\eta-1} + \gamma = 0$$

$$\gamma \geq 0, \quad e - k_{i,e}^U \geq 0, \quad \gamma(e - k_{i,e}^U) = 0$$

Assume $\gamma = 0$, then let $\hat{k}_{i,e}^U$ be the unique positive solution of the first order condition:

$$\hat{k}_{i,e}^U = \left(\frac{W_i'u + z}{\theta\sigma_i}\right)^{1/\eta}$$

The optimal choice of capital for the entrepreneur in the autarky case will be:

$$k_{i,e}^U = \min\{\hat{k}_{i,e}^U, e\}$$

Lemma 4.2:

First, let $t = (k_{i,e})^\eta$ and $\beta = 1/\eta$ $(1 < \beta \leq 2)$, so we rewrite the problem as:

$$\max_{t \geq 0} (e - x_{i,e}'p)s + (W_i'u + z)t - st^\beta + x_{i,e}'A'\mu - w_{i,e} - \frac{\theta}{2} \left(\sigma_t^2 + x_{i,e}'\Omega x_{i,e} + 2tx_{i,e}'\lambda\right)$$  \hspace{1cm} (C.0.2)
We prove the Lemma (4.2) in two steps. First, under the Assumption (4.1) and $s > 0$, we prove the objective function (C.0.2) is strictly concave. Then we only need to examine the first order condition. Let $y_{ie} = [x_{ie} \quad t]' \in \mathbb{R}^{H+1}$, so

$$\text{Var}(n_{ie}) = y_{ie}' \Lambda_i y_{ie} \geq 0 \quad \forall y_{ie} \in \mathbb{R}^{H+1}$$

where

$$\Lambda_i = \begin{bmatrix} \Omega_A & \lambda_i \\ \lambda_i' & \sigma_i \end{bmatrix}$$

So, we have $\Lambda_i$ is positive-semidefinite. Moreover $\Omega_A$ is positive definite, then the Schur complement of $\Omega_A$ in $\Lambda_i$ is

$$S = \sigma_i - \lambda_i' \Omega_A^{-1} \lambda_i \geq 0.$$ Under the Assumption 4.1, we have $(\sigma_i - \lambda_i' \Omega_A^{-1} \lambda_i) > 0$.

Now we are ready to prove the objective function is strictly concave. Let $H_i$ be the Hessian matrix of the objective function (C.0.2). Then:

$$-H_i = \theta \begin{bmatrix} \Omega_A & \lambda_i \\ \lambda_i' & \sigma_i + \frac{s}{\theta} \beta (\beta - 1) t^{\beta - 2} \end{bmatrix}$$

With $1 < \beta \leq 2$, $s > 0$ and $(\sigma_i - \lambda_i' \Omega_A^{-1} \lambda_i) > 0$ we have the Schur complement of $\Omega_A$ in the matrix $(-1/\theta)H_i$ is positive as:

$$\sigma_i + \frac{s}{\theta} \beta (\beta - 1) t^{\beta - 2} - \lambda_i' \Omega_A^{-1} \lambda_i > 0$$

So $(-H_i)$ is positive definite or $H_i$ is negative definite. We finish the first part of the proof such that the objective function is strictly concave when $t \geq 0$.

Take the first order condition with respect to $t$ and $x_{ie}$ (then replace $t = (k_{ie})^\eta$):

$$(W_i' \mu + z) - s \beta (k_{ie})^\eta (\beta - 1) - \theta \sigma_i (k_{ie})^\eta - \theta x_{ie}' \lambda_i = 0$$

$$(A' \mu - ps) - \theta (\Omega_A x_{ie} + (k_{ie})^\eta \lambda_i) = 0$$

**Lemma 4.3:**

The mean and variance of financiers’ net worth:

$$E(n_F) = (-x'_F p) s + x'_F A' \mu + w_{ie}$$

$$\text{Var}(n_F) = x'_F \Omega_A x_F$$

The financier’s problem can be rewritten as:

$$\max_{x_F} \left( (-x'_F p) s + x'_F A' \mu + w_{ie} \right) - \frac{\theta}{2} x'_F \Omega_A x_F$$  \hspace{1cm} (C.0.3)

As $\Omega_A$ is positive definite, the objective function in (C.0.3) is strictly concave. Take the first order
condition of (C.0.3) with respect to $\mathbf{x}_F$, we have:

$$-\mathbf{p}s + \Lambda'\mu - \theta \Omega_A \mathbf{x}_F = 0$$

$$\iff \mathbf{x}_F = \frac{\Omega_A^{-1}(\Lambda'\mu - \mathbf{p}s)}{\theta}$$

**Lemma 4.4:**

Let $\chi = \exp(\theta w_{i,e})$. As the values $\hat{V}_{i,e}, V_{i,e}^U$ and $V_{i,e}^N$ are independent of $w_{i,e}$ and negative, we can rewrite the problem as:

$$\min_{\chi} (\chi \hat{V}_{i,e} - V_{i,e}^U) \alpha \left( \frac{1}{\chi} - 1 \right)^{1-\alpha}$$

Take the first order condition of the above function with respect to $\chi$:

$$\alpha \hat{V}_{i,e} (\chi \hat{V}_{i,e} - V_{i,e}^U) \alpha^{-1} \left( \frac{1}{\chi} - 1 \right)^{1-\alpha} + (1 - \alpha) \left( \frac{1}{\chi^2} \right) (\chi \hat{V}_{i,e} - V_{i,e}^U) \alpha \left( \frac{1}{\chi} - 1 \right)^{-\alpha} = 0$$

$$\iff \alpha \hat{V}_{i,e} \left( \frac{1}{\chi} - 1 \right) + (1 - \alpha) \left( \frac{1}{\chi^2} \right) (\chi \hat{V}_{i,e} - V_{i,e}^U) = 0$$

$$\iff -\alpha \hat{V}_{i,e} \chi^2 + (2\alpha - 1) \hat{V}_{i,e} \chi + (1 - \alpha) V_{i,e}^U = 0 \quad (C.0.4)$$

We have $-\alpha(1 - \alpha) \hat{V}_{i,e} V_{i,e}^U < 0$, so the equation (C.0.4) always have an unique positive solution $\chi^*$, which is:

$$\chi^* = \frac{2\alpha - 1}{2\alpha} + \sqrt{\left( \frac{2\alpha - 1}{2\alpha} \right)^2 + \frac{(1 - \alpha)}{\alpha} \left( \frac{V_{i,e}^U}{\hat{V}_{i,e}} \right)}$$

In the case $\alpha = 0.5$, we have

$$\chi^2 = \frac{V_{i,e}^U}{\hat{V}_{i,e}}$$

In all case, the wage is increasing with the difference between certainty equivalent net worth of matched entrepreneur and unmatched entrepreneur. To see it clearly, let $\hat{n}_{i,e}$ be the net worth of a matched entrepreneur if he can access to the financial market and pay nothing to financiers:

$$\frac{V_{i,e}^U}{\hat{V}_{i,e}} = \exp\left\{ \theta \left[ (E(\hat{n}_{i,e}) - \frac{\theta}{2} \text{Var}(\hat{n}_{i,e})) - (E(n_{i,e}^U) - \frac{\theta}{2} \text{Var}(n_{i,e}^U)) \right] \right\}$$

**Lemma 4.5:**

Let $t_i = (k_i)^\eta \geq 0$ and $1 < \beta = 1/\eta \leq 2$. We can rewrite the the social planner’s problem as:

$$\min_{t_i \geq 0, \mathbf{x}_i} \mathcal{Y} = -\left[ \sum_{i=1}^{I} (W_i^\mu + z)t_i - \frac{\theta}{2} \left( \sigma_t^2 + \mathbf{x}_i' \Omega_A \mathbf{x}_i + 2t_i \mathbf{x}_i' \lambda_t \right) \right] + \frac{\theta I}{2} \mathbf{x}_F' \Omega_A \mathbf{x}_F \quad (C.0.5)$$
subject to

\[ \sum_{i=1}^{I} (t_i)^\beta \leq I\bar{e} \quad \text{(C.0.6)} \]

\[ \sum_{i=1}^{I} x_i + Ix_F = 0 \quad \text{(C.0.7)} \]

First we prove the objective function is strictly convex. Recall the matrix \( \Lambda_i \) we define from the Lemma (4.2). Let \( H \) be the Hessian matrix of the objective function (C.0.5). Then:

\[
H = \begin{pmatrix}
\theta\Lambda_1 & 0 & \cdots & 0 & 0 \\
0 & \theta\Lambda_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \theta\Lambda_I & 0 \\
0 & 0 & \cdots & 0 & \theta I\Omega_A
\end{pmatrix}
\]

From the Lemma (4.2), all \( \Lambda_i \) are positive definite under the Assumption (4.1). We also have \( \Omega_A \) is positive definite. So \( H \) is positive definite. The objective function is strictly convex. We also have the feasible set \( C \) is convex and closed (set is created by the intersection of two constraints (C.0.6) and (C.0.7)). Now, we prove the objective function is coercive on \( C \).

From the constraint (C.0.6), \( 0 \leq t_i \leq (I\bar{e})^\eta, \forall i \). So:

\[
\left( \sum_{i=1}^{I} (W_i^\mu + z) t_i \right)
\]

is bounded

Let \( y_i = [x_i' \quad t_i]' \) and recall positive definite matrix \( \Lambda_i \) from the Lemma (4.2). Let \( f \) be the objective function in (C.0.5)

\[
f(y_1, \ldots, y_I, x_F) = \frac{\theta}{2} \left( \sum_{i=1}^{I} y_i'\Lambda_i y_i + Ix_F'\Omega_A x_F \right) - \left( \sum_{i=1}^{I} (W_i^\mu + z) t_i \right)
\]

\[
\rightarrow +\infty \text{ when } ||(y_1, \ldots, y_I, x_F)|| \rightarrow +\infty \text{ bounded on } C
\]

So we have the objective function is coercive on \( C \) as \( \forall(y_1, \ldots, y_I, x_F) \in C, f(y_1, \ldots, y_I, x_F) \rightarrow +\infty \) whenever \( ||(y_1, \ldots, y_I, x_F)|| \rightarrow +\infty \).

To shorten the notation we denote \( z = (y_1, \ldots, y_I, x_F) \). As \( f \) is coercive on \( C \), we prove the set \( S = \{ z \in C | f(z) \leq 0 \} \) is non-empty and compact. First, if we set \( z = 0 \), then \( f(z) = 0 \), so the set \( \{ z \in C | f(z) \leq 0 \} \) is non empty. Function \( f \) is continuous on \( C \), implying the set \( S \) is closed. So to prove the compactness, we only need to prove \( \{ z \in C | f(z) \leq 0 \} \) is bounded. Assume it is not bounded, then there must exist a sequence \( \{ z^\nu \} \subset C \) with \( ||z^\nu|| \rightarrow \infty \). By the coercivity of \( f \), we must also have \( f(z^\nu) \rightarrow +\infty \). This contradicts the fact that \( f(z) \leq 0 \). So the set \( S \) must be bounded. Then \( S \) must be compact.

Now apply the Weierstrass’s Theorem, as \( f \) is continuous in the compact set \( S \), there must exist \( z^* \)
such that \( f \) attains the minimum value in \( S \). So \( f \) also attains the minimum value in \( C \) at \( z^* \). Moreover, \( f \) is strictly convex in \( C \), so \( z^* \) is unique.

**Theorem 4.1:**

We prove the existence and uniqueness of market equilibrium in three steps. First, we prove every market equilibrium’s allocation (if exists) is also the solution of social planner’s problem in Lemma (4.5). Second, we prove the unique solution of social planner’s problem in Lemma (4.5) is one of market equilibria. From the first two steps and Lemma (4.5), we can conclude the market equilibrium exists and be unique.

**Step 1:** Under the Assumption (4.1)-(4.2), every market equilibrium’s allocation (if exists) is also the solution of social planner’s problem in Lemma (4.5).

Consider again the social planner problem in Lemma (4.5). Let \( \nu \geq 0 \) be the Lagrangian multiplier with the resource constraint (C.0.6) and a \( \gamma = (\gamma_1, ..., \gamma_H) \) be the Lagrangian multipliers with the portfolio constraints (C.0.7).

Take the first order condition (for \( t_i \) after we take the FOC w.r.t \( t_i \) then we replace \( t_i = (k_i)\)\( ^\eta \)) , we have:

\[
- (W_i'\mu + z) + \nu \beta (k_i)^{\eta(\beta - 1)} + \theta \sigma_i (k_i)^{\eta} + \theta x_i' \lambda_i = 0, \quad i = 1, ..., I \tag{C.0.8}
\]

\[
\theta \Omega_A x_i + \theta (k_i)^{\eta} \lambda_i - \gamma = 0, \quad i = 1, ..., I \tag{C.0.9}
\]

\[
\theta I \Omega_A x_F - I \gamma = 0 \tag{C.0.10}
\]

\[
\nu \geq 0, \quad \sum_{i=1}^{I} k_i \leq I \bar{e}, \quad \nu \left( \sum_{i=1}^{I} k_i - I \bar{e} \right) = 0 \tag{C.0.11}
\]

\[
\sum_{i=1}^{I} x_i + I x_F = 0 \tag{C.0.12}
\]

From the Lemma (4.5), every allocation satisfies the system of equations from (C.0.8)-(C.0.12) will be the solution of social planner’s problem.

If we set \( \nu = s \geq 0, \gamma = (A'\mu - p_x) \), then from Lemma (4.2), every market equilibrium allocation will be the solution of the system (C.0.8)-(C.0.12), so they are solutions of the social planner’s problem.

**Step 2:** Under the Assumption (4.1)-(4.2), the planner’s solution is one of the market equilibrium.

Let \( \{k_i^*, x_i^*, x_F^*, \nu^*, \gamma^*\} \) be the solution of the social planner problem. We will prove \( \nu^* > 0 \) under the Assumption (4.1)-(4.2).

Sum the equations (C.0.9) across \( i \), then add with (C.0.10):

\[
\theta \Omega_A (\sum_{i=1}^{I} x_i^* + I x_F^*) + \theta \sum_{i=1}^{I} (k_i^*)^{\eta} \lambda_i = 2I \gamma
\]

\[
\rightarrow \gamma = \frac{\theta}{2I} \sum_{i=1}^{I} (k_i^*)^{\eta} \lambda_i \quad \text{(Use C.0.12)} \tag{C.0.13}
\]
Substitute (C.0.13) back into (C.0.9):

$$x_i^* = -(k_i^*)^\eta \Omega^{-1}_A \lambda_i + \frac{\Omega^{-1}_A}{2I} \sum_{i=1}^I (k_i^*)^\eta \lambda_i$$  \hspace{1cm} (C.0.14)

We put (C.0.14) into (C.0.8):

$$(W'_i \mu + z) - \nu \beta (k_i^*)^{1-\eta} - \theta \sigma_i (k_i^*)^\eta = -\theta (k_i^*)^\eta \Omega^{-1}_A \lambda_i + \frac{\theta}{2} \sum_{j=1}^I (k_j^*)^\eta \lambda_j \Omega^{-1}_A \lambda_i$$  \hspace{1cm} (C.0.15)

Sum (C.0.15) across $i$:

$$\sum_{i=1}^I (W'_i \mu + z) - \nu \beta (k_i^*)^{1-\eta} - \theta \sigma_i (k_i^*)^\eta = -\theta \sum_{i=1}^I (k_i^*)^\eta \lambda_i \Omega^{-1}_A \lambda_i$$  \hspace{1cm} (C.0.16)

We prove under the Assumption (4.2), (C.0.16) fails to happen when $\nu = 0$. Assume $\nu = 0$, the RHS of (C.0.16) is negative in the feasible set, we prove the LHS is positive. Consider the following subproblem:

$$\min g(k_1, ..., k_I) = \sum_{i=1}^I (W'_i \mu + z) - \theta \sigma_i (k_i^*)^\eta$$

subject to

$$k_i \geq 0, \quad \sum_{i=1}^I k_i \leq \bar{I} \hat{e}$$

Solve the above subproblem, we have:

$$\min g = \sum_{i=1}^I (W'_i \mu + z) - \theta \sigma_i (\hat{\sigma}_i \bar{I} \hat{e})^\eta; \quad \text{where} \quad \hat{\sigma}_i = \frac{\sigma_i^{1/(1-\eta)}}{\sum_{j=1}^I \sigma_j^{1/(1-\eta)}}$$

Under the Assumption (4.2), $\min g > 0$. So LHS of (C.0.16) is positive. We must have $\nu > 0$.

Then there is a allocation with $k_{i,e} = k_i^*$, $x_{i,e} = x_i^*$, $s = \nu^* > 0$ and $p = (A' \mu - \gamma')/\nu^*$ which is a market equilibrium.

As from Lemma (4.5), the social planner’s solution exists and be unique. Then from the proof in Step 1 and 2, the market equilibrium exists and be unique.

**Theorem 4.2:**

(i) From the Theorem 4.1, we know that the social planner problem is identical to the market solution under two assumptions. For the economy $\xi'$, the certainty equivalent net worth of the whole economy $Y'$ with $(H + 1)$ asset is:

$$Y' = \max_{k_i, x_i} \left[ \sum_{i=1}^I (W'_i \mu + z) (k_i)^\eta - \frac{\theta}{2} \left( \sigma_i (k_i^*)^{2\eta} + x_i' \Omega_A x_i + 2 (k_i^*)^\eta x_i' \lambda_i \right) \right] - \frac{\theta I}{2} x_F' \Omega_A x_F$$
subject to

\[ \sum_{i=1}^{I} k_i \leq I \bar{e} \]
\[ \sum_{i=1}^{I} x_i + I x_F = 0 \]

If we impose the another constraint to restrict the use of asset \((H + 1)th\) such that \(x_i^{H+1} = 0 \quad \forall i \in I\), and \(x_F^{H+1} = 0\) then the problem become the one when the social planner faces with the economy with only \(H\) assets. So we must have \(Y' \geq Y\) as we have less constraints.

(ii) Let \(Y_N\) be the sum of certainty equivalent net worth for all entrepreneurs if they cannot access to the financial market. We have \(Y_N\) be a constant, in detail:

\[ Y_N = \sum_{i=1}^{I} \bar{e} \int \left( E(n_{i,e}^U) - \frac{\theta}{2} \text{Var}(n_{i,e}^U) \right) f(e) de \]

Use the equation wage bargaining in case \(\alpha = 0.5\) and take log both sides:

\[ w_{i,e} = \frac{1}{2} \left[ (E(\hat{n}_{i,e}) - \frac{\theta}{2} \text{Var}(\hat{n}_{i,e})) - (E(n_{i,e}^U) - \frac{\theta}{2} \text{Var}(n_{i,e}^U)) \right] \]

So the difference between \(Y\) and \(Y_N\) could be consider as the value added of the financial sector and \(Y - Y_N = 2Y_F\). As \(Y' \geq Y\) and \(Y_N\) does not change when we add a new asset, we have \(Y'_F \geq Y_F\).

(iii) We rewrite:

\[ \varphi = \frac{Y_F}{Y} = \frac{Y_F}{2Y_F + Y_N} \]

When we add a new asset, \(Y_F\) increases, \(Y_N\) is a positive constant so \(\varphi\) also increases.

**Proposition 4.1:**

If the risks of two sectors are perfectly negative correlated, \(\lambda_1 = -\lambda_2\), \(\sigma_1 = \sigma_2 = \sigma\) and \(\lambda_1' \Omega_A^{-1} \lambda_1 = \lambda_2' \Omega_A^{-1} \lambda_2 = \lambda' \Omega_A^{-1} \lambda\). From the equation (C.0.15) in the proof of theorem 4.1, we have:

\[ z - \nu \beta(k_1^*)^{1-\eta} - \theta(\sigma - \lambda' \Omega_A^{-1} \lambda)(k_1^*)^\eta = z - \nu \beta(k_2^*)^{1-\eta} - \theta(\sigma - \lambda' \Omega_A^{-1} \lambda)(k_2^*)^\eta \quad \text{(C.0.17)} \]

Under the Assumption (4.1), in the proof of Lemma (4.2), we have \(\sigma - \lambda' \Omega_A^{-1} \lambda > 0\). Consider the function:

\[ f(k) = z - \nu \beta(k)^{1-\eta} - \theta(\sigma - \lambda' \Omega_A^{-1} \lambda)(k)^\eta \]

This function is decreasing in \(k\), therefore the equation (C.0.17) only happens when \(k_1^* = k_2^*\). Now substitute this result in the equation (C.0.14), we have \(x_1 = -x_2\), therefore \(x_F = 0\). The financiers’ transaction income is 0.
Proposition 4.2:

Consider the economy with only one production sector, then \( k_1^* = \tilde{e} \). Substitute this result into (C.0.13) and (C.0.14), we get:

\[
x_1^* = -\frac{1}{2} \tilde{e} \eta \Omega_A^{-1} \lambda_1
\]
\[
\gamma = \frac{\theta}{2} \tilde{e} \lambda_1
\]

From \( x_F = -x_1 \) and \( \gamma = A'\mu - sp \), the expected financiers’ transaction income:

\[
x_F'(A'\mu - sp) = \frac{\theta}{4} \tilde{e}^2 \eta \lambda_1 \Omega_A^{-1} \lambda_1
\]

Using the direct result from the Lemma (4.6), we have the expected financiers’ transaction income is weakly increasing when new asset is added into the economy.

Lemma 4.6:

First, we prove the following result:

\[
\left( \Omega_A^{-1} - \begin{bmatrix} \tilde{\Omega}_A^{-1} & 0 \\ 0' & 0 \end{bmatrix} \right)
\]

is positive semidefinite

Both \( \tilde{\Omega}_A \) and \( \hat{\Omega}_A \) are positive definite matrices. We denote \( A \) as the asset matrix in (\( \xi \)) and \( A_{H+1} \) as the new asset:

\[
Q = \tilde{\Omega}_A^{-1} - \begin{bmatrix} \tilde{\Omega}_A^{-1} & 0 \\ 0' & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\kappa} \tilde{\Omega}_A^{-1} b b' \hat{\Omega}_A^{-1} - \frac{1}{\kappa} \hat{\Omega}_A^{-1} b \\ -\frac{1}{\kappa} b' \hat{\Omega}_A^{-1} 1 \end{bmatrix}
\]

where \( b = A' \Omega A_{H+1} \); \( \kappa = A_{H+1}' \Omega A_{H+1} - b' \hat{\Omega}_A^{-1} b \)

Now we prove \( \kappa > 0 \). As \( \tilde{\Omega}_A \) is invertible, \( \kappa \neq 0 \). We have:

\[
\kappa = A_{H+1}' \left( \Omega - \Omega A (A' \Omega A)^{-1} A' \Omega \right) A_{H+1} = D
\]

Consider a matrix \( M \):

\[
M = \begin{bmatrix} A' \Omega A & A' \Omega \\ \Omega A & \Omega \end{bmatrix}
\]

\( M \) is positive-semidefinite as \( \Omega \) is positive definite and the Schur complement of (\( \Omega \)) in \( M \) is \( A' \Omega A - A' \Omega A^{-1} \Omega A = 0 \). So we have the Schur complement of \( A' \Omega A \) in \( M \) is positive semidefinite, and it is \( D \). So we have \( \kappa \geq 0 \) and \( \kappa \neq 0 \) then \( \kappa > 0 \).

The matrix \( (\kappa Q) \) is positive definite as \( 1 > 0 \) and the Schur complement of 1 in \( (\kappa Q) \) is \( 0 \). As \( \kappa > 0 \), \( Q \)
is also positive semidefinite.

One direct result we get from the above result is:

\[
\hat{\lambda}' \hat{\Omega}_A^{-1} \hat{\lambda} = \left[ \begin{array}{c} \hat{\lambda} \\ A'_{H+1} \Omega W \\ A'_{H+1} \Omega W \end{array} \right] \left[ \begin{array}{c} \hat{\lambda} \\ \Omega^{-1} A' \\ 0 \end{array} \right] \left[ \begin{array}{c} \hat{\lambda} \\ A'_{H+1} \Omega W \\ A'_{H+1} \Omega W \end{array} \right] \leq \tilde{\lambda} \tilde{\Omega}_A^{-1} \tilde{\lambda}
\]

**Theorem 4.3**

Under the Assumption (4.1)-(4.4), \( k^*_i = \tilde{e} \). Substitute this result in the equation (C.0.15):

\[
(W'_{i\mu} + z) - s \beta (\tilde{e})^{1-\eta} = \theta \sigma_i (\tilde{e})^\eta - \frac{\theta}{2} (\tilde{e})^\eta \lambda_i \Omega^{-1} \lambda_i, \quad i = 1, 2
\]

Under the Lemma (4.6) and the above equation, we have: \( \tilde{s} \geq \hat{s} \)

Now we prove the Gini-coefficient is increasing with \( s \). Change the variable from \( y \) to \( e \) and recall \( F(.) \) as the cumulative distribution of initial wealth, we can rewrite the Gini-coefficient as:

\[
G = \frac{1}{2z\tilde{e}} \int_{\tilde{e}} \int_{\tilde{e}} F(e)(1-F(e)) sde
\]

We can see that \( G \) is increasing in \( s \); therefore, \( \hat{G} \geq \tilde{G} \)
Appendix D

Numerical Method

D.1 Inequality Constraints

There are 5 occasionally binding inequality constraints in our model: the reserve requirement, the capital requirement, the ZMD-in-advance, the household’s borrowing constraint and the Taylor rule of the central bank.

For the reserve requirement and the ZMD-in-advance, we apply the method in Zangwill and Garcia (1981) and Schmedders, Judd and Kubler (2002) to transform the inequality constraints into the equality constraints. Here is an example for the reserve requirement:

\[ n_t - \phi m_t = \max \{-\mu'_r, 0\}^2 \]
\[ \gamma_t = \frac{\beta R^n_t \gamma_{t+1}}{\pi_t + 1} + \mu'_c + \max\{\mu'_r, 0\}^2 \]

For the capital requirement and the household’s borrowing constraint, we apply the penalty method in McGrattan (1996) to avoid the ill-conditioned of the system and deal with occasionally binding constraints. So the utility of banker and the capital constraint will be changed as:

\[ U = \log c_t - \frac{\rho_e}{3} \max\{\mu'_c, 0\}^3 \]
\[ n_t + b'_f + (1 - \kappa_t)b^h_t - m_t = -\mu'_c \]

where \( \rho_e = 1000 \) is the penalty coefficient. When the capital constraint is violated, banker will lose the utility. However, when they get positive net worth, they do not get reward for that. The household’s utility also is changed to deal with the borrowing constraint.

For the Taylor rule of the central bank, we use the soft max constraint to deal with the lower bound on \( R^f_{min} = \bar{R}^f + \varepsilon_f \) so we can still take derivative to solve the system of equations:

\[ u_t = \bar{R}^f \left( \frac{\pi_{t+1}}{\pi} \right)^{\phi_t} \]
\[ R^f_t = \begin{cases} 
  u_t + \frac{\log(1+\exp(s_{max}(u_t - u_t)))}{s_{max}} , & \text{if } u_t \geq R^f_{min} \\
  R^f_{min} + \frac{\log(1+\exp(s_{max}(u_t - R^f_{min})))}{s_{max}} , & \text{if } u_t < R^f_{min}
\end{cases} \]
When $s_{\text{max}} \to \infty$, the soft max constraint converges to the hard max constraint. We choose the coefficient $s_{\text{max}} = 1e4$.

\section*{D.2 Dynamics of Economy}

We solve the perfect foresight equilibrium with the unexpected shock by assuming that after $T = 300$ quarters, the economy will converge back to the initial steady state. The initial position before the unexpected shocks is the steady state. Basically, we need to solve a large system of equations to determine the dynamic path of the economy. The transform of occasionally inequality constraints in the previous section ensures that every equation is continuous and differentiable.

For every application, we use homotopy method (by gradually increasing the size of shocks) for solving this large system of equation, with the initial point starting from the steady state or the previous result. We use Ipopt written by \cite{WachterBiegler2006} with the linear solver HSL\footnote{HSL. A collection of Fortran codes for large scale scientific computation. http://www.hsl.rl.ac.uk/} to conduct homotopy.