1. A firm has produced \( x > 2 \) units of (divisible) output at cost \( c \). The firm can sell all or part of its output now, in period 1, or it can wait to sell it in period 2. However, we assume first that output depreciates between periods, so that if \( x_2 > 1 \) units are stored at time 1, then only \( \sqrt{x_2} \) are available for sale at time 2.

a. Initially, suppose the selling prices in the two periods are \( p^0_1 \) and \( p^0_2 \), respectively, and are known, and that \( p^0_2 > 2p^0_1 \). Assuming the firm wishes to maximize aggregate profits and there is no discounting, characterize the optimal amount to sell in each period. Discuss the effects of changes in \( x \), \( p^0_1 \) and \( p^0_2 \) on the optimal quantities.

b. Next, suppose \( p^0_1 \) is known with certainty but it is only known that \( p^0_2 \) can take one of two values, \( p^0_2 + \varepsilon \) or \( p^0_2 - \varepsilon \), with probabilities \( \theta \) and \( 1 - \theta \), respectively, where \( \theta \) is known. Assuming the firm maximizes expected profits, determine the amount it will sell in each period as a function of \( \theta \). Again assume there is no discounting. Show that the amount the firm sells in period 2 is increasing in \( \theta \).

c. Characterize when the firm would sell more in period 2 under the conditions of part b than in part a.

d. Returning to the case in which the prices in both periods are known, suppose the firm discounts future profits by the factor \( \rho < 1 \). In this case, determine the firm’s optimal quantities. How does the optimal allocation vary with \( \rho \)?

e. Again for the case in which the prices are known, suppose the firm does not discount future profits but it must pay interest \( i \) on the inventory it keeps until period 2. This has the effect of imposing an additional cost on the firm based on the value (at period 1 prices) of the period 2 stock. Again determine the optimal quantities to sell in each period. Characterize the relationship between \( \rho \) and \( i \) under which the firm will sell more in period 2 under the conditions of part e than in part d.

f. Finally, suppose that instead of depreciating between periods, output appreciates. In particular, if \( x_2 > 1 \) units are stored at time 1, then \( (x_2)^2 \) units are available in period 2. Under the conditions of part a, characterize the optimal amounts to sell in each period.

2. Let \( M \) and \( W \) be non-empty finite and disjoint sets of men and women. A function \( \mu : M \cup W \to M \cup W \) is a matching if for each \( m \) in \( M \), \( \mu(m) \in W \) or \( \mu(m) = m \); for each \( w \) in \( W \), \( \mu(w) \in M \) or \( \mu(w) = w \); and for any pair \( (m,w) \in M \times W \), \( \mu(w) = m \) if and only if \( \mu(m) = w \). Assume that each man \( m \) has a complete, transitive and strict preference \( \succsim_m \) over \( W \cup \{m\} \). Each woman \( w \) has a complete, transitive and strict preference over \( M \cup \{w\} \). We assume that all men and women prefer being matched (to anyone of the opposite sex) to remaining single (matched to themselves). A non-empty subset \( S \) of \( M \cup W \) blocks \( \mu \) if there is a matching \( \mu' : M \cup W \to M \cup W \) such that for each \( i \in S \), \( \mu'(i) \in S \) and \( \mu'(i) \succsim \mu(i) \). A matching is in the core if it is not blocked by any set \( S \). A matching \( \mu \) is stable if it cannot be blocked by an individual or by a pair.
a. Show that the set of stable matchings is equal to the set of matchings in the core.
b. Consider the preference profile

\[
\begin{array}{cccc}
w_1 & w_2 & w_3 & w_4 \\
m_1 & 1,3 & 3,2 & 2,1 & 4,3 \\
m_2 & 1,4 & 2,3 & 3,2 & 4,4 \\
m_3 & 3,1 & 1,4 & 2,3 & 4,2 \\
m_4 & 2,2 & 3,1 & 1,4 & 4,1 \\
\end{array}
\]

where, \((m_1, w_1) = (1,3)\) means \(m_1\) ranks \(w_1\) first among women and \(w_1\) ranks \(m_1\) third among men, etc. Apply the men \((M)\) proposing deferred acceptance algorithm to the above preference profile and obtain the resulting match.
c. Show directly (without resorting to a theorem) that the resulting match in part b is stable.
d. Start with the match you obtain in part b and then suppose that women apply the Top Trading Cycle algorithm to trade men, ignoring the preferences of men. What would be the outcome? Describe and explain the outcome.

3. A government needs to procure 10 units of a product. There are two firms it can procure the product from. Each firm independently draws constant marginal cost 1 with probability \(\frac{3}{4}\) and constant marginal cost 2 with probability \(\frac{1}{4}\). There is no fixed cost. These draw probabilities are common knowledge. The firms learn their marginal costs but the information remains private. The government announces its compensation plan that is contingent on reports of the firms of their marginal costs. The firms report their marginal costs truthfully or not. The payoff of a firm when it is assigned \(q\) units of procurement quota at procurement price \(p\) per unit is \(pq - cq\), where \(c\) is the firm’s true marginal cost.

a. If both firms report 2, the government pays 2 per unit and splits the order evenly. If both report 1, government pays \(x\), \(1 \leq x \leq 2\) per unit and splits the order evenly. If one firm reports 1 and the other reports 2, the government orders the whole procurement from the lower cost (reporting) firm and pays \(y\), \(1 \leq y \leq 2\) per unit. Find the set of \(x, y\) that would induce the firms to report their true marginal costs in a Bayesian Nash equilibrium.
b. Find the set of \(x, y\) among those given in part a that would minimize the expected procurement cost of the government. What is the minimum expected procurement cost?
c. In part a, find the set of \(x, y\) that would make the firms’ reporting true marginal costs a dominant strategy equilibrium.
d. In part c, find \(x\) and \(y\) that would minimize the expected government procurement cost. What is the minimum procurement cost in this case?
e. Argue directly (without computation) that the minimum expected expenditure of the government in the case of dominant strategy equilibrium cannot be lower than that in the Bayesian Nash equilibrium case.
4. Consider an economy with one firm, one consumer, two inputs (capital and labor) and one output, food. The consumer owns the firm and initially owns 2 units of capital, 4 units of time for leisure or labor and has a strictly increasing, strictly quasiconcave utility function $u(\ell, f)$, where $\ell \geq 0$ is leisure time consumed and $f \geq 0$ is food consumption. The firm produces $F(K, L)$ units of food when it uses $K \geq 0$ units of capital and $L \geq 0$ units of labor time. The production function $F$ is **homogeneous of degree 1**, with strictly positive first order partial derivatives $F_1$ and $F_2$ and continuous second order partial derivatives $F_{ij}, i, j = 1, 2$, satisfying $F_{22}(2, L) > 0$ for $L < 2$ and $F_{22}(2, L) < 0$ for $L > 2$.

a. Interpret the restrictions on $F_{22}$ in the last sentence above. Why might a production function estimated for a real firm have these properties?

b. Justify the claim that the production function $F$ is not concave.

c. Show that $F(K, L)$ is determined by the function $F(2, \cdot)$. You can do this by deriving a formula for $F(K, L)$ in terms of $K$, $L$ and the function $F(2, \cdot)$.

d. Use the given information to draw a possible feasible consumption set for this economy (the set of all feasible consumption vectors $(\ell, f)$ for the consumer). Label the axes and intercepts of the upper boundary of the set and explain why you drew the set the way you did.

e. Add a possible indifference curve of the consumer to the graph in part d in such a way that no competitive (Walrasian) equilibrium exists in this economy. Justify the claim that no competitive equilibrium exists in this case and explain what accounts for that fact.

f. Does the conclusion of the second welfare theorem hold in the economy represented by your graph of part e? Explain how you can tell.

g. Justify the claim that in a competitive (Walrasian) equilibrium of this economy, the firm’s profit is 0.

h. Draw a second graph of the feasible consumption set you drew in part d. Assume now that the economy has a competitive equilibrium and draw, in the graph, a possible indifference curve through the consumer’s equilibrium consumption vector $(\bar{\ell}, \bar{f}) \gg 0$. Draw the consumer’s competitive equilibrium budget line in the space of $(\ell, f)$. Explain why the equilibrium value of the consumer’s food consumption in this equilibrium is greater than the value of the labor time the consumer supplies.

i. Suppose that the consumer consumes a positive amount of food in a competitive equilibrium. Justify the claim that the consumer supplies at least 2 units of labor time in equilibrium.

j. What properties of the fundamentals of the economy in part h account for the existence of competitive equilibrium and differ from the case in part e, where no competitive equilibrium exists? Explain.