Instructions: Answer any **three** of the four numbered problems. **Justify your answers whenever possible.** Write your answer to each numbered question in a separate stapled booklet. Write the number of the question AND NOTHING ELSE on the cover of the booklet. **No electronic devices may be used.** The exam lasts 4 hours.

1. Consider two islands, A and B. On each island there are 3 units of input \( L \) which can be used to produce goods \( x \) and \( y \). On island A, \( x \) and \( y \) are produced according to the technologies \( x = \sqrt{L} \) and \( y = L \), respectively, whereas on island B they are produced according to \( x = L \) and \( y = \sqrt{L} \). On each island, there is one individual with preferences for \( x \) and \( y \) represented by \( u(x, y) = xy \). Hint: Use the symmetry of the problem.
   a. Determine the respective production possibility sets on islands A and B.
   b. Assuming each island is isolated and there is no trade between them, determine a Pareto optimal allocation on each.
   c. Suppose markets are used to solve the allocation problem on each island. Assume the \( x \) and \( y \) firms maximize profits and the consumer maximizes utility, all taking prices as given. The consumer is endowed with the input and owns the firms. In this case determine the prices that would occur in equilibrium on each island.
   d. Next, suppose the inhabitants of the islands discover that it is possible to trade \( x \) and \( y \); \( L \) is nontradable. Determine the competitive equilibrium prices and quantities that would occur in this case.
   e. What would be the pattern of trade in equilibrium? That is, which country would import and export each good? Explain why this pattern emerges.
   f. Show that the inhabitants of both islands would be better off with trade across islands than without it.
   g. Suppose island A were to specialize entirely in the production of \( y \) and B were to specialize in the production of \( x \), after which they would each trade half of their output with the other island. Would this be better or worse than the outcome in part d. Explain why.

2. Consider a firm that uses input, \( L \), to produce output, \( y \), in each of two periods. The prices of \( L \) and \( y \) are \( w \) and \( p \), respectively, and are the same in each period.
   a. First, assume the technology is the same in each period and is given by \( y = \sqrt{L} \). Assuming there is no discounting, determine the firm’s per period supply function, input demand function and its total profit function.
   b. Next, suppose the firm can set aside part of its profits in period 1 to improve the technology in period 2. Specifically, if it sets aside \( r \) in period 1, the new technology would be \( y = (1 + \sqrt{r})\sqrt{L} \). Determine conditions on \( w \) and \( p \) under which the firm would choose to invest \( r > 0 \).
   c. If \( w = p = 1 \), determine the firm’s optimal level of investment.
   d. Next, suppose the outcome of R&D expenditure is uncertain. With probability \( \rho \in (0,1) \), the effort will succeed and the technology will be \( (1 + \sqrt{r})\sqrt{L} \), but with probability \( (1 - \rho) \), it will fail and the technology will remain \( y = \sqrt{L} \), despite spending \( r \). Assume the outcome of the R&D expenditure is known at the start of the second period. In this case, determine the firm’s total profit function if it were to invest \( r > 0 \).
e. Show that when \( w = p = 1 \), the firm’s optimal level of investment is increasing in \( \rho \).

f. Returning to the case in which the outcome of R&D is certain, suppose the firm did discount future profits at the rate \( \delta \in (0, 1) \). Compare the role of discounting versus uncertainty (part d) in deciding the optimal level of investment.

g. Is it possible to say whether the firm is more or less likely to invest when it discounts future profits versus when it does not discount?

3. Two researchers try to complete scientific projects in a single period. Each is endowed with 1 unit of time. They can either work alone or together but they cannot split their time and do both. Each researcher can accomplish at most one project whether alone or jointly during the period. Researcher 1 working alone for \( e_1 \) units of time (which represents his ‘effort’) has probability \( p(e_1) \) of successfully completing his work. If researcher 1 completes his work alone, he consumes the credit (monetary or not) of \( x_1 = 1 \). If he fails to complete the project, he consumes the credit \( x_1 = 0 \). Similarly, researcher 2 is successful working alone with probability \( q(e_2) \) if he puts in time \( e_2 \), and he gets credit \( x_2 = 1 \) if successful and \( x_2 = 0 \) otherwise. If the researchers work together and researcher \( i \) spends \( r_i \) units of time on the joint work, then the probability of success for their joint work is \( q(r_1 + r_2) \), but each \( i = 1, 2 \) gets the credit \( x_i = c, \) 0 < \( c < 1 \) if they succeed. Each gets 0 if they fail. Each researcher \( i \) has a von Neumann-Morgenstern utility function \( u(l_i, x_i) = l_i x_i \), where \( l_i = 1 - e_i \) is the amount of leisure time \( i \) consumes. All variables are nonnegative. The researchers maximize their expected utilities, with

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p(z) \equiv \begin{cases} 
- z^2 + 2z & \text{for } 0 \leq z \leq 1 \\
1 & \text{for } z > 1
\end{cases}
\]

and \( q(z) \equiv \begin{cases} 
z & \text{for } 0 \leq z \leq 1 \\
1 & \text{for } z > 1
\end{cases}.

a. Find the researchers’ optimal efforts and expected utilities if they work alone.

b. Now, suppose that efforts of the researchers are not directly observable and not contractible. The researchers choose their labor inputs independently, but they can only do joint work and this can be enforced. Derive strictly positive Nash equilibrium levels of effort of the two researchers and the corresponding expected utilities. Show why there cannot be a Nash equilibrium involving a researcher devoting either 0 or 1 of his time.

c. Now consider the following game: Researcher 1 either decides to work alone (S) or proposes to Researcher 2 to work together (T). If S is chosen, the researchers work alone. If T is chosen, then researcher 2 either decides to work alone (s) or to work together (t). If researcher 2 chooses s, both researchers work alone. If t is chosen, the workers work together but their efforts are chosen simultaneously and the efforts are neither observable nor contractible. Draw an extensive form game tree describing this game and find pure strategy subgame perfect Nash equilibria for each possible value of \( c \). (When a researcher faces the same expected utility working alone (s or S) and working jointly (t or T), assume that he chooses to work alone).

d. Suppose the researchers’ efforts are observable and contractible. What are the expected utility maximizing cooperative efforts for joint work in which the researchers contribute the same effort levels?
4. A monopoly produces one unit of a perishable good per period at 0 cost to itself. The only potential buyer of the good, attaches value \( v_1 = v > 0 \) to consuming one unit of it in period 1 and the monopoly knows it. In period 1, the monopoly and the consumer believe that the consumer will attach value \( v_2 \) to consuming one unit of the good in period 2, where \( v_2 \) is uniformly distributed over the interval \([v-s,v+s]\), with \( 0 \leq s \leq v \). The monopoly offers one unit of the good to the consumer at the price \( p_1 \geq 0 \) in period 1 and the consumer accepts or rejects the offer. If the consumer rejects the offer, then the unit offered disappears and the monopoly offers the unit of the good produced in period 2 at price \( p_2 \) in period 2. The consumer learns the value of \( v_2 \) in \([v-s,v+s]\) and accepts or rejects the offer. This interaction can be treated as a game of common knowledge in which the players maximize their expected payoffs. The monopoly gets payoff \( a_1p_1 + a_2p_2 \) and the consumer gets \( a_1(v_1 - p_1) + a_2(v_2 - p_2) \), where \( a_t \) equals 1 or 0 depending on whether the consumer accepts or rejects the offer made in period \( t \). Note that the consumer might buy the good in both periods and the agents do not discount period 2 payoffs.

a. Give a complete description of a pure strategy for the monopoly.

b. Under what (if any) conditions on the parameters does the period 2 price offer satisfy \( p_2 \geq v \) in a subgame perfect equilibrium (SPE) of the game?

c. Under what (if any) conditions on the parameters does the monopoly charge a price \( p_2 \) that it believes will be accepted with probability 1 in SPE?

d. Find a pure SPE of the game, assuming \( v < 3s \). Be sure to specify the entire joint strategy.

e. Is it possible that the monopoly could obtain a higher expected payoff than in the SPE of part d if it could commit to prices \( p_1 \) and \( p_2 \) at the beginning of the game?

f. Is it possible that the monopoly could get higher expected payoff by playing a different, possibly more complex, strategy, rather than simply offering a price in each period? Explain.

In the remaining parts of the problem, assume that the consumer’s value \( v_1 \) of consuming the good in period 1 equals either \( v \) or \( u \) \((0 < u < v)\) and that the value \( v_2 \) in period 2 equals \( v_1 \). The consumer knows this and knows its value in period 1. Initially, the monopoly believes \( v_1 = v \) with probability \( \lambda \) and \( v_1 = u \) with probability \( 1 - \lambda \). The agents interact as described above.

g. Can there be a pure sequential equilibrium (SE) of the new game in which, with positive probability, when period 2 is reached the monopoly is sure that \( v_2 = u \) and offers \( p_2 = u \), which the consumer accepts for sure? If so, under what conditions on the parameters? If not, why not?

h. Can there be a sequential equilibrium (SE) in which the monopoly offers a first-time discount in period 2 (so that in period 2 the price is higher if the consumer bought in period 1 than if the consumer did not buy)? Explain.