Answer any three of the following four numbered problems. Justify your answers whenever possible. Write your answer to each numbered problem in a separate bluebook. Write the number of the problem AND NOTHING ELSE on the cover of the bluebook. No electronic devices may be used. The exam lasts 4 hours.

1. Consider a monopolist who produces output $Q$ under constant returns to scale, with marginal cost of 10. The monopolist produces during $T > 2$ periods. Demand for its product in each period is $D(P) = 100 - 2P$, for $P \leq 50$. However, the monopolist does not know the demand at the outset. It only knows that the demand is linear and stationary.
   
   a. One recourse is for the monopolist to sample the market by announcing prices $P_1$ and $P_2$ in periods 1 and 2 and producing whatever quantities $Q_1$ and $Q_2$ consumers demand at these prices. It would then determine demand and behave optimally in subsequent periods. Show how this procedure works for the sample prices $P_1 = 20$ and $P_2 = 40$ and determine the optimal supply and pricing strategy for each period $t > 2$. (Assume there is no discounting of future profits.)

   b. An alternative recourse would be for the monopolist to hire a market research firm at a cost of $s$ to determine the exact demand it faces. What is the value of knowing the exact demand prior to the commencement of operations (i.e., prior to period 1) rather than having to announce the sample prices $P_1 = 20$ and $P_2 = 40$ in periods 1 and 2? Would the firm be willing to pay an amount $s$ up to this value? Is it capable of making this determination ex ante? Discuss.

   c. Instead of announcing specific prices $P_1 = 20$ and $P_2 = 40$, suppose the monopolist were to draw prices randomly from a uniform distribution on the interval $[10, 50]$. Explain how to determine the firm’s expected profit in the single period when it sets this randomly chosen price. Is the firm capable of performing this calculation ex ante? Explain.

   d. Alternatively, suppose the firm knows that its demand is of the form $D(P) = mP + b$. The firm believes that $m$ and $b$ are independently distributed; $m$ has density function $f$ on $[-1, -4]$ and $b$ has density $g$ on $[80, 200]$. In this case, write an expression for the firm’s expected single-period profit from announcing price $P$. Write an expression for its expected profit if it were to announce an optimal $P$. Is it capable of evaluating these?

   e. Discuss how the sampling procedure described above involving one observation per period might be generalized beyond the linear case to allow for an arbitrary demand curve. What considerations might be relevant in deciding whether to hire a market research firm to determine the firm’s demand?

Answer:  

1a. In response to $P_1 = 20$ and $P_2 = 40$, the quantities demanded would be $Q_1 = 60$ and $Q_2 = 20$. From these the firm can determine that inverse demand is $P = -\frac{1}{2}Q + 50$.

1b. If the firm knew the demand, it could compute the profit maximizing $P^* = 30$, $Q^* = 40$, $\pi^* = 800$. Therefore, if it knew $D(P)$, its total profit for periods 1 and 2 would be $1600$. At the sample prices, its profits would be $600$ in each period, or $1200$ overall. Therefore, the value of knowing $D(P)$ vs. sampling is $400$. However, ex ante, the firm is not able to determine this since it does not know $Q_1, Q_2$ or $D(P)$. 
1c. \( \int_{10}^{50} (P(100 - 2P) - 10(100 - 2P)) dP. \) The firm cannot evaluate this ex ante since it doesn’t know \( D(P) \).

1d. For each \( P \), \( E\pi(P) = \int_{-1}^{-4} \int_{80}^{200} (P(mP + b) - 10(mP + b)) g(b)f(m) db dm. \)

   Solving for optimal profits as a function of \( m \) and \( b \), \( \pi = -\frac{(10m+b)^2}{4m} \). Therefore, \( E\pi = \int_{-1}^{-4} \int_{80}^{200} \frac{(10m+b)^2}{4m} g(b)f(m) db dm. \) The firm is able to evaluate these.

1e. This would simply amount to fitting a curve to the data generated by sampling, where a new data point/observation is generated in each period. Each sample price other than the true profit maximizing price would result in a loss of potential profits.

2. Consider a pure exchange economy with two consumers and two goods. When consumer \( i \) consumes \( x_{\ell i} \geq 0 \) units of good \( \ell \), consumer 1 gets utility \( x_{11} + \phi(x_{21}) \) and consumer 2 gets utility \( \phi(x_{12}) + x_{22} - \theta x_{21}^2 \), where \( \theta \geq 0 \) and \( \phi' > 0, \phi'' < 0 \). Consumer 1 initially owns \( e_1 > 0 \) units of good 1 and consumer 2 owns \( e_2 > 0 \) units of good 2.

   a. Describe in English the consumers’ preferences and demand functions, being as specific and complete as possible. For what goods might real consumers have preferences similar to those in the model?

   b. What is the set of feasible allocations in this economy?

   c. Characterize the set of interior Pareto efficient (PE) allocations (in which both consumers consume positive amounts of both goods). In what way does this set depend on \( \theta \)? Starting from a PE allocation, does a small transfer of good 1 from consumer 1 to consumer 2 leave the allocation Pareto efficient?

   d. Let \( \theta = 0 \). Characterize a competitive (Walrasian) equilibrium (CE) for this economy. Is the allocation Pareto efficient?

   e. Let \( \theta = 1/2 \). Characterize a CE when consumer 2 acts as if the equilibrium consumption of good 2 by consumer 1 is fixed, unaffected by actions of consumer 2.

   f. Under what, if any, condition is the CE allocation in part e Pareto efficient?

   g. Is it possible that in the CE allocation in part e consumer 1 consumes inefficiently little of good 2, given consumer 1’s consumption of good 1? Consider the case in which \( \phi(t) = \ln t \). Explain and interpret your answer.

Answer: 2a. The consumers have strictly convex preferences with utility functions quasilinear, but with respect to different goods. This implies that when their wealth is high enough, small variations in wealth, with prices fixed, have no effect on the demand for good 2 by consumer 1 and on the demand for good 1 by consumer 2. There is also an externality if \( \theta > 0 \). Consumption of good 2 by consumer 1 reduces utility of consumer 2, given its consumption of good 2. If good 2 is connected with social status such as a certain type of clothing or “high class” durable (car or house), consumer 1, who is not much concerned with status might buy an amount that does not vary with wealth, while consumer 2 spends all additional wealth on it. Consumer 2 is concerned with status when \( \theta > 0 \), so is worse off the more consumer 1 spends on good 2.

b. \( x_{\ell i} \geq 0, x_{\ell 1} + x_{\ell 2} = e_\ell \), for \( \ell, i = 1, 2 \).

c. A PE allocation \( \bar{x} = (\bar{x}_{\ell i})_{\ell, i} \), with \( \bar{\bar{u}}_2 = \phi(\bar{x}_{12}) + \bar{x}_{22} - \theta \bar{x}_{21}^2 \), maximizes \( x_{11} + \phi(x_{21}) \) subject to the feasible conditions and \( \phi(x_{12}) + x_{22} - \theta x_{21}^2 \geq \bar{\bar{u}}_2 \). The Lagrange function can be written as
\[ \mu_1[x_{11} + \phi(x_{21})] + \mu_2[\phi(x_{12} + x_{22} - \theta x^2_{21} - \bar{u}_2] - \rho_1(x_{11} + x_{12} - e_1) - \rho_2(x_{21} + x_{22} - e_2). \]

Then the first order conditions necessary for an interior solution imply

\[ \rho_1 \phi'(x_{21}) - 2\rho_2 \theta x_{21} - \rho_2 = 0 \quad \text{and} \quad \rho_2 \phi'(x_{12}) - \rho_1 = 0, \]

where \( \rho_1 > 0 \), since at least one Lagrange multiplier must be nonzero. Therefore, \( \rho_2 > 0 \)
and \( \phi'(x_{12})\phi'(x_{21}) = 1 + 2\theta x_{21} \). There are infinitely many interior PE allocations. For given \( x_{12} > 0 \), the PE level of \( x_{21} \) is lower the higher \( \theta \) is. (Stronger negative externality reduces the efficient level of \( x_{21} \).) A small transfer of good 1 from consumer 1 to consumer 2 raises \( x_{12} \) without changing \( x_{21} \), so the last equation no longer holds and the allocation is no longer PE.

d., e., and f. Since the utilities are strictly increasing, neither price can be 0 and we can let the price of good 1 be \( p_1 = 1 \). Suppose \( x_{21} > 0 \) in CE. Utility maximization for consumer 1 implies \( \phi'(x_{21}) = p_2 \). Feasibility requires \( x_{22} = e_2 - x_{21} \). If \( x_{12} = 0 \), then \( x_{11} = e_1 \), which implies \( x_{21} = 0 \) by consumer 1’s budget constraint. So \( x_{12} > 0 \) and utility maximization by consumer 2 requires \( \phi'(x_{12}) = 1/p_2 \). The last four equations determine \( x_{12}, x_{21}, x_{22}, \) and \( p_2 \), with \( x_{11} \) determined by feasibility. These equations also imply \( \phi'(x_{12})\phi'(x_{21}) = 1 \), which shows that the CE allocation is not PE if \( x_{21} > 0 \) and \( \theta > 0 \). The same characterization applies for all \( \theta \).

The first welfare theorem implies that CE allocations are PE if \( \theta = 0 \), since the utilities are strictly increasing. A CE allocation with \( \theta > 0 \) and \( x_{21} = 0 \) is also PE. This can be seen since a CE allocation with \( x_{21} = 0 \) and \( \theta > 0 \) is also CE with \( \theta = 0 \), since the optimization problems of both consumers are the same as when \( \theta = 0 \). Suppose there is a feasible Pareto improvement from such an allocation when \( \theta > 0 \). Since the change in allocation cannot reduce \( x_{21} \), it is also a feasible Pareto improvement when \( \theta = 0 \). But there is no such Pareto improvement since the CE allocation is PE when \( \theta = 0 \). The CE allocation is more likely to have \( x_{21} = 0 \) when \( \phi'(0) \) is small. This makes good 2 less valuable for consumer 1 and more valuable for consumer 2.

g. The answer above shows that a CE allocation with \( x_{21} = 0 \) or \( \theta = 0 \) is PE. In a CE allocation with \( x_{21} > 0 \) and \( \theta > 0 \), \( \phi'(x_{12})\phi'(x_{21}) = 1 \). The CE quantity \( x_{11} \) determines the CE \( x_{21} \) by feasibility. The corresponding PE quantity \( \hat{x}_{21} \) satisfies \( \phi'(x_{12})\phi'(\hat{x}_{21}) = 1 + 2\theta \hat{x}_{21} \), as in the answer to c. The only solution \( \hat{x}_{21} \) to this last equation must be less than the CE level \( x_{21} \) since the right side of the equation is increasing in \( \hat{x}_{21} \) and \( \phi'' < 0 \). So when \( \theta > 0 \), consumer 1 consumes an inefficiently large amount of good 2 in CE. If \( \phi(t) = \ln t \), then \( x_{21} > 0 \) in CE.

3. A government devises an optimal tax schedule for a population, half of whom are of type \( H \), half of type \( L \). A type \( H \) agent produces \( q_H = 2e_H \) units of output (income) when it provides \( e_H \in [0, 1] \) units of effort. A type \( L \) agent produces \( q_L = (3/2)e_L \) units of output when it provides \( e_L \in [0, 1] \) units of effort. Outputs produced by different agents are physically homogeneous, hence indistinguishable, and (except in part a, below) the government cannot directly observe the type of any particular agent or its effort level. The government observes agents’ incomes (outputs) and assigns a tax to each income level. The government knows that if a type \( i \) agent produces \( q_i \) units of output, providing \( e_i \) units of effort and paying \( t_i \leq q_i - e_i \) in tax, the agent gets utility \( (q_i - t_i - e_i)^{1/2} \). (If \( t_i < 0 \), then \( |t_i| \) is the amount of subsidy type \( i \) receives.) The government wants to maximize the average utility \( (1/2)[(q_H - t_H - e_H)^{1/2} + (q_L - t_L - e_L)^{1/2}] \) subject to a tax revenue constraint (net tax revenue must be nonnegative) and subject to incentive
constraints, where \( q_i \) and \( e_i \) are the output and effort level chosen by type \( i \) and \( t_i \) is the corresponding tax. The incentive constraints reflect the fact that each type of agent is free to choose any effort level in \([0, 1]\), produce what it can produce and pay the corresponding tax.

a. For this part of the problem only, consider the case of perfect information, where the government can observe the effort level of each agent and tax accordingly. Explain why, under an optimal tax schedule, the chosen \((q_i, e_i, t_i)\) for types \( i = H, L \) maximize average utility subject only to the constraint that net tax revenue from these choices is nonnegative. Compute the solution to this optimization problem.

b. From now on, we return to the case in which the government cannot observe agents’ types and effort levels. Explain why, under an optimal tax schedule, the chosen \((q_i, e_i, t_i)\) for types \( i = H, L \) maximize average utility subject to the tax revenue constraint and to an incentive constraint for each type, which specifies that that type gets at least the utility it would get by producing as much output as the other type chooses. Write this constrained optimization problem using the notation above.

c. Show that if the incentive constraint for type \( L \) is omitted from the optimization problem in part b, then the incentive constraint for type \( H \) binds at a solution.

d. Solve the optimization problem in part b with the incentive constraint for type \( L \) omitted.

e. Show that at the solution to the optimization problem in part d, the incentive constraint for type \( L \) in part b is satisfied. What can be concluded about the allocation and the utilities of the different types under an optimal tax schedule in part b? Be as specific as possible.

Answers: 3a.

4. Candidates for a job with a single employer, E, are either of type \( H \) or \( L \). Knowing their own type, they choose whether or not to take a training course. An untrained candidate (who has not completed the course) produces output valued at 2 units working for E. A type \( H \) candidate who is trained (has completed the course) produces a value of 6 units working for E. A trained type \( L \) produces 5 units working for E. Untrained workers can get jobs with other firms paying 1 unit. Trained workers of type \( H \) [respectively, type \( L \)] can get jobs with other firms paying 4 units [respectively, 2 units]. Training costs type \( H \) a value of 1 unit and costs type \( L \) 2 units, a cost that must be fully paid before the course is completed. The interaction between the candidate and E can be represented as a game in which E’s payoff equals the value of the candidate’s output minus the wage E pays if the candidate accepts E’s offer and equals 0 otherwise. An offered wage can be any real number. The candidate’s payoff is its wage (from E or else from another firm) minus its training cost (no cost if it does not take the training course). In the game, the candidate first chooses whether to get training, then E offers a wage that can depend on whether the candidate is trained. The candidate accepts E’s offer or else rejects it and works for another firm. The interaction described above is common knowledge.

a. Why might it make sense to model a type \( L \) candidate as having a higher cost of training than a type \( H \) candidate even if all students pay the same price for training courses?

b. Suppose that the job candidate is of type \( L \) and E knows it. Explain what a pure strategy for E is. How many pure strategies does E have? Give an example of a pure
strategy of the type $L$ candidate. Be sure to specify or describe it completely.

c. For the game in part b with a type $L$ candidate, find a pure subgame perfect Nash equilibrium (SPE) of the subgame after the candidate gets training.

d. Find a pure SPE for the entire game in part b, where $E$ knows that the candidate is type $L$. Interpret the outcome. Does the outcome maximize total surplus over all allocations that are feasible with a type $L$ candidate?

In the remaining parts of the problem, assume that $E$ initially does not know the candidate’s type and believes it is $H$ with probability $\lambda$, where $\frac{1}{2} < \lambda < 1$.

e. Draw a game tree representing the interaction between $E$ and the job candidate.

f. In the game described above, how many pure strategies does the employer $E$ have? Give an example of one.

g. In the game described above, how many pure strategies does the job candidate have? Give an example of one.

h. Show that there is a weak sequential equilibrium (WSE, also called WPBE) in which neither type candidate gets training. What can be said about the plausibility of $E$’s beliefs in such an equilibrium?

i. Show that there is a pure WSE in which both candidate types get training. Compare the efficiency of the allocation for type $L$ in this WSE to that in the SPE of part d. Use the comparison to describe the impact of $E$’s uncertainty about the candidate’s type.

Answer: 4a. Type $L$ might have to work longer or harder than $H$ to complete the course. Since $L$ types have lower productivity and wages from other firms than $H$ types, transportation costs or other costs associated with the training could be higher for them than for $H$ types.

b. A pure strategy for $E$ is a wage offer $w$ for an untrained candidate and a wage $W$ for a trained one. $E$ has infinitely many possible pure strategies ($w, W$). A pure strategy for the $L$ type is a choice to train, $t$, or not train, $n$, and a reaction to all pairs of wage offers. The reaction can be denoted by $a_n : w \mapsto \{0, 1\}$, $a_t : W \mapsto \{0, 1\}$, where $a_n(w) = 1$ means that the offer $w$ when $L$ is untrained is accepted and $a_n(w) = 0$ means it is rejected, while $a_t(W) = 1$ or 0 when $W$ is accepted or rejected when $L$ is trained. An example of a pure strategy for $L$ is the strategy $n$ and $a_n(w) = 1 \iff w \geq 1$, $a_t(w) = 1 \iff w \geq 2$.

c. In the subgame after $t$ (L gets training), $E$ offers wage $W$. Type $L$’s best response is to accept any wage with a payoff $W - 2 > 2 - 2$, hence any $W > 2$ and to reject any $W < 2$. The only best response by $E$ is $W = 2$, which $L$ must accept in SPE. (If $L$ rejects $W = 2$ with positive probability, then $E$ raises its expected payoff by offering $W$ slightly higher than 2.) Thus the only SPE in the subgame after $t$ is $W = 2$ and $a_t(W') = 1 \iff W' \geq 2$.

d. If $E$ offers $W = 2$ after $t$, then type $L$ gets payoff 0. But $L$ gets payoff 1 by choosing $n$ (no training) since in the SPE after $n$, by the same argument as in the answer to c, $L$ chooses $a_n(w) = 1 \iff w \geq 1$, $E$ chooses $w = 1$, and $L$ chooses $n$ (no training). The outcome is inefficient in the sense that $t$ (getting training) yields higher total surplus: output $- \text{training cost} = 5 - 2 = 3$, higher than the output 2 when $L$ chooses $n$. This is an example of a holdup problem. It is efficient for type $L$ to invest in training, but once it does, $E$ can get the candidate to accept a wage that does not fully compensate it for the training cost.

e. In the game tree, $T$ and $t$ represent taking the training course if the candidate is type
\( H \) or \( L \). \( N \) and \( n \) represent not taking the course in those two cases. \( W \) and \( w \) are E’s wage offers to a trained and untrained candidate. \( A, A', a \) and \( a' \) are accept in each of the candidate’s information sets and \( R, R', r \) and \( r' \) are reject in those information sets.

f. E has infinitely many pure strategies, the same wage pairs \((w, W)\) as in part b. An example: \( w = 1, W = 2 \).

g. The candidate has infinitely many pure strategies of the form in the answer to d for each possible type \( H \) and \( L \).

h. There is a WSE in which E believes the candidate is of type \( L \) if trained. If the candidate is untrained, E believes the type is \( H \) with probability \( \lambda \). The beliefs are rational as long both types choose no training. Sequential rationality requires that untrained candidates accept wage offers of 1 or higher and that trained type \( H \) accepts of 4 or more and trained type \( L \) offers of 2 or more. The best response by E is the offer pair \((w, W) = (1, 2)\) and the best response by both types is no training. This WSE is also a sequential equilibrium. Different beliefs are more plausible. The productivity gain and gain in wages from other firms is bigger for type \( H \) than for type \( L \), yet after observing that the candidate is trained, E believes that there is less chance the type is \( H \) than if E had no information about the candidate’s training. An alternative argument follows the equilibrium dominance argument that justifies “intuitive criteria”: If E believes the trained type is \( H \), then it offers at least 4, giving type \( H \) a payoff above its WSE payoff. If E believes the trained type is \( L \), it offers wage 2, giving type \( L \) a payoff below its WSE payoff. Thus, E should not believe there is no chance that a trained candidate’s type is \( H \).

i. If, in WSE, both types get trained, then E believes the candidate is of type \( H \) with probability \( \lambda \geq 1/2 \). E can have the same belief if the candidate is untrained. Then E offers an untrained candidate wage 1 and a trained candidate wage 4. Both types accept these offers. E’s expected payoff from wage 4 is \( 6\lambda + 5(1 - \lambda) - 4 = \lambda + 1 \). E’s next best alternative is to attract type \( L \) with a wage of 2, yielding E an expected payoff of \((5 - 2)(1 - \lambda)\), which is less than \( 1 + \lambda \) when \( \lambda > 1/2 \). Because of E’s uncertainty, type \( L \) is better off (with training and wage 4) than if E knew its type as in part d.