1. Suppose an individual’s “health preferences” are represented by the utility function
\[ u(s, e, h) = s(2 - e) + h, \]
where \( s \) is sugar, \( e \) is exercise, and \( h \) is its general state of health. In turn, \( h \) is adversely (i.e., negatively) affected by \( s \) and positively affected by \( e \). Specifically, suppose \( h(s, e) = \ln e - (s^2/2) \). Overall, \( s \) and \( e \) are subject to the respective constraints \( 0 \leq s \leq 2 \) and \( 1/2 \leq e \leq 2 \).

a. How much \( s \) and \( e \) would the individual choose to consume? Verify that this is indeed a maximum.

Next, suppose that while the direct effects of \( s \) and \( e \) on \( u \) are certain, the effects of these on health are not. Rather, each of them may or may not have the purported (i.e., claimed) effect on health. For \( x = s \) or \( e \), let \( \pi_x \) denote the probability that factor \( x \) does have the purported health effect and \( (1 - \pi_x) \) denote the probability that \( x \) does not, and assume \( \pi_s \) and \( \pi_e \) are independent. In the event that \( s \) does affect health but \( e \) does not, then \( h \) is given by \( h(s) = -s^2/2 \). Similarly, if \( e \) is effective but \( s \) is not, then \( h(e) = \ln e \); and if neither is effective, then \( h = 0 \).

b. For each case, determine the optimal choices of \( s \) and \( e \) if the individual knew whether or not the purported health effects were true.

c. Returning to the case in which the true effects on health are uncertain, write the decision problem facing the individual and characterize an interior solution assuming the individual maximizes expected utility. What, if any, restrictions on \( \pi_s \) and \( \pi_e \) ensure that an interior solution exists?

d. Discuss the comparative static effects of \( \pi_s \) and \( \pi_e \) on the optimal \( s \) and \( e \) in part c.

e. Suppose that regardless of whether or not the purported health effects of \( s \) and \( e \) are true, health is subject to an exogenous adverse shock that occurs with probability \( \pi_o \). In that event, \( h(s, e, \varepsilon) = \ln e - (s^2/2) - \varepsilon \). What effect would this have on the optimal values of \( s \) and \( e \)?

f. Suppose the individual is currently at an interior solution as in part c. The government is considering a proposal to ban sugar entirely (\( s \equiv 0 \)). Would this be beneficial or detrimental to the individual?

g. Finally, suppose that sugar cannot be obtained directly by the individual. However, in order to encourage people to exercise, insurers provide one unit of sugar for each unit of \( e \), i.e., \( s \equiv e \). How will this affect the choice of \( s \) and \( e \) at an interior solution?

Answers: 1a. It is simplest to assume that sugar is free since the problem does not mention any cost of sugar other than the health cost. The first order conditions necessary for an interior solution are \( 2 - e - s = 0 \) and \( -s + (1/e) = 0 \), which imply \( 2e - e^2 - 1 = 0 \) and \( e = s = 1 \). The corresponding utility is \( 1/2 \). If instead \( s = 2 \), then the optimal \( e \) is \( e = 1/2 \), but, given this \( e \), \( s = 2 \) is not optimal (see part b). If \( s = 0 \), then the optimal \( e \) is 2 and the utility is \( \ln 2 > 1/2 \), so \( (s, e) = (0, 2) \) is optimal.
b. If $s$ alone affects health, then $u = s(2 - e) - (s^2/2)$. If $s > 0$, then $e = 1/2$ is uniquely optimal and the optimal $s$ satisfies $2 - e - s = 0$, so $s = 3/2$. In that case, $u = (3/2)^2/2 = 9/8$ is greater than 0, the value of $u$ if $s = 0$, so $(s, e) = (3/2, 1/2)$ is optimal. If $e$ alone affects health, then $u = s(2 - e) + \ln e$. If $e < 2$, then $s = 2$ is optimal. If $s = 2$, then $e = 1/2$ is optimal and $u = 3 - \ln 2$, greater than $\ln 2$, the value of $u$ if $s = 0, e = 2$. So $(s, e) = (2, 1/2)$ is optimal in this case and is optimal if neither variable affects health.

c. The problem is to maximize $s(2 - e) + \pi_s \pi_e (\ln e - (s^2/2) + \pi_s (1 - \pi_e)(-s^2/2) + (1 - \pi_s) \pi_e \ln e = s(2 - e) + \pi_e \ln e - \pi_s (s^2/2)$. First order conditions necessary at an interior solution are $2 - e - \pi_s s = 0$ and $-s + (\pi_e/e) = 0$, which imply $2e - e^2 - \pi_s \pi_e = 0$. This equation has two real roots when $\pi_s \pi_e < 1$. To check the second order conditions, note that the optimal $s$, given $e$, is $s = (2 - e)/\pi_s$. Expected utility as a function of $e$, given the corresponding optimal $s$, is $[(2 - e)^2/(2\pi_s)] + \pi_e \ln e$, which is maximized at the lower of the two roots, $e = 1 - \sqrt{1 - \pi_s/\pi_e}$. An interior must satisfy $e = 1 - \sqrt{1 - \pi_s/\pi_e} > 1/2$ and $s = \pi_e/\left\{1 - \sqrt{1 - \pi_s/\pi_e}\right\} < 2$. These inequalities imply $\pi_e/\pi_s > 3/4$ and $4\pi_s + \pi_e > 4$. Expected utility $\hat{u} \equiv \left\{(1 + \sqrt{1 - \pi_s/\pi_e})^2/(2\pi_s)\right\} + \pi_e \ln(1 - \sqrt{1 - \pi_s/\pi_e})$ at an interior solution must also be at least as great as at any boundary point of the constraint set. If $e = 2$, then the optimal $s$ is 0 (and $e = 2$ is optimal if $s = 0$) and expected utility is $\pi_e \ln 2$. If $e = 1/2$, then the optimal $s$ is 3/(2$\pi_s$) and expected utility is $[9/(8\pi_s)] - \pi_e \ln 2$, since $\pi_s > 3/4$. So $\hat{u}$ must be at least as great as these levels for there to be an interior solution. No other case need be considered since $e = 1/2$ is optimal when $s = 2$. [It can be shown that the solution is interior if $\pi_s = \pi_e = 7/8$, but this was not part of the problem.] The boundary point $(s, e) = (0, 2)$ is optimal if $\pi_s$ and $\pi_e$ are near 1. The boundary point $(s, e) = (2, 1/2)$ is optimal if $\pi_s$ is small.

d. If the optimization problem has an interior solution for $\pi_s$ and $\pi_e$, as in part c, then higher $\pi_s$ or $\pi_e$ increases optimal $e = 1 - \sqrt{1 - \pi_s/\pi_e}$ and reduces $s = (2 - e)/\pi_s$.

e. If an unknown constant that is perceived as uncorrelated with the other parameters is added to the objective function, it has no effect on the optimal values of the control variables. Whatever maximizes the original objective function maximizes the new one, no matter what the constant is.

f. Adding a constraint to a constrained maximization problem typically reduces and cannot increase the optimal value of the objective function.

g. Now the consumer maximizes $2s - s^2 - \pi_s (s^2/2) + \pi_e \ln s$ with respect to $s$ and lets $e = s$. The derivative is $2 - (2 + \pi_s)s + (\pi_e/s)$, which is decreasing, positive at $s = 0$ and negative at $s = 2$. So there is an interior solution at $s = e = [1 + \sqrt{1 + \pi_e(\pi_s + 2)}/(\pi_s + 2)$. This value of $e$ is greater than its unconstrained interior optimal value in part e. To show why, we can compare the two solutions directly. But it is easier to compare the polynomials $\psi(e) \equiv 2e - e^2 - \pi_e \pi_s = 0$ and $\phi(e) \equiv 2e - (2 + \pi_s)e^2 + \pi_e = \psi(e) + (1 + \pi_s)(\pi_e - e^2)$, the roots of which are the optimal levels of $e$. At the unconstrained optimal $e$, $\psi(e) = 0$ and $e^2 < \pi_e$, so $\phi(e) > 0$. This implies that the constrained problem in part g has a higher optimal level of $e$. The same method shows that the constrained optimal value of $s$ in part g is smaller than the unconstrained optimal value in part c. The insurers' plan leads the consumer to exercise more and eat less sugar.
2. In a pure exchange economy with two consumers and two goods, when consumer $i$ ($i = 1, 2$) consumes $x_{\ell i} \geq 0$ units of good $\ell$, consumer 1 gets utility $x_{11} + \phi(x_{21}) - \theta x_{22}^2$ and consumer 2 gets utility $x_{12} + \phi(x_{22})$, where $\phi' > 0$, $\phi'' < 0$, and $\theta \in \{0, 1/2\}$. Consumer 1 initially owns the entire endowment $e_1 > 0$ of good 1 in the economy and consumer 2 owns the entire endowment $e_2 > 0$ of good 2. The problems below that ask for “conditions on the economy” are asking for restrictions on what is given exogenously.

a. Describe the consumers’ preferences in English.

b. Characterize all the Pareto efficient (Pareto optimal) allocations in which both consumers consume positive amounts of both goods. Under what conditions on the economy is there a Pareto efficient allocation with $x_{22} = 0$?

c. Characterize the competitive (Walrasian) equilibria in which consumer 1 treats the consumption of good 2 by consumer 2 (consumer 2’s choice) as given, unaffected by actions of consumer 1. Can there be more than one competitive equilibrium allocation in this economy? Under what conditions on the economy is a competitive equilibrium allocation Pareto efficient when $\theta = 0$? Explain your answer. Under what conditions on the economy is a competitive allocation Pareto efficient when $\theta = 1/2$?

d. Consider a competitive equilibrium in which the price of good 2 is 1 and consumer 2 is charged a fee of $\theta x_{22}^2$ in addition to the purchase cost of good 2. The fee paid by consumer 2 is given as a lump sum transfer to consumer 1 in compensation for consumer 1’s utility loss due to consumer 2’s consumption. (Consumer 1 acts as if the fee consumer 2 pays cannot be affected by anything consumer 1 does.) Characterize a resulting competitive equilibrium allocation. Is it necessarily Pareto efficient?

e. Suppose, instead, that consumer 2 is free to bargain with consumer 1 without being charged the fee in part d. Consumer 2, knowing the utility function of consumer 1, chooses a consumption bundle $(x_{12}, x_{22}) \gg 0$, and offers consumer 1 two alternatives. Either

(1) consumer 2 will give consumer 1 $e_2 - x_{22}$ units of good 2 in return for receiving $x_{12} > 0$ units of good 1 from consumer 1, or

(2) consumer 1 will consume its endowment $e_1$ of good 1 and get none of good 2. Consumer 1 must choose one of the alternatives, (1) or (2). Characterize a pure subgame perfect Nash equilibrium (SPE) in which the consumers’ payoffs are their utilities. Which alternative does consumer 1 choose? Show that consumer 1 gets the same payoff as if it chose the other alternative. Is a SPE allocation necessarily Pareto efficient?

f. Compare the quantity of good 2 consumed by consumer 2 in a competitive equilibrium allocation in part d with that in a SPE allocation in part e. How do the entire allocations in parts d and e differ from each other (if they do)? Explain why the differences arise if they do.

Answers: 2a. Each consumer’s willingness to pay in units of good 1 for an amount of good 2 does not depend on how much of good 1 the consumer has. Both consumers care about the goods they consume and want more of them, but consumer 1 would be willing to give up some of either good in order to have consumer 2 consume less of good 2. (Consumer 2’s consumption of good 2 imposes a negative externality on consumer 1.

b. A Pareto efficient (PE) allocation in which consumer 2 gets utility $\bar{u}_2$ maximizes the utility of consumer 1 over the set of feasible allocations giving consumer 2 at least utility $\bar{u}_2$. Otherwise a Pareto improvement is feasible. This optimization has the Lagrange
function
\[ L = \mu_1(x_{11} + \phi(x_{21}) - \theta x_{22}^2) + \mu_2(x_{12} + \phi(x_{22}) - \bar{u}_2) - \rho_1(x_{11} + x_{12} - e_1) - \rho_2(x_{21} + x_{22} - e_2). \]

First order conditions necessary for an interior solution are \( \mu_i = \rho_1 > 0, \ i = 1, 2, \mu_1\phi'(x_{21}) - \rho_2 = 0, \mu_2\phi'(x_{22}) - 2\mu_1\theta x_{22} - \rho_2 = 0. \) These imply \( \phi'(x_{22}) = \phi'(x_{21}) + 2\theta x_{22}. \) If \( x_{22} = 0 \) and \( x_{12} > 0, \) then \( x_{21} = e_2 > 0. \) The first order conditions become \( \mu_2 = \rho_1 \geq \mu_1 \) and \( \mu_2\phi'(0) \leq \mu_1\phi'(e_2), \) contradicting the assumption \( \phi'' < 0. \) Therefore, the only PE allocation with \( x_{22} = 0 \) also has \( x_{12} = 0, \) \( x_{1i} = e_i, \ i = 1, 2. \) Every other feasible allocation gives consumer 1 less utility, so this allocation is PE without any additional restriction on what is exogenously given in the problem.

c. The consumers’ optimization problems have Lagrange functions \( x_{1i} + \phi(x_{2i}) - \lambda_i(p_i x_{1i} + p_2 x_{2i} - p_i e_i) \) and first order conditions \( 1 - \lambda_i p_i \leq 0, = 0 \) if \( x_{i1} > 0, \phi'(x_{2i}) - \lambda_i p_2 \leq 0, = 0 \) if \( x_{i2} > 0. \) This implies positive prices in CE, so that each consumer consumes a positive amount of at least one good. At an interior solution, \( \phi'(x_{2i}) = p_i/p_1, \) so \( x_{21} = x_{22} = e_2/2, \) \( p_1 x_{11} + p_2(e_2/2) = p_1 e_1, x_{11} = e_1 - (p_2/p_1)(e_2/2), \) and \( x_{12} = (p_2/p_1)(e_2/2), \) hence \( \phi'(e_2/2) = p_2/p_1 < 2 e_1/e_2. \)

If this last inequality is violated, then the allocation is not interior. An argument similar to the one in part b implies that in such an allocation, \( x_{i1} = 0 \) and \( x_{i2} > 0 \) if \( \ell \) or \( i \) is not 1. In this allocation, \( x_{21} = e_1 \) and \( e_1 + (p_2/p_1)x_{22} = e_2/p_1. \) Since \( p_2/p_1 = \phi'(x_{22}), \) \( e_1 = (e_2 - x_{22})\phi(x_{22}). \) The right side of this equation is decreasing in \( x_{22}, \) so \( x_{22} \) is uniquely determined and the rest of the allocation is determined by budget identities. Thus, there cannot be more than one CE allocation.

If \( \theta = 0, \) then each competitive equilibrium (CE) allocation is PE, by the first welfare theorem, since the consumers are locally nonsatiated. If \( \theta = 1/2, \) then the CE allocation cannot be PE, since \( x_{21} > 0 \) and the first order conditions for interior PE allocations are violated.

d. The budget identity for consumer 2 changes to \( p_1 x_{11} + p_2 x_{22} + \theta x_{22}^2 \leq p_2 e_2 \) and the first order conditions imply \( \lambda_1 = 1/p_1 = \lambda_2 = \phi'(x_{21}) \) and \( \phi'(x_{22}) = \phi'(x_{21})(1 + 2\theta x_{22}). \) As before, the allocation is PE if \( \theta = 0. \) If \( \theta = 1/2, \) the allocation is PE only if \( \phi'(x_{21}) = 1 \) at the equilibrium. Since \( \phi'(x_{22}) - \phi'(e_2 - x_{22})(1 + 2\theta x_{22}) \) is decreasing in \( x_{22}, \) it has a unique 0, given \( e_2, \) and, by the implicit differentiation, this equilibrium value of \( x_{22} \) increases with \( e_2. \) So there is a unique level of \( e_2 \) for which the equilibrium is PE. When \( e_2 \) is higher [respectively, lower] than this level, \( \phi'(x_{21}) < [>] 1 \) and the equilibrium value of \( x_{21} \) is higher [resp., lower] and the value of \( x_{22} \) is lower [higher] than the PE level.

e. In SPE, consumer 1 accepts any offer giving utility above \( \bar{u}_1 \equiv e_1 + \phi(0) - \theta e_2^2. \) Consumer 2’s SPE offer must solve the problem \( \max x_{12} + \phi(x_{22}) \) s.t. \( e_1 - x_{12} + \phi(e_2 - x_{22}) - \theta x_{22}^2 - e_1 - \phi(0) - \theta e_2^2 \geq 0. \) Otherwise, the offer is rejected or else it gives consumer 1 utility above \( \bar{u}_1. \) In either case, consumer 2 can get a higher payoff by making an offer with the constraint satisfied with a small strict inequality. This argument shows that the offer by consumer 2 is accepted in SPE and gives consumer 1 utility \( \bar{u}_1. \) The fact that the SPE allocation solves the optimization problem implies that it is not feasible to raise the utility of consumer 2 without reducing the utility of consumer 1. Since the Lagrange multiplier on the utility constraint in the optimization problem is positive, it is not feasible to raise the utility of consumer 1 without reducing the utility of consumer 2. Therefore the SPE allocation is PE. The characterization is as in part b.

f. As shown in the answer to d, there is a level of \( e_2 \) such that the allocation in d is PE. If \( e_2 \) is higher [respectively, lower], then \( x_{22} \) in d is less [more] than in e. By treating \( \theta \)
as a continuous variable, it can be shown that \( x_{12} \) is higher in e than in d. Consumer 2 gets less of good 1 when it has to pay compensation for the negative externality it imposes on consumer 1.

3. Two roommates are considering buying an air-conditioner that costs $100. It is common knowledge that the utility function of roommate \( i \) is of the form \( u_i(\theta_i, q, t_i) = \theta_i q - 50q + t_i \), where \( q \in \{0, 1\}, \theta_i \in \{0, 60\} \). Here, \( q = 1 \) represents buying the air conditioner, \( q = 0 \) represents not doing so, and the utility is measured in dollars. The values of \( \theta_i, i = 1, 2 \), are private. The roommates want to buy the air conditioner if and only if their joint valuation of \( \theta_1 + \theta_2 \) exceeds the $100 cost. There is no external financial help. An outcome can be expressed formally by a social choice function \( (q(\theta_1, \theta_2), t_1(\theta_1, \theta_2), t_2(\theta_1, \theta_2)) \), such that \( q(\theta_1, \theta_2) = 1 \) if and only if \( \theta_1 + \theta_2 > 100 \) and such that \( t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) \leq 0 \).

a. Consider a direct mechanism where each roommate \( i \) is asked for a report \( \hat{\theta}_i \) of his \( \theta_i \) and where \( q(\hat{\theta}_1, \hat{\theta}_2) = 1 \) if and only if \( \hat{\theta}_1 + \hat{\theta}_2 > 100 \). Derive conditions that symmetric \( \{t_i(\hat{\theta}_1, \hat{\theta}_2)\}, \hat{\theta}_i \in \{0, 60\} \), need to satisfy to truthfully implement the social choice function \( f \) in dominant strategies. (Symmetry means \( t_1(\alpha, \beta) = t_2(\beta, \alpha) \) for each \( \alpha, \beta \in \{0, 60\} \).) Using the symmetry assumption, express all conditions on \( \{t_i\} \) in terms of \( t_1(\alpha, \beta), \alpha, \beta \in \{0, 60\} \) only and simplify the conditions.

b. Give the range of values of \( t_1(0, 60) \) compatible with \( \{t_1(0, 0) = 0, t_1(60, 0) = 0, t_1(60, 60) = 0\} \) and with all the conditions in part a.

c. Let 

\[
q^*(\hat{\theta}_1, \hat{\theta}_2) \in \text{arg max}_{q \in \{0, 1\}} (\hat{\theta}_1 + \hat{\theta}_2)q - 100q \quad \text{and}
\]

\[
q^*_{-i}(\hat{\theta}_{-i}) \in \text{arg max}_{q \in \{0, 1\}} \hat{\theta}_{-i}q - 50q, \quad i = 1, 2.
\]

The Clarke-Groves pivot mechanism employs \( t_1(\hat{\theta}_1, \hat{\theta}_2) = \hat{\theta}_2 q^*(\hat{\theta}_1, \hat{\theta}_2) - 50 q^*(\hat{\theta}_1, \hat{\theta}_2) - [\hat{\theta}_2 q^*_{-1}(\hat{\theta}_2) - 50 q^*_{-1}(\hat{\theta}_2)] \), with \( t_2(\hat{\theta}_1, \hat{\theta}_2) \) defined symmetrically. Show that in the Clarke-Groves mechanism, reporting the truth is always (for all \( \hat{\theta}_i \geq 0, i = 1, 2 \)) a dominant strategy and the mechanism implements the social choice \( f \) given at the start of the problem.

d. In a Clarke-Groves pivot-mechanism, give the value of \( t_1(\theta_1, \theta_2) \) for each profile of \( (\theta_1, \theta_2) \), where \( \theta_i \in \{0, 60\}, i = 1, 2 \).

e. Verify that the \( \{t_i\} \) in the Clarke-Groves mechanism in part d satisfy the conditions in part a.

Answers: 3a. When \( \theta_1 = 0 \), 

\[
\begin{bmatrix}
\hat{\theta}_2 = 0 & \hat{\theta}_2 = 60 \\
\hat{\theta}_1 = 0 & t_1(0, 0) & t_1(0, 60)
\end{bmatrix}.
\]

When \( \theta_1 = 60 \), 

\[
\begin{bmatrix}
\hat{\theta}_1 = 0 & \hat{\theta}_1 = 60 & \hat{\theta}_1 = 60 & \hat{\theta}_1 = 60 & \hat{\theta}_2 = 0 & \hat{\theta}_2 = 60 \\
t_1(0, 0) & t_1(0, 60) & t_1(60, 0) & t_1(60, 0) & 10 + t_1(60, 60) & 10 + t_1(60, 60)
\end{bmatrix}.
\]

So, \( t_1(0, 0) \geq t_1(60, 0), t_1(0, 60) \geq t_1(60, 0) \geq \frac{10}{10 + t_1(60, 60)}\). These simplify to: \( t_1(0, 0) = t_1(60, 0) \geq t_1(60, 60) \geq -50 \). Moreover, \( t_1(0, 0) + t_2(0, 0) \leq 0, t_1(60, 60) + t_2(60, 60) \leq 0, t_1(60, 0) + t_2(60, 0) \leq 0 \).
\[t_1(0, 60) + t_2(0, 60) \leq 0. \text{ Then, } t_1(0, 0) = t_2(0, 0) \leq 0, t_1(60, 60) = t_2(60, 60) \leq 0, t_1(60, 0) + t_1(0, 60) \leq 0.\]

b. \(-50 \leq t_1(0, 0) \leq 0\)

c. From the definition of \(q^*(\theta_1, \theta_2), u_1(\theta_1, q^*(\theta_1, \theta_2), t_1(\theta_1, \theta_2)) = \theta_1 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) + \theta_2 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) - 50q^*_2(\theta_2)\]
\[\geq \theta_1 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) + \theta_2 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) - 50q^*_2(\theta_2)\]
\[= u_1(\theta_1, q^*(\theta_1, \theta_2), t_2(\theta_1, \theta_2)) \text{ for any } \theta_1. \] Similarly, for roommate 2. Also, ignoring a tie, \(q^*(\theta_1, \theta_2) = 1\) if and only if \(\theta_1 + \theta_2 > 100. \) Since \(\theta_i = \theta_j, i = 1, 2, q^*(\theta_1, \theta_2) = 1\) if and only if \(\theta_1 + \theta_2 > 100. \) \(t_1(\theta_1, \theta_2) = \theta_2 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) - 50q^*_2(\theta_2) \leq 0. \) Thus, \(t_1(\theta_1, \theta_2) \leq 0. \) Similarly, \(t_2(\theta_1, \theta_2) \leq 0.\)

d. \(t_1(\theta_1, \theta_2) = \theta_2 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) - 50q^*_2(\theta_2) \leq 0,\)
\(t_2(\theta_1, \theta_2) = \theta_1 q^*(\theta_1, \theta_2) - 50q^*(\theta_1, \theta_2) - 50q^*_2(\theta_1) \leq 0.\)
We have: \(t_1(0, 0) = 0, t_1(60, 60) = 0, t_1(60, 0) = -10, t_1(60, 0) = 0.\)

e. The pivot mechanism gives \(\{t_1(0, 0) = 0, t_1(60, 0) = 0, t_1(60, 60) = 0\}\) and \(-50 \leq t_1(0, 60) = -10 \leq 0.\) From part b, it satisfies the conditions in a.

4. Two men and two women have preference profile \(\succeq = (\succeq_{m_1}, \succeq_{m_2}, \succeq_{w_1}, \succeq_{w_2}),\) where \(\succeq_{m_1} = (w_1, w_2, m_1), \succeq_{m_2} = (w_2, w_1, m_2), \succeq_{w_1} = (m_2, m_1, w_1), \succeq_{w_2} = (m_1, m_2, w_2).\) Here, \(\succeq_{m_1} = (w_1, w_2, m_1)\) means \(w_1 \succ m_1, w_2 \succ m_1, m_1, \) etc. The preferences are transitive and strict. Let \(M = \{m_1, m_2\}, W = \{w_1, w_2\}.\) A matching is a function \(f : M \cup W \rightarrow M \cup W\)
such that
\[(1) \text{ for each } m \in M, f(m) \in W \text{ or } f(m) = m, \text{ and}\]
\[(2) \text{ for each } w \in W, f(w) \in M \text{ or } f(w) = w, \text{ and}\]
\[(3) f(m) = w \text{ if and only if } f(w) = m.\]

a. Give the matching that the men-proposing deferred acceptance algorithm produces (call this matching \(\alpha)).\)

b. Give the matching that the women-proposing deferred acceptance algorithm produces (call this matching \(\beta)).\)

c. Define a stable matching.

d. Show directly (without resorting to a theorem) that the matchings produced in parts a and b are stable.

e. Show that there are no other stable matchings.

f. Suppose \(\phi\) is a social choice function that always selects a stable matching, given any (strict) preference profile. If \(\phi\) chooses \(\alpha\) given the above preference profile \(\succeq,\) show that \(w_2\) can gain by (unilaterally) misrepresenting her preference.

g. In part f, suppose \(\phi\) chooses \(\beta,\) given the above preference profile \(\succeq.\) Show that there is an agent who can gain by (unilaterally) misrepresenting his/her preference.

h. Carefully formulate a conclusion from the above findings.
Answers: 4a. $\alpha = \{\{m_1, w_1\}, \{m_2, w_2\}\}$

b. $\beta = \{\{m_1, w_2\}, \{m_2, w_1\}\}$.

c. A matching $f$ is stable if (1) there does not exist a pair $\{m, w\}$ such that $w \succ_m f(m)$ and $m \succ_w f(w)$ and (2) there does not exist $i$ such that $i \succ_i f(i)$.

d. In each matching, one side (men or women) get their most preferred match, so there is no alternative match that can block the given matching.

e. Every other matching includes some pair for which each member is unmatched (matched to itself). The members of that pair prefer being matched to each other, so every other matching is unstable.

f. If $w_2$ reports $\{m_1, w_2, m_2\}$ while everyone else reports his/her true preference, the only possible stable matching would match $w_2$ to $w_2$ or to $m_1$. We show that $w_2 - w_2$ is not part of a stable matching. If $w_1$ is matched to $m_2$, both $m_1$ and $w_2$ would prefer to be matched to each other than their current matches. If $w_1$ is matched to $m_1$, $w_1$ and $m_2$ would prefer to be matched to each other. Since a stable match exists, the stable match must match $w_2$ to $m_1$ and $w_2$ prefers $m_1$ to $m_2$.

g. $m_1$ can report $\{w_1, m_1, w_2\}$. Then, a stable matching has to match $m_1$ to either $w_1$ or $m_1$. If $m_1$ is matched to $m_1$, $m_2$ is matched to $w_1$ or $w_2$. In the former case, $\{m_2, w_2\}$ can block the matching. The latter case, $\{m_1, w_1\}$ can block the matching. Thus, stable matching will match $m_1$ to $w_1$ improving the welfare of $m_1$.

h. There is no matching rule that always selects a stable matching and that is always incentive compatible (reporting one’s true preference is a dominant strategy).