Instructions: Answer any three of the four numbered problems below. Justify your answers whenever possible. Write your answer to each problem in a separate bluebook. Write the number of the problem AND NOTHING ELSE on the cover of the bluebook. You may not use any electronic devices. The exam lasts 4 hours.

1. Suppose an individual’s “health preferences” are represented by the utility function \( u(s, e, h) = s(2 - e) + h \), where \( s \) is sugar, \( e \) is exercise, and \( h \) is its general state of health. In turn, \( h \) is adversely (i.e., negatively) affected by \( s \) and positively affected by \( e \). Specifically, suppose \( h(s, e) = \ln e - (s^2/2) \). Overall, \( s \) and \( e \) are subject to the respective constraints \( 0 \leq s \leq 2 \) and \( 1/2 \leq e \leq 2 \).

a. How much \( s \) and \( e \) would the individual choose to consume? Verify that this is indeed a maximum.

Next, suppose that while the direct effects of \( s \) and \( e \) on \( u \) are certain, the effects of these on health are not. Rather, each of them may or may not have the purported (i.e., claimed) effect on health. For \( x = s \) or \( e \), let \( \pi_x \) denote the probability that factor \( x \) does have the purported health effect and \((1 - \pi_x)\) denote the probability that \( x \) does not, and assume \( \pi_s \) and \( \pi_e \) are independent. In the event that \( s \) does affect health but \( e \) does not, then \( h \) is given by \( h(s) = -s^2/2 \). Similarly, if \( e \) is effective but \( s \) is not, then \( h(e) = \ln e \); and if neither is effective, then \( h = 0 \).

b. For each case, determine the optimal choices of \( s \) and \( e \) if the individual knew whether or not the purported health effects were true.

c. Returning to the case in which the true effects on health are uncertain, write the decision problem facing the individual and characterize an interior solution assuming the individual maximizes expected utility. What, if any, restrictions on \( \pi_s \) and \( \pi_e \) ensure that an interior solution exists?

d. Discuss the comparative static effects of \( \pi_s \) and \( \pi_e \) on the optimal \( s \) and \( e \) in part c.

e. Suppose that regardless of whether or not the purported health effects of \( s \) and \( e \) are true, health is subject to an exogenous adverse shock that occurs with probability \( \pi_o \). In that event, \( h(s, e, \varepsilon) = \ln e - (s^2/2) - \varepsilon \). What effect would this have on the optimal values of \( s \) and \( e \)?

f. Suppose the individual is currently at an interior solution as in part c. The government is considering a proposal to ban sugar entirely \((s \equiv 0)\). Would this be beneficial or detrimental to the individual?

g. Finally, suppose that sugar cannot be obtained directly by the individual. However, in order to encourage people to exercise, insurers provide one unit of sugar for each unit of \( e \), i.e., \( s \equiv e \). How will this affect the choice of \( s \) and \( e \) at an interior solution?

2. In a pure exchange economy with two consumers and two goods, when consumer \( i \) \((i = 1, 2)\) consumes \( x_{i1} \geq 0 \) units of good \( \ell \), consumer 1 gets utility \( x_{11} + \phi(x_{21}) - \theta x_{22}^2 \) and consumer 2 gets utility \( x_{12} + \phi(x_{22}) \), where \( \phi' > 0, \phi'' < 0, \) and \( \theta \in \{0, 1/2\} \). Consumer 1 initially owns the entire endowment \( e_1 > 0 \) of good 1 in the economy and consumer 2 owns the entire endowment \( e_2 > 0 \) of good 2. The problems below that ask for “conditions on the economy” are asking for restrictions on what is given exogenously.
2. a. Describe the consumers' preferences in English.
b. Characterize all the Pareto efficient (Pareto optimal) allocations in which both consumers consume positive amounts of both goods. Under what conditions on the economy is there a Pareto efficient allocation with $x_{22} = 0$?
c. Characterize the competitive (Walrasian) equilibria in which consumer 1 treats the consumption of good 2 by consumer 2 (consumer 2's choice) as given, unaffected by actions of consumer 1. Can there be more than one competitive equilibrium allocation in this economy? Under what conditions on the economy is a competitive equilibrium allocation Pareto efficient when $\theta = 0$? Explain your answer. Under what conditions on the economy is a competitive allocation Pareto efficient when $\theta = 1/2$?
d. Consider a competitive equilibrium in which the price of good 2 is 1 and consumer 2 is charged a fee of $\theta x_{22}^2$ in addition to the purchase cost of good 2. The fee paid by consumer 2 is given as a lump sum transfer to consumer 1 in compensation for consumer 1's utility loss due to consumer 2's consumption. (Consumer 1 acts as if the fee consumer 2 pays cannot be affected by anything consumer 1 does.) Characterize a resulting competitive equilibrium allocation. Is it necessarily Pareto efficient?
e. Suppose, instead, that consumer 2 is free to bargain with consumer 1 without being charged the fee in part d. Consumer 2, knowing the utility function of consumer 1, chooses a consumption bundle $(x_{12}, x_{22}) \succ 0$, and offers consumer 1 two alternatives. Either

(1) consumer 2 will give consumer 1 $e_2 - x_{22}$ units of good 2 in return for receiving $x_{12} > 0$ units of good 1 from consumer 1, or

(2) consumer 1 will consume its endowment $e_1$ of good 1 and get none of good 2. Consumer 1 must choose one of the alternatives, (1) or (2). Characterize a pure subgame perfect Nash equilibrium (SPE) in which the consumers' payoffs are their utilities. Which alternative does consumer 1 choose? Show that consumer 1 gets the same payoff as if it chose the other alternative. Is a SPE allocation necessarily Pareto efficient?
f. Compare the quantity of good 2 consumed by consumer 2 in a competitive equilibrium allocation in part d with that in a SPE allocation in part e. How do the entire allocations in parts d and e differ from each other (if they do)? Explain why the differences arise if they do.

3. Two roommates are considering buying an air-conditioner that costs $100. It is common knowledge that the utility function of roommate $i$ is of the form $u_i(\theta_i, q, t_i) = \theta_i q - 50q + t_i$, where $q \in \{0, 1\}$, $\theta_i \in \{0, 60\}$. Here, $q = 1$ represents buying the air conditioner, $q = 0$ represents not doing so, and the utility is measured in dollars. The values of $\theta_i, i = 1, 2$, are private. The roommates want to buy the air conditioner if and only if their joint valuation of $\theta_1 + \theta_2$ exceeds the $100$ cost. There is no external financial help. An outcome can be expressed formally by a social choice function $f(\theta_1, \theta_2) = (q(\hat{\theta}_1, \theta_2), t_1(\theta_1, \theta_2), t_2(\theta_1, \theta_2))$, such that $q(\hat{\theta}_1, \theta_2) = 1$ if and only if $\hat{\theta}_1 + \theta_2 > 100$ and such that $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) \leq 0$.

a. Consider a direct mechanism where each roommate $i$ is asked for a report $\hat{\theta}_i$ of his $\theta_i$ and where $q(\hat{\theta}_1, \hat{\theta}_2) = 1$ if and only if $\hat{\theta}_1 + \hat{\theta}_2 > 100$. Derive conditions that symmetric $(t_1(\hat{\theta}_1, \hat{\theta}_2), \hat{\theta}_i \in \{0, 60\})$, need to satisfy to truthfully implement the social choice function $f$ in dominant strategies. (Symmetry means $t_1(\alpha, \beta) = t_2(\beta, \alpha)$ for each $\alpha, \beta \in \{0, 60\}$.) Using the symmetry assumption, express all conditions on $\{t_i\}$ in terms of $t_1(\alpha, \beta), \alpha, \beta \in \{0, 60\}$ only and simplify the conditions.
b. Give the range of values of \( t_1(0,60) \) compatible with \( \{t_1(0,0) = 0, t_1(60,0) = 0, t_1(60,60) = 0\} \) and with all the conditions in part a.

c. Let

\[
q^*(\hat{\theta_1}, \hat{\theta_2}) \in \arg \max_{q \in (0,1)} (\hat{\theta_1} + \hat{\theta_2})q - 100q \quad \text{and} \\
q^*_i(\hat{\theta_i}) \in \arg \max_{q \in (0,1)} \hat{\theta_i}q - 50q, \quad i = 1, 2.
\]

The Clarke-Groves pivot mechanism employs \( t_1(\hat{\theta_1}, \hat{\theta_2}) = \hat{\theta_2}q^*(\hat{\theta_1}, \hat{\theta_2}) - 50q^*(\hat{\theta_1}, \hat{\theta_2}) - \hat{\theta_2}q^*_1(\hat{\theta_2}) - 50q^*_1(\hat{\theta_2}) \), with \( t_2(\hat{\theta_1}, \hat{\theta_2}) \) defined symmetrically. Show that in the Clarke-Groves mechanism, reporting the truth is always (for all \( \theta_i \geq 0, i = 1, 2 \)) a dominant strategy and the mechanism implements the social choice \( f \) given at the start of the problem.

d. In a Clarke-Groves pivot-mechanism, give the value of \( t_1(\theta_1, \theta_2) \) for each profile of \( (\theta_1, \theta_2) \), where \( \theta_i \in \{0, 60\}, i = 1, 2 \).

e. Verify that the \( \{t_i\} \) in the Clarke-Groves mechanism in part d satisfy the conditions in part a.

4. Two men and two women have preference profile \( \succ \equiv (\succ_{m1}, \succ_{m2}, \succ_{w1}, \succ_{w2}) \), where \( \succ_{m1} = (w_1, w_2, m_1) \), \( \succ_{m2} = (w_2, w_1, m_2) \), \( \succ_{w1} = (m_2, m_1, w_1) \), \( \succ_{w2} = (m_1, m_2, w_2) \). Here, \( \succ_{m1} = (w_1, w_2, m_1) \) means \( w_1 \succ_{m1} w_2 \succ_{m1} m_1 \), etc. The preferences are transitive and strict. Let \( M = \{m_1, m_2\} \), \( W = \{w_1, w_2\} \). A matching is a function \( f : M \cup W \rightarrow M \cup W \) such that

1. for each \( m \) in \( M \), \( f(m) \in W \) or \( f(m) = m \), and
2. for each \( w \) in \( W \), \( f(w) \in M \) or \( f(w) = w \), and
3. \( f(m) = w \) if and only if \( f(w) = m \).

a. Give the matching that the men-proposing deferred acceptance algorithm produces (call this matching \( \alpha \)).

b. Give the matching that the women-proposing deferred acceptance algorithm produces (call this matching \( \beta \)).

c. Define a stable matching.

d. Show directly (without resorting to a theorem) that the matchings produced in parts a and b are stable.

e. Show that there are no other stable matchings.

f. Suppose \( \phi \) is a social choice function that always selects a stable matching, given any (strict) preference profile. If \( \phi \) chooses \( \alpha \) given the above preference profile \( \succ \), show that \( w_2 \) can gain by (unilaterally) misrepresenting her preference.

g. In part f, suppose \( \phi \) chooses \( \beta \), given the above preference profile \( \succ \). Show that there is an agent who can gain by (unilaterally) misrepresenting his/her preference.

h. Carefully formulate a conclusion from the above findings.