1. Consider an economy with two consumers, 1 and 2. Each consumer consumes only grapes
and wine and can use grapes as an input to produce wine. Grapes used as input cannot be
consumed directly as grapes. Consumer 1 produces 1 unit of wine for each unit of grapes it
uses as input. Consumer 2 produces 2 units of wine for each unit of grapes it uses as input.
Consumer 1 initially owns 1 unit of grapes and no wine. Consumer 2 initially owns 1 unit of
grapes and 2 units of wine. Consumer 1 gets utility
\[ u_1(g_1, w_1) = g_1w_1 \]
from consuming \( g_1 \geq 0 \) units of grapes and \( w_1 \geq 0 \) units of wine. Consumer 2 gets utility
\[ u_2(g_2, w_2) = g_2(w_2 - 1) \]
from consuming \( g_2 \geq 0 \) units of grapes and \( w_2 > 1 \) units of wine, and gets utility \( u_2(g_2, w_2) = 0 \) otherwise.

a. Give a verbal explanation of the way the consumers’ preferences differ from each other.
b. Find the optimal production and consumption of each consumer separately, assuming
that the consumers do not trade with each other.
c. Consider the private ownership economy consisting of the two consumers, each owning a
competitive firm that uses that consumer’s wine-making technology. Prove that in a com-
petitive (Walrasian) equilibrium neither good can have a price of 0.
d. Find every competitive equilibrium for this economy, letting grapes be numeraire. How
many such equilibria are there? Compare the utility levels of the two consumers to what
they receive in part b, when they do not trade.
e. Find a Pareto efficient (Pareto optimal) allocation for this economy that is not a com-
petitive equilibrium allocation and that gives positive utility to consumer 1. (Try to find an
allocation that is easy to characterize with little calculation.) Describe how this allocation
differs from a competitive equilibrium allocation.
f. Is there any Pareto efficient allocation in which both firms produce positive quantities of
wine? Is every competitive equilibrium allocation in this economy Pareto efficient?
g. In a competitive equilibrium in this economy, the consumer with the more productive
technology supplies more input and the consumer who has more need of output for con-
sumption gets more of the output. Is this property (more productive agents supply more
input and agents more in need of a good get to consume more of it) a characteristic of all
competitive equilibria in general equilibrium models? Explain and justify your answer by
checking whether the property continues to hold in the economy above no matter what the
consumers’ initial endowments are.

Answers:
a. Consumer 1 prefers having a small amount of grapes and of wine to having none of both
goods. Consumer 2 is no better off having small amounts of each of the goods (less than 1)
than having none of both goods.
b. Consumer \( i \), having endowment \( e_i \) of wine and using \( G_i \) units of grapes as input, maximizes
\[ g_i(w_i - c_i) \text{ s.t. } g_i + G_i \leq 1 \text{ and } w_i \leq iG_i + c_i, \text{ where } c_1 = e_1 = 0, \ c_2 = 1, \ e_2 = 2. \]
Substituting \( 1 - g_i \) for \( G_i \), the focs are \( w_i - c_i - \lambda_i = 0 \) and \( g_i - \lambda_i = 0 \) at interior solutions. There are
no other solutions because each consumer can get positive utility. The focs are solved by
\( g_1 = 1/2, \ w_1 = 1/2, \ g_2 = 3/4, \ w_2 = 5/2. \) The focs are sufficient for a solution for each \( i \),
since the constraints are linear and the objective function is quasiconcave.
c. If the price of grapes is 0, then the price of wine must be positive, and neither firm has a
profit maximizer. If the price of wine is 0, then the price of grapes is positive. Consumer 2
has positive wealth and no utility maximizer. No matter what it consumes, it can afford a
positive amount of grapes and an amount of wine big enough to raise its utility.
d. Let $p$ be the price of wine. If $p > 1/2$, then firm 2 has no profit maximizer. (Higher input $G$ always raises its profit $2pG - G$.) Therefore in equilibrium, $p \leq 1/2$, so firm 1, owned by consumer 1 does not produce. Since firm 2 has constant returns to scale, its equilibrium profit is 0. Thus each consumer gets 0 profit. Each consumer $i$ maximizes utility subject to the budget constraint $g_i + pw_i \leq 1 + pe_i$. Each consumer is able to obtain positive utility with $w_i > 3/2$ and $g_i = 1/4$, so the solution must be interior. The focs are $w_i - c_i - \gamma_i = 0$, $g_i - \gamma_i p = 0$, and the budget constraint. The solutions are $g_1 = 1/2$, $w_1 = 1/(2p)$, $g_2 = (1 + p)/2$, $w_2 = (1 + 3p)/(2p)$. Since total wine demand $w_1 + w_2 = (2 + 3p)/(2p)$ is more than the total endowment, firm 2 produces in equilibrium, which implies $p = 1/2$, $g_1 = 1/2$, $w_1 = 1$, $g_2 = 3/4$, $w_2 = 5/2$. Then $2 - g_1 - g_2 = 3/4$ is supply of grapes to firm 2 and is also the firm’s demand for grapes. The wine output of 3/2 plus the endowment of 2 equals the consumers’ demand for wine.

With grapes as numeraire, there is only one equilibrium price vector and allocation. In the competitive equilibrium, consumer 1 gets the same amount of grapes, but twice as much wine as in the autarky solution of part b. Consumer 1 benefits from the superior technology of consumer 2. Consumer 2 gets the same bundle as in autarky, so gets the same utility, not gaining from trade.

e. The allocation must be Pareto efficient if it maximizes the utility of consumer 1 subject to feasibility alone and maximizes the utility of consumer 2 subject to consumer 1 getting the maximal utility for consumer 1. To find this allocation, we maximize $g_1 w_1$ subject to $w_1 \leq 2 + 2(2 - g_1)$ and get $g_1 = 3/2$ and $w_1 = 3$. Then feasibility implies $g_2 = 0$, $w_2 = 0$. Firm 1 does not produce. Firm 2 uses 1/2 unit of grapes to produce 1 unit of wine. Raising the utility of consumer 1 is not feasible and raising the utility of consumer 2 requires giving it more than 1 unit of wine, which reduces the utility of consumer 1. Thus, the allocation is Pareto efficient. Consumer 1 consumes more of both goods and consumer 2 consumes less of both goods than in competitive equilibrium.

f. If firm 1 produces, then taking away its input and giving it to firm 2 produces more wine and raises the utility of at least one consumer, so there is no Pareto efficient allocation in which firm 1 produces. The competitive allocation is Pareto efficient. Consumer 2 is not locally nonsatiated, but both consumers are locally nonsatiated at the competitive equilibrium allocation. This is enough for the conclusion of the first welfare theorem to hold.

g. That property is not generally satisfied in competitive equilibrium. Who owns the more productive technology does not matter in competitive equilibrium. It is also not true that having higher productivity makes a consumer work more in competitive equilibrium. If consumer 2 cared relatively more about grapes, say had utility $g_2 (w_2 - 1)^\alpha$, with $\alpha$ very near 0, then it would work less than consumer 1. If consumer 1 had a bigger endowment of wine, it would consume more wine than consumer 2 in competitive equilibrium even though consumer 2 is more needy.

2. There are two types of consumer who have differing tastes. A firm faces a consumer. It is known that the probability of the consumer being of type $H$ is $\frac{1}{2}$ and being type $L$ is $\frac{1}{2}$. The $H$ type has utility function $U_H = 4\sqrt{q} - t$ and the $L$ type has utility function $U_L = \sqrt{q} - t$, where $q$ is the quality of a particular good the firm sells and $t$ is the monetary payment for the good. The firm can choose any level of quality $q \geq 0$ at a cost of $q$ dollars. There is no fixed cost. The reservation utility of both types is zero.

a. Suppose the firm can identify the type of the consumer before offering a two part tariff $(T, p)$; i.e., $t = T + pq$. What are the optimal two part tariffs and what is the corresponding ex-ante (before the firm learns the type of the consumer) expected profit?
b. From now on, assume that the firm cannot identify the types directly. What is the expected profit maximizing two-part tariff \((T, p)\) for the firm? What is the corresponding expected profit?

c. Suppose now that the firm can offer any (non-linear) contract. What is an optimal contract and what is the corresponding expected profit of the firm? Compare your answer with your answer in part a.

d. Now, suppose there are two consumers whose types are drawn independently from the identical distribution in the above. The consumers know their types but not the other consumer’s type. The producer knows only their distributions. Suppose the producer produces \(q\) and auctions it off to the two consumers according to the ascending (English) auction. What is the expected revenue of the firm? What is the optimal \(q\) and corresponding profit of the firm?

**Answers:**

a. For \(L\) type, \(\frac{d}{dq}(\sqrt{q} - T - pq) = \frac{1}{2\sqrt{q}} - p = 0\). For \(H\) type, \(\frac{d}{dqh}(4\sqrt{qh} - T - pqh) = \frac{2}{\sqrt{qh}} - p = 0\). So, \(q_h = \left(\frac{2}{p}\right)^2\) and \(q_l = \left(\frac{1}{2p}\right)^2\). The firm solves \(\max T + pqh - q_h\) subject to \(4\sqrt{qh} - T - pqh \geq 0\), \(q_h = \left(\frac{2}{p}\right)^2\), \(T = 4\sqrt{qh} - pqh = \frac{4}{p}\). So, \(\max T + pqh - q_h = \frac{4}{p} + \frac{4}{p} - \left(\frac{2}{p}\right)^2 = \frac{8}{p} - \left(\frac{4}{p}\right)\). Optimal \(p = 1, T = 4\) and the corresponding profit is 4. Similarly, for the low type, \(p = 1, T = \frac{1}{4}\) and the profit is 1/4. So, the ex-ante expected profit is \([4 + (1/4)]/2 = 17/8\).

b. There are two possibilities. When both types are served, the participation constraint is \(\sqrt{q} - T - pq \geq 0\), since \(4\sqrt{qh} - T - pqh \geq 0\) is redundant. The firm solves \(\max \frac{1}{2}(T + pqh - q_h + T + pq_l - q_l)\) subject to \(\sqrt{q} - T - pq \geq 0\). \(T = \sqrt{q} - pq = \frac{1}{2p} - p \left(\frac{1}{2p}\right)^2 = \frac{1}{2p} - \frac{1}{4p} = \frac{1}{4p}\).

max\(p\) \(\frac{1}{2p} + (p - 1) \left(\frac{2}{p}\right)^2 + (p - 1) \left(\frac{1}{2p}\right)^2 = \frac{1}{2p} + (p - 1) \left(\frac{4}{p^2} + \frac{1}{4p^2}\right) = \frac{1}{2p} + (p - 1) \left(\frac{17}{4p^2}\right)\). Then \(p = \frac{34}{19}, T = \frac{10}{19}\), profit = \(\frac{1}{2}\left(\frac{1}{2p} + (p - 1) \left(\frac{17}{4p^2}\right)\right) = \frac{1361}{2272}\). If the firm sells only to the high type, then from part \(a\), \(p = 1, T = 4\) and the corresponding expected profit is \(\frac{1}{2} \ast 4 = 2\). So, the optimal two part tariff is \((T, p) = (4, 1)\) with expected profit of 2.

c. The firm solves \(\max(1/2)(t_H - q_H + t_L - q_L)\) subject to \(4\sqrt{q} - t_h \geq 4\sqrt{q} - t_l, \sqrt{q} - t_l \geq \sqrt{q} - t_h, 4\sqrt{q} - t_h \geq 0, \sqrt{q} - t_l \geq 0\), which is equivalent to maximizing \(t_H - q_H + t_L - q_L\) subject to \(4\sqrt{q} - t_h = 4\sqrt{q} - t_l = 2\sqrt{q} - t_l = 0\). Solution: Offer a menu \(\{(q_H, t_H), (q_L, t_L)\} = \{(4, 8), (0, 0)\}\). Expected profit is 2. Comparison: The two part tariff gave the optimal contract. \(t_H = 8 = T + pq = 4 + 1 \cdot 4\). The low type chooses \((0, 0)\), which is equivalent to non-participation.

d. Since English auction is equivalent to second price auction, the expected revenue of firm is \(\frac{1}{4}4\sqrt{q} + \frac{1}{4}\sqrt{q} + \frac{1}{2}\sqrt{q} = \frac{7}{4}\sqrt{q}\). The firm maximizes \(\frac{7}{4}\sqrt{q} - q\), choosing \(q = \frac{49}{64}\) with expected profit \(\frac{49}{64}\).

3. The following is a three period model of a financial contract involving a lender (bank) and a borrower (firm). In the first period, the firm either builds a new facility at cost \(c \geq 0\) or it does not. If the firm builds the new facility, its subsequent production function in value terms becomes \(4\sqrt{T}\); otherwise it is \(2\sqrt{T}\). Here, \(L\) is the amount of loan used in the production. In the second period, the bank offers a contract \((L, t)\) to the firm. A contract consists of \(L \geq 0\) the amount of loan and \(t \geq 0\), the gross amount the borrower has to repay. Assume that the firm does not default. In the third period, the firm either accepts the contract or rejects the offer. If a contract \((L, t)\) is offered and accepted, the bank receives the payoff of \(t - RL\), where \(R > 1\), while the payoff to the firm is \(4\sqrt{T} - t - c\) if it built.
the facility in the first period and $2\sqrt{L} - t$, if it did not. If the firm rejects the contract, the payoff to the bank is zero and the payoff to the firm is $-c$ if the firm built the facility in the first period and is zero otherwise. Assume that the firm accepts the contract if accepting and rejecting the contract result in the same payoffs. Both the bank and the firm are risk neutral. Assume that $\frac{2}{R} > c$.

a. Suppose the bank knows whether the firm built the new facility or not in the first period when it offers a contract. Draw the game tree of this game.

b. Find all sequential equilibria of the game in (a).

c. Suppose from now on that the firm’s possible investment in the first period is in software and the bank does not know whether or not the firm made the investment when it offers the contract. Draw the game tree of this game.

d. What is the optimal contract offer of the bank as a function of its beliefs? The bank believes that the firm invested in the software with probability $p$.

e. Find all sequential equilibria of this game.

**Answers:**

a. In the tree, b stands for build, n for not build, A and R for accept and reject when the firm has built, and a and r for accept and reject when the firm has not built. The firm’s payoff is listed above the bank’s.

b. After the firm builds, it accepts a contract if and only if $4\sqrt{L} - t - c \geq -c$ or $4\sqrt{L} - t \geq 0$. The bank maximizes $t - RL$ subject to $4\sqrt{L} - t \geq 0$. Solution: $\frac{d}{dL}(4\sqrt{L} - RL) = 2\frac{1}{\sqrt{L}} - R = 0$, $\sqrt{L} = \frac{2}{R}$, $t = \frac{8}{R}$. Bank’s profit: $\frac{8}{R} - \frac{4}{R} = \frac{4}{R}$. Firm’s profit: $-c$. If the firm does not build, it accepts a contract if and only if $2\sqrt{L} - t \geq 0$. The bank maximizes $t - RL$ subject to $2\sqrt{L} - t \geq 0$. Solution: $\frac{d}{dL}(2\sqrt{L} - RL) = \frac{1}{\sqrt{L}} - R = 0$, $\sqrt{L} = \frac{1}{R}$, $t = \frac{2}{R}$. Bank’s profit: $\frac{2}{R} - \frac{1}{R} = \frac{1}{R}$. Firm’s profit: $0$. The firm does not build.

c. In this case, the bank has only one information set and chooses only one contract.
d. When the firm builds, the firm accepts if and only if $4\sqrt{L} - t \geq 0$. When the firm does not build, the firm accepts if and only if $2\sqrt{L} - t \geq 0$. The bank maximizes $t - RL$ subject to the following constraints: If the bank wants the firm to accept whether it built or not, the constraint is: $2\sqrt{L} - t \geq 0$. If the bank wants the firm to accept if it built but not if it did not, the constraint is $4\sqrt{L} - t \geq 0$ (When $4\sqrt{L} - t = 0$, the firm that did not build rejects it since $2\sqrt{L} - t < 0$ so long as $L > 0$). Both constraints bind at the optimum. In the latter case, the maximum expected profit of the bank is $pR$. Under the constraint of $2\sqrt{L} - t = 0$, the profit of bank is $\frac{1}{R}$. So the bank chooses $(\sqrt{L}, t) = (\frac{2}{R}, \frac{8}{R})$ if $p > \frac{1}{4}$ and chooses $(\sqrt{L}, t) = (\frac{1}{R}, \frac{2}{R})$ if $p < \frac{1}{4}$. If $p = \frac{1}{4}$, the bank chooses $(\sqrt{L}, t) = (\frac{2}{R}, \frac{8}{R})$ with probability $\alpha$ and $(\frac{1}{R}, \frac{2}{R})$. probability $1 - \alpha$.

e. Since the bank’s information set is reached for sure, its equilibrium belief is correct and the firm builds with probability $p$. If $p > 1/4$, then $(\sqrt{L}, t) = (\frac{2}{R}, \frac{8}{R})$. The firm rejects this contract if it did not build. The firm’s expected payoff is $p(-c) + (1 - p)0 = -pc < 0$. The optimal response of the firm is $p = 0$, so this is not a sequential equilibrium (SE). For $p < 1/4$, $(\sqrt{L}, t) = (\frac{1}{R}, \frac{2}{R})$. The firm’s expected payoff is $p(4\frac{1}{R} - \frac{2}{R} - c) + (1 - p)(2\frac{1}{R} - \frac{2}{R}) = p(\frac{2}{R} - c)$. Since $\frac{2}{R} > c$, this payoff is maximized at $p = 1$, so there is no SE with $p < 1/4$.

Let $p = \frac{1}{4}$. From part d, the bank chooses $(\sqrt{L}, t) = (\frac{2}{R}, \frac{8}{R})$ with some probability $\alpha$ and chooses $(\frac{1}{R}, \frac{2}{R})$ with probability $1 - \alpha$. is $\frac{1}{4}$ and the payoff is $\frac{1}{4}(\frac{2}{R} - c)$. If the firm does not build, it rejects the former offer and accepts the latter. Its payoff is 0 either way. If the firm builds, its expected payoff is $\alpha(-c) + (1 - \alpha)(\frac{2}{R} - c)$. For $p = \frac{1}{4}$ to be an optimal choice for the firm, it is necessary and sufficient that $\alpha(-c) + (1 - \alpha)(\frac{2}{R} - c) = 0$ and therefore, $\alpha = 1 - \frac{R}{2}c > 0$. There is a unique sequential equilibrium. The firm chooses $p = \frac{1}{4}$ and accepts a contract if and only if $4\sqrt{L} - t \geq 0$ if it invested in the first period and accepts if and only if $2\sqrt{L} - t \geq 0$ if it did not. The lender offers $(\sqrt{L}, t) = (\frac{2}{R}, \frac{8}{R})$ with probability $\alpha = 1 - \frac{R}{2}c$ and $(\sqrt{L}, t) = (\frac{1}{R}, \frac{8}{R})$ with probability $1 - \alpha$. 
4. Consider a firm with the technology \( q = (x_1 x_2)^{\frac{1}{3}} \) that purchases inputs \( x_1 \) and \( x_2 \) in competitive factor markets at the prices \( w_1 = w_2 = 1 \). Initially, the firm faces the demand curve \( q(p) = 120 - p \) for its output.

a. Determine the firm’s optimal supply decision and profits.

b. Next, suppose the firm’s profits were taxed at the constant rate of 20%. How would this affect the firm’s optimal decisions? How much would the firm pay in taxes and what would be its post tax profits?

c. Suppose that, in addition to producing \( q \), the firm could also contribute money, \( y \), to charity and thereby reduce its overall tax rate. The rate would then become \( \tau(y) = \max\{(0.2 - 0.001y), 0\} \). Assuming the contribution is from pretax earnings, i.e., it can be treated as an additional cost, determine the optimal supply decision and charitable contribution of the firm. What are the firm’s profits? Compare the amount of taxes paid in part (b) to the amount of taxes and charity paid here. Also compare the firm’s post tax profits.

d. Suppose again that the tax rate is fixed at \( t = 0.2 \). Now, while charitable contributions do not affect the tax rate, they garner publicity for the firm and increase demand for its product. Thus, the firm faces the demand \( q(p) = 120 - p + 2\sqrt{y} \). Again, assume the contribution is from pretax earnings. Determine the firm’s optimal supply decision in this case. Also determine the amounts of taxes and charity paid and the post tax profits.

e. Assuming that tax revenues and charitable contributions are devoted to precisely the same uses, rank the cases (b), (c) and (d) in terms of the resources (tax revenues and charity) generated and post tax profits.

**Answers:**

a. Derive cost function: \( c(q) = 2q^2 \), the value of the objective function at a solution to \( \min x_1 + x_2 \) s.t. \( (x_1 x_2)^{\frac{1}{3}} \geq q \). Then maximize \( p(q)q - c(q) \) at \( q = 20 \), with profit \( \pi = 1200 \).

b. The tax does not affect decisions. Max \( p(q)q - c(q) \) vs Max \( (1 - t)[p(q)q - c(q)] \). Firm pays \( t\pi = 240 \). Post-tax profits are 960.

c. Max \( (1 - \tau(y))[p(q)q - c(q) - y] \). Solution is \( q^* = 20, y^* = 200 \). Hence, \( \tau(y^*) = 0 \). Pays 0 taxes, 200 in charity. Post tax profits are 1000.

d. Max \( (1 - \tau)[p(q,y)q - c(q) - y] \). Solution is \( q' = 30, y' = 900, p' = 150, \pi' = 1800, \tau\pi' = 360, (1 - \tau)\pi' = 1440. \tau\pi' + y' = 1260. \)

e. Ranking of tax revenues plus charitable contributions: 1260 for (d) > 240 for (b) > 200 for (c). Ranking of post tax profits: 1440 for (d) > 1000 for (c) > 960 for (b).