1. Consider a couple consisting of a husband ($H$) and wife ($W$). Each is endowed with one unit of time which has three possible uses: market production ($m$), home production ($h$), and leisure ($\ell$). Labor devoted to market production, $L^{im}$, is paid the wage $w^i$, $i = H, W$, where $w^W < w^H$. The market price of $m$ is normalized to 1; all of the couple’s earnings are devoted to purchasing $m$. The household good is produced according to the technology $h = \sqrt{L^{Hh}} + \sqrt{L^{Wh}}$, where $L^{ih}$ denotes the amount of time $i$ devotes to home production. $H$ and $W$ both consume the entire amount of $m$ purchased and the entire amount of $h$ produced. However, they consume only their own leisure. Agent $i$’s utility is given by $u^i(m, h, \ell)$, which is increasing, strictly concave and twice continuously differentiable.

a. What variables must be determined by the couple? $m, h, \ell^H, L^Hm, L^Hh, \ell^W, L^Wm, L^Wh$, but $L^Hm, L^Hh, L^Wm, L^Wh$ determine the other variables.

b. Assuming the couple’s objective is to maximize joint welfare, $u^H + u^W$, characterize an interior optimum.

They should solve:

$$\max_{m, h, \ell^H, L^Hm, L^Hh, \ell^W, L^Wm, L^Wh} u^H(m, h, \ell^H) + u^W(m, h, \ell^W)$$

subject to the following constraints:

1. $1 = \ell^H + L^Hm + L^Hh$
2. $1 = \ell^W + L^Wm + L^Wh$
3. $m = w^H L^Hm + w^W L^Wm$
4. $h = \sqrt{L^{Hh}} + \sqrt{L^{Wh}}$

Or substituting for $m, h, \ell^H, \ell^W$,

$$\max_{L^Hm, L^Hh, L^Wm, L^Wh} u^H(w^H L^Hm + w^H L^Hh, \sqrt{L^{Hh}} + \sqrt{L^{Wh}}, 1 - L^Hm - L^Hh) + u^W(w^H L^Hm + w^W L^Wm, \sqrt{L^{Hh}} + \sqrt{L^{Wh}}, 1 - L^Wm - L^Wh)$$

FOCs:

$$L^{im} : \frac{u^H_m}{2\sqrt{L^{Hh}}} - \frac{u^W_m}{2\sqrt{L^{Wh}}} = 0$$
$$L^{ih} : \frac{u^H_h}{2\sqrt{L^{Hh}}} - \frac{u^W_h}{2\sqrt{L^{Wh}}} = 0$$

c. Argue that if the agents have identical preferences, i.e., if $u^H = u^W$, then as long as $w^W < w^H$, it is optimal for $H$ to consume less leisure and to supply more market labor than $W$ but for $W$ to devote more time to home production than $H$.

From the FOCs for $L^{im}$, $u^H = u^H(u^H + u^W) > w^W(u^H + u^W) = u^W$. Since $u$ is concave, $\ell^H < \ell^W$. From this and the FOC for $L^{ih}$, $u^W = \frac{u^H}{2\sqrt{L^{Hh}}} > \frac{u^W}{2\sqrt{L^{Wh}}} < \frac{u^H}{2\sqrt{L^{Hh}}} + \frac{u^W}{2\sqrt{L^{Wh}}} = u^H$. Therefore $L^{Wh} > L^{Hh}$ and $L^{Wm} < L^{Hm}$.

d. For the general case in which the agents’ preferences might differ, could there be an interior optimum in which they devote equal amounts of time to home production even though $w^W < w^H$? Explain.

No. Reversing the findings in part c, if $L^{Wh} = L^{Hh}$, then $u^W = u^H$. From the FOCs for $L^{im}$, this requires $w^H = w^W$.

For parts e through g, below, assume $H$ and $W$ decide separately how to allocate their time rather than jointly, as above. They each maximize their own utility, taking the other’s choices as given.

e. Write the decision problem facing each of the two agents and describe the principle difference between this and the previous joint welfare maximization.
Agent $i$ solves:
\[
\max_{L^i,m,L^h} u^i(w^i L^i + w^j L^j, \sqrt{L^i h} + \sqrt{L^j h}, 1 - L^i m - L^j h) \text{ treating } L^i, L^j \text{ as given. This is a game and the solution concept asked for is Nash equilibrium.}
\]

f. Argue that in this case too little time will be devoted to both home production and market production, and too much will be devoted to leisure.

The idea is that there is an externality in the Nash equilibrium. By working more, an agent increases $m$ or $h$, which helps the other agent. However, the conclusion does not follow without further assumptions. For example, it can be shown that without further restrictions on the utility functions, there can be a Nash equilibrium in which an agent spends more than the optimal amount of time on home production.

Suppose that the agents’ utility functions are the same and are additively separable. This means that there are strictly concave functions $M$, $N$, and $L$ such that $u^i(m, h, \ell^i) = M(m) + H(h) + L(\ell^i)$. The FOCs for an interior Nash equilibrium are
\[
w^i u^i_m - u^i = w^i M'(m) - L'(\ell^i) = 0 \quad \text{and} \quad 2\sqrt{L^h} u^i_\ell - u^i_h = 2\sqrt{L^h} L'(\ell^i) - H'(h) = 0, \forall i.
\]
The FOCs in part b become $2w^i L^i - \hat{L}^i = 0$ and $2\sqrt{L^h} \hat{L}^i - 2H' = 0$, where $\hat{L}^i = L'(\ell^i)$ and where $\hat{\cdot}$ denotes functions evaluated at the optimal allocation in part b.

Suppose that $m \geq \hat{m}$, so that $M' \leq \hat{M}' < 2\hat{M}'$. The FOCs imply $\ell^i > \hat{\ell}$, $\forall i$. It follows that $h < \hat{h}$. Otherwise, the Nash equilibrium yields higher utility for both agents than does the solution to part b, a contradiction. The FOCs also imply that the ratio $\hat{L}^h / L^h$ is independent of $i$, so $\hat{L}^h > L^h$, $\forall i$. Replacing $\hat{L}^i$ and $\hat{L}^i$ in the FOCs yields $2w^i \sqrt{L^h} M' = H'$ and $2w^i \sqrt{L^h} \hat{M} = \hat{H}'$, but this is false since $M' \leq \hat{M}'$, $L^h < \hat{L}^h$ and $H' < \hat{H}'$. This proves that $m < \hat{m}$.

The same argument starting with $h \geq \hat{h}$ implies $L^h \geq \hat{L}^h$, $\ell^i > \hat{\ell}$ and $m < \hat{m}$, which again leads to a contradiction. Thus $h < \hat{h}$ and $L^h < \hat{L}^h$, $\forall i$. Since $m < \hat{m}$, we must have $\ell^i > \hat{\ell}$ for some agent $j$. But the FOCs imply that $\hat{L}^i / \hat{L}^i$ is the same for both $i$, so both agents have more leisure in the Nash equilibrium than in the solution to part b.

g. Again, suppose the agents have identical preferences and that $w^W < w^H$. Can you infer in this case that $H$ will supply more market labor than $W$? Explain.

No. One can conclude $\frac{w^H}{w^m} > \frac{w^W}{w^m}$, but due to the cross effects of $\ell^i$ on $u^j_\ell$ one cannot infer the relationship between the (Nash) equilibrium values of $L^{Hm}$ and $L^{Wm}$. It was the fact that the external effects of $L^i m$ and $L^j h$ on $u^i$ were taken into consideration in solving the joint problem which enabled one to infer the relative impact on $\ell^i$, $L^i m$, $L^j h$, for $i = H, W$.

h. State whether the following statement is true, false or uncertain, and explain your answer:

Since each agent has more discretion over its own decisions when they allocate their time separately, they will be better off.

The claim is false. Consider the special case in the answer to part f, where the agents have the same separable utility function. If $w^H = w^W$, then the problem is completely symmetric and there is a symmetric Nash equilibrium in which the agents’ consumption vectors are equal. Their consumption vectors in the optimal allocation that solves part b are also equal. It follows that their utility the Nash equilibrium is less than in the optimal allocation. If it was higher, the latter allocation would not be optimal. The implicit function theorem can be used to show that the Nash equilibrium allocation and optimal allocation are differentiable functions of the wages, so increasing $w^H$ and reducing $w^W$ slightly changes the allocations only slightly and leaves the utilities in the Nash equilibrium below the utilities in the optimal allocation.
2. A firm manager has already offered to hire a worker. The worker knows that working for the firm is either pleasant or unpleasant and initially believes that it is pleasant with probability \( p \in (0, 1) \). The manager knows if the job is pleasant or unpleasant and has no way to affect the pleasantness of the job for the worker. Before the worker decides whether to accept or reject the job offer, the manager can give the worker a gift.

Suppose first that the manager does not give the gift. If the worker does not work for the firm, then both the manager and worker get payoff 0 from the interaction. If the worker works for the firm, then both get payoff 10 if working for the firm is pleasant; if it is not, then the manager gets payoff 5 and the worker gets payoff -4. If, instead, the manager gives the gift, then it raises the worker’s payoff by 4 units no matter what the worker decides. (The worker keeps the gift whether she accepts or rejects the job offer.) Giving the gift reduces the manager’s payoff by \( c > 0 \) no matter what. In other words, no matter what the worker decides, \( c \) is subtracted from the manager’s payoff specified above.

Before deciding whether to accept the job, the worker knows if a gift has been given. The manager and worker are both rational expected payoff maximizers with common knowledge of the entire interaction described above. (In particular, the manager knows the value of \( p \).)

a. The above interaction can be represented as a sequential (extensive form) game. Draw a complete game tree representing this game. The tree should include all the possible choices of the manager and worker given their respective information. Define all the notation you introduce in labeling the tree.
b. How many pure strategies does the manager have? Give an example of one.
c. How many pure strategies does the worker have? Give an example of one.
d. Suppose that, for some values of the parameters \( p \) and \( c \), there is a pure strategy sequential equilibrium (SE) in which the manager never gives a gift under any circumstances. To formulate the SE, it is still necessary to specify the worker’s beliefs in case the manager did give a gift. Show that sequential equilibrium places no restriction on those beliefs.
e. Find all parameter values of the game in part a such that there is a pure strategy SE in which the manager never gives a gift under any circumstances. Describe the SE completely when it exists and show that it is SE.
f. Find all parameter values of the game in part a such that there is a pure strategy SE in which the manager gives a gift if and only if working for the firm is pleasant. Describe the SE completely when it exists and show that it is SE. Explain why the equilibrium strategies of manager and worker are equilibrium strategies.
g. Find some set of parameter values and a corresponding SE such that the manager gives a gift under all circumstances. Show that the SE is SE.
h. Whenever the manager chooses to give a gift, it might seem that the same purpose could have been achieved by making the conditions in the original job offer more attractive. Why might it make economic sense to model the interaction so that there is a difference between the manager giving the gift and making the original offer more attractive?

2a. In the game tree, C denotes Chance or Nature, M denotes the manager and W the worker. P represents a pleasant job and U an unpleasant job. G and g represent giving a gift. N and n represent not giving a gift. A and R represent accepting and rejecting the job offer when a gift is given, and a and r represent accepting and rejecting the job offer when no gift is given. In the payoff pairs, the manager’s payoff is on top.

b. The manager has 4 pure strategies. An example is \((G, n)\).
c. The worker has 4 pure strategies. An example is \((A, r)\).
d. Suppose M plays \((N, n)\) in some SE. Let W believe with probability \( \mu \in (0, 1) \) that the
job is pleasant in the information set following G or g. We must show that \( \mu \) is the limit of conditional probabilities of \( P \) derived from a sequence of totally mixed strategies for \( M \) converging to the pure strategy \((N, n)\). Let \( M \) play G with probability \( \alpha/k \) and N with probability \( 1 - (\alpha/k) \) after \( P \) (with \( k > \alpha \)) and play g with probability \( 1/k \) and n with probability \( 1 - (1/k) \) after U. The corresponding strategy for \( M \) is totally mixed. As \( k \) goes to \( \infty \), the information set following G or g is reached with probability \( (\alpha p + 1 - p)/k \) and the conditional probability that the job is pleasant is \( \alpha p/(\alpha p + 1 - p) \), which equals \( \mu \) if \( \alpha = \mu (1 - p)/[p(1 - \mu)] > 0 \). If \( \mu = 0 \), we replace \( \alpha \) by \( \alpha_k > 0 \) approaching 0. If \( \mu = 1 \), we can replace \( \alpha \) by \( k - (1/k) \) in the above argument. In each case, \( \alpha_k p/(\alpha_k p + 1 - p) \) approaches \( \mu \) as \( k \to \infty \). This shows that W’s belief about the information set after G or g is consistent no matter what that belief is.

e. If the manager plays \((N, n)\), then rationality requires that W believes the job is pleasant with probability \( p \). W’s expected payoff from \( a \) is \( 10p - 4(1 - p) = 14p - 4 \), which is nonnegative if \( p \geq 2/7 \). Suppose \( p \geq 2/7 \). \((A, a)\) is sequentially rational for W with the belief that the job is pleasant with probability \( p \) in each information set. From the answer to part d, these beliefs are consistent. Suppose instead that \( p < 2/7 \). If M plays \((N, n)\), then \((R, r)\) is sequentially rational with the belief that the job is pleasant with probability \( p \) in each information set. The belief is consistent and \((N, n)\) is sequentially rational, so this is SE. This shows that for all values of \( p \) and \( c \) there is a pure SE in which M plays \((N, n)\).

f. Suppose that the manager plays \((G, n)\). With consistent beliefs, the worker believes the job is pleasant with probability \( 1 \) if the manager gives a gift and believes that the job is pleasant with probability \( 0 \) otherwise. Sequential rationality requires that the worker plays \((A, r)\). For \((G, n)\) to be sequentially rational in response to \((A, r)\), we need \( 5 \leq c \leq 10 \). Otherwise, the manager prefers N or g. Thus, for any of these values of \( c \) and any \( p \in (0, 1) \), there is a pure SE in which M plays \((G, n)\).

g. In SE in which M plays \((G, g)\), consistency requires that if a gift is given the worker believes the job is pleasant with probability \( p \). Sequential rationality of \((G, g)\) in a pure SE
requires that the worker plays \((A, r)\) and that \(c \leq 5\). Sequential rationality of \(A\) requires \(p \geq 2/7\). Sequential rationality of \(r\) requires that the worker believes the job is pleasant with probability no more than \(2/7\). But by answer d, any belief is consistent in the information set following no gift. Thus there is a pure strategy in which \(M\) plays \((G, g)\) as long as \(p \geq 2/7\) and \(c \leq 5\).

h. Giving a gift has the same effect on the worker’s payoff as offering a wage that is higher by that amount. The only difference between these actions has to do with the manager’s information. It makes sense to model the gift as separate from the original job offer if, when making the job offer, the manager does not know if the job would be pleasant for this particular worker. This could happen, for example, if there is a standard wage policy for all job applicants, but the manager has additional information about a particular applicant. After learning that the job is pleasant for this worker, the manager can gain by giving the gift in the SE of part f. In that case, the manager has less flexibility and gets a lower expected payoff if the gift is part of the original wage offer.

3. Consider a pure exchange private ownership economy with two consumers and two goods. Consumer 1 has utility function \(u_1(x_{11}, x_{21}) = x_{11}x_{21}\) and endowment \(e_1 = (3, 1)\), and consumer 2 has utility function \(u_2(x_{12}, x_{22}) = x_{12}^2x_{22}\) and endowment \(e_2 = (3, 3)\), where \(x_{li} \geq 0\) is the quantity of good \(l\) consumed by consumer \(i\), for \(i = 1, 2\).

a. Find all the Pareto efficient allocations in this economy.

b. Find a competitive (Walrasian) equilibrium for this economy. Is there any other competitive equilibrium allocation?

c. Is the allocation in part b Pareto efficient? Justify your answer.

d. Could the welfare of consumer 2 increase if the consumer could change one of the prices by a small amount from its equilibrium level in part b? For this assume that consumer 1 would act as a competitive price taker at the new prices and consumer 2 would get whatever goods are available after satisfying consumer 1’s demand. If your answer is yes, show that there is a small price change that would make consumer 2 better off. If you answer no, explain why the welfare of consumer 2 could not increase with a small change in a price. Would the allocation resulting from a small change in a single price be Pareto efficient? Explain.

e. Suppose that the original economy is modified so that consumer 2 now owns a competitive firm. (The consumers’ utility functions and endowment vectors remain as before.) The firm has constant returns to scale production, producing 2 units of good 2 for each unit of good 1 it uses as input. Find competitive equilibrium prices for this economy and compare them to the prices in the equilibrium of part b. Find the corresponding consumption vector of consumer 2 and compare that consumer’s utility in the new equilibrium to the utility in the equilibrium of part b. Interpret and explain the comparison.

d. Could the welfare of consumer 2 increase if the consumer could change one of the prices by a small amount from its equilibrium level in part b? For this assume that consumer 1 would act as a competitive price taker at the new prices and consumer 2 would get whatever goods are available after satisfying consumer 1’s demand. If your answer is yes, show that there is a small price change that would make consumer 2 better off. If you answer no, explain why the welfare of consumer 2 could not increase with a small change in a price. Would the allocation resulting from a small change in a single price be Pareto efficient? Explain.

3a. An allocation in which one consumer gets the entire aggregate endowment is Pareto efficient. Every feasible change in allocation hurts that consumer. An allocation in which a consumer gets none of one good and a positive amount of the other is Pareto inefficient. Giving some of the latter good to the other consumer raises the second consumer’s utility without hurting the first consumer.

The only remaining possible Pareto efficient allocations are interior, i.e., have positive consumption of each good by each consumer. A Pareto efficient allocation \(\bar{x} = (\bar{x}_1, \bar{x}_2)\), with \(\bar{x}_i = (x_{1i}, x_{2i})\) and \(\bar{u}_i \equiv u_i(\bar{x}_i)\), \(i = 1, 2\), maximizes \(u_1(x_1)\) over the set of allocations satisfying \(u_2(x_2) \geq \bar{u}_2\) and feasibility \((x_1 + x_2 = (6, 4), x_i \geq 0)\). The first order conditions imply that at an interior solution, the consumers’ marginal rates of substitution \([\partial u_i(x_i)/\partial x_{1i}]/[\partial u_i(x_i)/\partial x_{2i}]\) must be equal. Combined with feasibility, this implies \(x_{21}/x_{11} = 2x_{22}/x_{12} = 2(4 - x_{21})/(6 - x_{11})\), hence \(x_{21} = 8x_{11}/(6 + x_{11})\). The Pareto efficient
allocations are the ones satisfying this last equation and feasibility. Corner solutions in which one consumer gets the entire endowment is included among these.

b. At price vector \( p = (p_1, p_2) \), consumer 1 chooses \( x_{11} = (3p_1 + p_2)/(2p_2) \) units of good 2 and consumer 2 chooses \( x_{12} = 3(p_1 + p_2)/(3p_2) \). In Walrasian equilibrium, these demands are feasible, so \( [(3p_1 + p_2)/(2p_2)] + [(p_1 + p_2)/p_2] = 4 \) and \( 3p_1 + p_2 + 2(p_1 + p_2) = 8p_2 \), hence \( p_1 = p_2 \). Then the demands are \( x_{11} = (3p_1 + p_2)/(2p_2) = 2 \), \( x_{21} = 2 \), \( x_{12} = 6(p_1 + p_2)/(3p_1) = 4 \), \( x_{22} = 2 \). There are infinitely many Walrasian equilibrium prices, since \( p_1 \) can be any positive number. But there is a unique equilibrium price vector up to multiplication by a scalar. In other words, each equilibrium price vector is a scalar multiple of every other. There is a unique Walrasian equilibrium allocation.

c. Since each consumer has locally nonsatiated preferences, any competitive equilibrium allocation is Pareto efficient by the first welfare theorem. Alternatively, we can see that the allocation in part b satisfies the conditions for Pareto efficiency found in part a.

d. If consumer 2 can set the prices and the allocation is determined by the demand by consumer 1, then at prices \( (p_1, p_2) \), consumer 2 gets utility \( x_{12}^2x_{22} = \{6 - [(3p_1 + p_2)/(2p_2)]\}^2\{4 - [(3p_1 + p_2)/(2p_2)]\} \). Its derivative with respect to \( p_1 \), evaluated at \( p_1 = p_2 = 1 \) is
\[
2\{6 - [(3p_1 + p_2)/(2p_2)]\}\{4 - [(3p_1 + p_2)/(2p_2)]\}p_2/(2p_1^2) - \{6 - [(3p_1 + p_2)/(2p_2)]\}^2[3/(2p_2)]
\]
\[
= (2 \cdot 4 \cdot 2/2) - 4^2(3/2) = -16.
\]
Therefore, consumer 2 prefers a lower price of good 1 or a higher price of good 2. (The utility of consumer 2 only depends on the ratio \( p_1/p_2 \).) In the unique Walrasian equilibrium allocation, consumer 2 buys good 1 and sells good 2, so the consumer is made better off by a reduction in the price of good 1.

The resulting allocation cannot be Pareto efficient. If it were, then the consumers’ marginal rates of substitution would be equal, but then they would both be optimizing at the new prices, so the new prices and allocation would be a Walrasian equilibrium allocation. But we know that it is not since there is only one Walrasian equilibrium allocation in this economy.

e. If the firm does not produce, then the equilibrium allocation must be as it was before, with \( p_1 = p_2 \) since the rest of the economy is as before. But at these prices, the firm can make more profit by producing good 2 using good 1 as input. This shows that the firm produces in equilibrium. Since the firm has constant returns to scale, its equilibrium profit is \( 0 = 2p_2z - p_1z \), when it uses \( z > 0 \) units of good 1 as input. Therefore in equilibrium, \( p_1 = 2p_2 \). In the resulting allocation, \( x_{12} = 6(p_1 + p_2)/(3p_1) = 3 \) and \( x_{22} = 3(p_1 + p_2)/(3p_2) = 3 \), so consumer 2 gets utility level 27, whereas its utility was 32 in the equilibrium of part b. The reason is that the consumer gets no profit from the firm and the presence of the firm reduces the relative price of the good that consumer 2 supplied in the original equilibrium.

4. Two researchers with identical characteristics try to complete scientific projects in a single period. They can work separately and/or together. Each researcher can complete at most one project working alone and also at most one project jointly during the period. The endowment of time is 1 for each researcher. Researcher \( i \) working alone for \( e_i \) units of time has probability \( p(e_i) \) of completing his work (time invested represents the effort expended). Here, \( p(\cdot) \) is a continuous function satisfying \( p(0) = 0, p'(e_i) > 0 \) and \( p''(e_i) < 0 \) for \( 1 > e_i \geq 0 \). If researcher \( i \) completes his work alone, he consumes the credit (monetary or not) \( x_{is} = 1 \). If he fails at the project working alone, he consumes the credit \( x_{is} = 0 \). If the researchers work together and researcher \( i \) spends \( r_i \) units of time on the joint work, then the probability of the joint work being successful is \( p(r_1 + r_2) \), but each \( i = 1, 2 \) gets the credit \( x_{it} = c, 0 < c < 1 \) if they succeed. If they fail, \( x_{it} = 0, i = 1, 2 \). Each researcher \( i \) has a utility function \( u(l_i, x_i) = l_ix_i \), where \( l_i = 1 - e_i - r_i \) is the amount of leisure time \( i \) consumes and \( x_i = x_{is} + x_{it} \) is the total amount of credit \( i \) gets from the separate and joint
projects. All variables are nonnegative. The probabilities of success of different projects are independent. The researchers maximize their expected utilities.

a. Find the first order necessary condition for optimal effort of a researcher who only works alone for a general probability function \( p(z) \). Calculate the optimal effort \( e_i \) in this case when \( p(\cdot) \) is given by \( f(z) = \begin{cases} -z^2 + 2z & \text{for } 0 \leq z \leq 1 \\ 1 & \text{for } z > 1 \end{cases} \)

b. Suppose that the researchers only engage in joint work and that labor inputs are directly observable and contractible. Assuming that the researchers agree to devote the same amount of labor to their work, find the first order necessary condition for utility maximizing \( r_1 \) for scholar 1 for a general probability function \( p(z) \). Also, calculate the optimal effort \( r_i \) for the probability function \( f(z) \) in part a.

c. Now, suppose that efforts of the researchers are not directly observable and not contractible. The researchers choose their labor inputs independently, but they can only do joint work and this can be enforced. Derive the necessary conditions for strictly positive Nash equilibrium effort levels. Compare these effort levels with the effort level derived in b for general probability function \( p(\cdot) \). Compute effort levels \( r_i = 1, 2 \) for the probability function \( f(z) \).

d. What is the expected number of successfully completed projects for the two researchers together in cases a, b and c, respectively when the probability function is given by \( f(\cdot) \)? Interpret the results.

e. Suppose in part c, the researchers can also work on the side on their own if they so choose. Researcher \( i \) works \( r_i \) units of time for the partnership and \( e_i \) on his own private work. The effort inputs are not directly observable or contractible and the scholars choose their levels independently.

(i) What are the first order conditions for a symmetric (i.e., \( e_1 = e_2, r_1 = r_2 \)) and strictly positive Nash equilibrium under the general probability function \( p(\cdot) \) when \( 0 < c < 1 \)?

(ii) Suppose, instead, that the probability function is given by: \( g(z) = \begin{cases} z & \text{for } 0 \leq z \leq 1 \\ 1 & \text{for } z > 1 \end{cases} \). Compute a symmetric Nash equilibrium in this case.

4a.

Here, \( r_i = 0 \). Scholar \( i = 1, 2 \) solves: \( \max_{e_i} p(e_i)(1 - e_i) \) subject to \( 1 \geq e_i \geq 0 \). Since \( e_1 = 1 \) or \( e_1 \equiv 0 \) are not optimal, one can solve \( \max_{0 < e_i < 1} p(e_i)(1 - e_i) \). Since there is an optimal effort level on \([0, 1]\), there is an optimal effort level in the interior. The necessary condition is: \( p'(e_i)(1 - e_i) - p(e_i) = 0 \). Solution \( e_i \) satisfies: \( e_i = 1 - \frac{p(e_i)}{p'(e_i)} \). When \( p(\cdot) \) is given by \( f(z) \), optimal \( e_i = 1 - \frac{1}{\sqrt{3}} \).

b. Let \( r \equiv r_1 = r_2 \). Researchers solve: \( \max_r p(2r)(1-r)c \) subject to \( 1 \geq r \geq 0 \). Since \( r = 0 \) or \( r = 1 \) are not optimal, there is a solution to: \( \max_{0 < r < 1} p(2r)(1-r)c, 2p'(2r)(1-r)c - p(2r)c = 0, r = 1 - \frac{p(2r)}{2p'(2r)} \). For \( f(z) \), the optimal \( r = \frac{1}{3} < 1 - \frac{1}{\sqrt{3}} \).

c. Researcher 1 solves: \( \max_{r_1} p(r_1 + r_2)c(1 - r_1) \) subject to \( 1 \geq r_1 \geq 0 \). We can replace the problem with \( \max_{r_1} p(r_1 + r_2)c(1 - r_1) \) subject to \( 1 \geq r_1 \geq 0 \) since \( r_1 = 1 \) is not optimal. Let \( L = p(r_1 + r_2)c(1 - r_1) + \lambda r_1 \). \( L_r = p'(r_1 + r_2)c(1 - r_1) - p(r_1 + r_2)c + \lambda = 0 \). \( \lambda \geq 0, \lambda r_1 = 0 \). When \( r_1 > 0 \), \( r_1 = 1 - \frac{p(r_1 + r_2)}{p'(r_1 + r_2)} \). If \( r_2 > 0 \) as well, \( r_2 = 1 - \frac{p(r_1 + r_2)}{p'(r_1 + r_2)} \). From this \( r_1 = r_2 = r \) and \( r = 1 - \frac{p(2r)}{2p'(2r)} \). Compare with b, letting \( \hat{r} \) be the solution to \( b \), we show that \( r < \hat{r} \). \( \hat{r} = 1 - \frac{p(2\hat{r})}{2p'(2\hat{r})} > 1 - \frac{p(2r)}{p'(2r)} \). Since \( \frac{p(z)}{p'(z)} \) is a strictly increasing function of \( z \), \( r \geq \hat{r} \).
would imply \( r \geq \hat{r} > 1 - \frac{p(2r)}{p'(2r)} \geq 1 - \frac{p(2r)}{p'(2r)} \), a contradiction. For the probability function \( f(\cdot), r = \frac{1}{4} < \frac{1}{3} \).

d. a. \( \frac{4}{3} \), b. \( \frac{8}{9} \), c. \( \frac{3}{4} \). Since two researchers each working alone can work on two projects and the probability of success of each project is reasonably given enough efforts, case a gets the highest expected value. In c, researchers consider marginal contribution to success probability \( p'(r_1 + r_2) \), which is \( 2p'(2r) \) in a symmetric case, whereas in b, they consider marginal contribution to success probability, \( 2p'(2r) \). So, they put more time to the project in b.

e(i). \( U = p(e_1)p(r_1 + r_2)(1 + c)(1 - e_1 - r_1) + p(e_1)(1 - p(r_1 + r_2))(1 - e_1 - r_1) + (1 - p(e_1))p(r_1 + r_2)c(1 - e_1 - r_1) \)
\( = p(e_1)(1 - e_1 - r_1) + p(r_1 + r_2)c(1 - e_1 - r_1) \). \( U_{e_1} = p'(e_1)(1 - e_1 - r_1) - p(e_1) - p(r_1 + r_2)c = 0 \)
and \( U_{r_1} = p'(r_1 + r_2)c(1 - e_1 - r_1) - p(e_1) - p(r_1 + r_2)c = 0 \).

e(ii). Since \( p'(e_1) = p'(r_1 + r_2) = 1 \), \( U_{e_1} = p'(e_1)(1 - e_1 - r_1) - p(e_1) - p(r_1 + r_2)c > U_{r_1} = p'(r_1 + r_2)c(1 - e_1 - r_1) - p(e_1) - p(r_1 + r_2)c. \) So, \( e_1 = e_2 = \frac{1}{2} > 0 \) and \( r_1 = r_2 = 0 \) with \( U_{e_1} = 0 \) satisfy necessary conditions. \( U_{e_1} > 0, U_{r_1} = 0, e_1 = 1 \) cannot be a solution since \( U \) would be non-positive.