Partial Answers to Spring 2008 Micro Prelim

1. Consider an individual who is endowed with one unit of time which can be allocated to leisure (\( \ell \)) or to either of two types of labor. Generally, the person uses its labor income to purchase consumption goods \( c \) at the price \( p \). The first type of labor, \( L_1 \), pays a lower wage \( (w_1) \) but is easier to perform, whereas the second type, \( L_2 \), pays a higher wage \( (w_2 > w_1) \) but is more difficult. The person’s preferences are given by the strictly concave utility function \( u(c, \ell, L_1, L_2) \), where the partial derivatives are \( u_c > 0, u_\ell > 0, u_{L_1} < 0 \) and \( u_{L_2} < 0 \).

a. In terms of \( u \), propose an appropriate way to formalize the expressions “\( L_1 \) is easier than \( L_2 \)” or “\( L_2 \) is more difficult than \( L_1 \).”

Answer: \( u_{L_1} > u_{L_2} \) when \( L_1 = L_2 \). Given the strict concavity of \( u \), it is not appropriate to require \( u_{L_1} > u_{L_2} \) everywhere.

b. The specified utility function does satisfy the condition in part a.

c. Assume the person can do \( L_1 \) or \( L_2 \) but not both, and that it is only possible to choose \( L_i \in \{0, \frac{1}{3}\} \); that is, labor can only be supplied in the discrete amounts 0 or \( \frac{1}{3} \). Explain how the individual would choose which job, if either, to perform.

Answer: There are three possibilities: (1) \( L_1 = L_2 = 0, \ell = 1 \), (2) \( L_1 = \frac{1}{3}, L_2 = 0, \ell = \frac{2}{3} \), or (3) \( L_1 = 0, L_2 = \frac{1}{3}, \ell = \frac{2}{3} \).

Answer: \( u(0, 1, 0, 0) \), \( u\left(\frac{w_1}{3p}, \frac{2}{3}, 0, 0\right) \) and \( u\left(\frac{w_2}{3p}, \frac{2}{3}, 0, \frac{1}{3}\right) \).

d. Now assume that labor can be supplied continuously but that again it is not possible to do both jobs. In this case, explain how the individual would choose which job to accept.

Answer: Solve separately and compare. Let \( v_1 = \max_{L_1} u\left(\frac{w_1 L_1}{p}, 1 - L_1, L_1, 0\right) \) and \( v_2 = \max_{L_2} u\left(\frac{w_2 L_2}{p}, 1 - L_2, 0, L_2\right) \) and compare \( v_1 \) vs. \( v_2 \).

e. Next, assume it is possible to supply both types of labor. Formulate the individual’s decision problem and characterize the solution in terms of the appropriate first order conditions.

Answer: Solve \( \max_{L_1, L_2} u\left(\frac{w_1 L_1 + w_2 L_2}{p}, 1 - L_1 - L_2, L_1, L_2\right) \).

First order conditions:
\[
\begin{align*}
\frac{w_1}{p} u_c - u_\ell + u_{L_1} & \leq 0 \\
\left[\frac{w_1}{p} u_c - u_\ell + u_{L_1}\right] L_1 & = 0 \\
L_1 & \geq 0 \\
\frac{w_2}{p} u_c - u_\ell + u_{L_2} & \leq 0 \\
\left[\frac{w_2}{p} u_c - u_\ell + u_{L_2}\right] L_2 & = 0 \\
L_2 & \geq 0
\end{align*}
\]

f. Prove that in order for the agent to supply positive quantities of both types of labor, it is necessary that \( L_2 \) should be more difficult than \( L_1 \) according to your definition in part (a).

Answer: For \( \frac{w_1}{p} u_c - u_\ell + u_{L_1} = 0 \) and \( \frac{w_2}{p} u_c - u_\ell + u_{L_2} = 0 \) to both hold when \( w_2 > w_1 \), it is necessary that \( u_{L_1} > u_{L_2} \).

g. Finally, suppose that while \( L_1 \) is easier than \( L_2 \) (i.e., entails less exertion) it is also more tedious or boring. As presently formulated, is the above model sufficient to allow these two factors – exertion and tedium – to be taken into consideration? If so, explain. Otherwise, discuss how the model might be extended to incorporate them.

Answer: According to the present formulation, all that matters is the combined effect of the two as captured by \( u_{L_2} \). While both factors can play a role, only their combined effect is reflected in the values of the variables of the model. The separate effects cannot be distinguished. Intuitively, on the basis of exertion, \( u_{L_1} > u_{L_2} \) since “\( L_2 \) is more difficult than \( L_1 \),” but on the basis of tedium, \( u_{L_1} < u_{L_2} \) since “\( L_1 \) is more tedious than \( L_2 \).” The current formulation combines the two effects. In order to represent the effects separately, it would be necessary to distinguish the two factors in the utility function, such as \( u(c, \ell, f(e_1, t_1), f(e_2, t_2)) \), where \( e_i \) and \( t_i \) denote the effort expended and tedium associated with job/labor type \( i \).
2. Consider a Robinson Crusoe economy with a single agent and three commodities: a primary resource \( x \), a produced good \( y \), and time. The person is initially endowed with three units of \( x \) and one unit of time. The agent supplies labor \( (L) \) to produce \( y \). However, the productivity of \( L \) is affected by the agent’s health, \( h \). The technology for producing \( y \) is given by \( y = f(L, h) = \sqrt{L}h \). Health is produced from the primary resource and from exercise according to the production function \( h = g(x_h, e) = x_h e \), where \( x_h \) denotes the quantity of \( x \) allocated to health and \( e \) is time devoted to exercise. The agent’s preferences are described by \( u(x, y, \ell) \), where \( x \) is the amount of \( x \) consumed directly and \( \ell \) denotes leisure.

a. What is an allocation in this economy? What quantities must be determined? Identify the feasible allocations. Be precise.

Answer: An allocation specifies each of the following \(( (x_h, x_c), (\ell, L, e), y, h) \).

Feasibility requires:
\[ x_h + x_c \leq 3 \]
\[ \ell + L + e \leq 1 \]
\[ y \leq f(L, h) \]
\[ h \leq g(x_h, e) \]

b. Set up the choice problem facing the agent.

Answer: Choose \(( (x_h, x_c), (\ell, L, e), y, h) \) to maximize \( u(x_c, y, \ell) \) subject to the feasibility constraints or, embedding the constraints
Solve: \( \max_{x_h, x_c, \ell, L} u(3 - x_h, \sqrt{L}x_h e, 1 - L - e) \)

c. For the case in which \( u(x, y, \ell) = xy\ell \), solve the agent’s choice problem.

Answer: Solving the first order conditions yields:
\[ x_h = 1, x_c = 2, y = \frac{1}{3}, h = \frac{1}{3}, \ell = \frac{1}{2} \] and \( L = e = \frac{1}{4} \)

d. By definition of the optimization problem in part c, there is no feasible allocation giving the agent higher utility, so the solution in part c is Pareto efficient. (Note that it is not necessary to assume that the consumer is locally nonsatiated.)

e. The production of health exhibits increasing returns to scale. Even if all the production is combined into a single process using health as an intermediate good, there are increasing returns to scale. No competitive firm chooses a production vector with increasing returns if it is able to produce no output with no inputs. The answer to the question is no.

Next, suppose health is a binary variable taking on the value 0 if the agent is in poor health and 1 if it is in good health. Also, suppose the effect of \( x_h \) and \( e \) on health is stochastic rather than deterministic; no longer is \( h = g(x_h, e) \), but now \( p(h = 1|x_h, e) = \frac{x_h e}{3} \).

f. Set up the agent’s choice problem in this case.

Answer: embedding the constraints, solve:
\[ \max_{x_h, x_c, L} \frac{x_h e}{3} u(3 - x_h, \sqrt{L}, 1 - e - L) + (1 - \frac{x_h e}{3}) u(3 - x_h, 0, 1 - e - L) \]

g. Again assuming \( u(x, y, \ell) = xy\ell \), solve the problem in part (d) and compare the level of exercise and the amount of \( x \) devoted to health here versus in part (c).

Answer: \( x_h = \frac{3}{2}, x_c = \frac{3}{2}, y = \sqrt{3}, h = \frac{3}{2}, \ell = \frac{2}{3}, e = \frac{2}{3} \) and \( L = \frac{1}{2} \)
e and \( x_h \) are both greater here.

h. Do you think the qualitative comparison in part (e) is general or do you think it is peculiar to this particular specification? Discuss the issues involved. What role, if any, does risk aversion play?

The outcome in part g depends on the particular formulation. For example, if \( p(h = 1|x_h, e) = \sqrt{x_h e} \) instead of what it is above, then the solution in part g would be the same as in part c. If \( x_h \) and \( e \) have even less impact on the probability of staying healthy in part g, then their optimal levels can be less in part g than in c. This does not depend on the degree of risk aversion. But stronger risk aversion with respect to consumption \( y \) also plays a role. It
tends to raise the optimal investment in health (raising \( x_h \) and \( e \)) since it reduces the impact of higher \( x_h e \) in the \( y \) argument, whereas \( x_h e \) does not enter the argument \( y \) in the utility when health is good in part \( g \).

3. An economy consists of agents whose ability \( \theta \) takes two possible values, \( \theta_H \) and \( \theta_L \), with \( 0 < \theta_L < \theta_H < 2\theta_L \). The private cost to an agent of ability \( \theta \) of earning \( y \) is \( \frac{\theta}{2} \). The abilities are agents’ private information. When an agent of type \( \theta_i \) earns \( y \) and pays tax \( t \), his utility is \( u_i = y - \frac{\theta_i}{2} - t \), \( i = h, l \). Suppose that the proportion of each type in the (large) population is \( \frac{1}{2} \). The government can tax or subsidize. Assume that government tax/subsidy is purely for transfer of income. Participation in the tax/subsidy system is mandatory.

a. Formulate the government’s problem of choosing a tax/subsidy scheme that maximizes a social welfare function \( u_H + ru_L \), where \( r \geq 1 \), subject to agents’ incentive constraints and the budget constraint for the government.

Answer: \( \max \left( y_h - \frac{\theta_H^2}{2\theta_H} - t \right) + \left( y_l - \frac{\theta_L^2}{2\theta_L} + t \right) \) subject to \( y_h - \frac{\theta_H^2}{2\theta_H} - t \geq y_l - \frac{\theta_L^2}{2\theta_L} + t \) and \( y_l - \frac{\theta_L^2}{2\theta_L} + t \geq y_h - \frac{\theta_H^2}{2\theta_H} - t \).

b. Ignore the incentive constraint of the low ability person (you will be asked to justify this later) and show that, when \( r > 1 \), the incentive constraint of the high ability person binds at a solution.

Answer: If \( y_h - \frac{\theta_H^2}{2\theta_H} - t > y_l - \frac{\theta_L^2}{2\theta_L} + t \), one can increase \( t \) slightly and improve the objective function.

c. Characterize the set of government’s optimal schemes when \( r = 1 \). Assume that tax for the high ability type is non-negative.

Answer: \( L = \left( y_h - \frac{\theta_H^2}{2\theta_H} - t \right) + \left( y_l - \frac{\theta_L^2}{2\theta_L} + t \right) + \lambda \left( y_h - \frac{\theta_H^2}{2\theta_H} - t - y_l + \frac{\theta_L^2}{2\theta_L} - t \right) \).

Thus \( \lambda = 0 \), \( y_h = \theta_h \), \( y_l = \theta_l \). Also,

\[
y_h - \frac{\theta_H^2}{2\theta_H} - t - y_l + \frac{\theta_L^2}{2\theta_L} - t = \theta_h - \frac{\theta_H^2}{2\theta_H} - t - \theta_l + \frac{\theta_L^2}{2\theta_L} - t = \frac{1}{2} \theta_h - \theta_l + \frac{\theta_H^2}{2\theta_H} - 2t \geq 0.
\]

From this,

\[
\frac{1}{4} \theta_h - \frac{1}{2} \theta_l + \frac{\theta_H^2}{4\theta_H} \geq t \geq 0.
\]

(Note \( \frac{1}{4} \theta_H - \frac{1}{2} \theta_l + \frac{\theta_H^2}{4\theta_H} = \frac{1}{4\theta_H} (\theta_H - \theta_l)^2 > 0 \).) Corresponding frontier utilities are \( \{ \theta_h - \frac{\theta_H^2}{2\theta_H} - t, \theta_l - \frac{\theta_L^2}{2\theta_L} + t \} \) \( \frac{1}{4} \theta_h - \frac{1}{2} \theta_l + \frac{\theta_H^2}{4\theta_H} \geq t \geq 0 \}.

d. Describe the government’s optimal scheme when \( r > 1 \) and \( \theta_h = \frac{3}{2} \), \( \theta_l = 1 \).

Answer: \( L = \left( y_h - \frac{\theta_H^2}{2\theta_H} - t \right) + \left( y_l - \frac{\theta_L^2}{2\theta_L} + t \right) + \lambda (y_h - \frac{\theta_H^2}{2\theta_H} - t - y_l + \frac{\theta_L^2}{2\theta_L} - t) \).

One can solve these: \( y_h = \frac{1.5(1+t)}{2r+1} \), \( y_h = 1.5 \), \( t = \frac{0.375r^2}{4r^2+4r+1} \).

e. Show that the incentive constraint for the low ability is satisfied at the solutions in c. and d. In c., \( \theta_l - \frac{\theta_L^2}{2\theta_L} + t - \theta_h + \frac{\theta_H^2}{2\theta_H} + t = (\theta_h - \theta_l)^2 + 2t \geq 0 \). Similarly, one can check for d. when \( \theta_h = 1.5 \) and \( \theta_l = 1 \).

f. Now, vary \( r \geq 1 \) and trace out (qualitatively) the utility possibility frontier of the economy.

g. Now suppose by some government action (such as opening the market to foreign competition), government can change agents’ ‘productivity’. Give possible examples of the utility possibility frontiers of the economy before and after the governmental action that permits Pareto improvement of agents when government combines its action with suitable income.
transfer system. Give an example of such utility possibility frontiers where the Pareto improvement is not possible.

Answer: When the utility possibility curve corresponding to old parameters is strictly below the new utility possibility curve, such Pareto improvement is always possible. If they cross, however, there is some point corresponding to the old curve for which Pareto improvement is not possible.

4. A seller offers a potential buyer an item at a price $p \geq 0$. The seller’s payoff is $p$ if the buyer accepts, and 0 otherwise. A type $I$ buyer ($0 \leq I \leq 1$) gets payoff $v + I - (I^2/2) - p$ if it accepts the seller’s offer and $-I^2/2$ if it rejects the offer. Both agents know $v$. The buyer knows its type before the seller makes the offer, but the seller does not. The seller’s initial belief about the buyer’s type $I$ is represented by a uniform probability distribution on [0, 1].

a. Draw a game tree that can represent the interaction described above.

b. Explain what a pure strategy is for the buyer and give an example of such a strategy.

A pure strategy for the buyer is a choice of acceptance or rejection for every combination of $I \in [0, 1]$ and $p \geq 0$. A pure strategy can be represented as an acceptance correspondence that gives for each $I$ a set of $p$ that are accepted or rejected. An example is reject everything except a price $p = 0$ no matter what $I$ is.

c. Find a subgame perfect equilibrium (SPE) of the game. Is it possible that there is a SPE in which the seller offers a price $p < v$?

In SPE a buyer of type $I$ rejects every $p > v + I$. Suppose it accepts every other offer. Then the seller gets $p$ whenever $I \geq p - v$. The seller believes that this occurs with probability 1 if $p < v$ and with probability $1 - (p - v)$ otherwise, since $I$ is uniformly distributed on $[0, 1]$. This implies that it can never be optimal for the seller to charge $p < v$, since the expected payoff is $p$ in that case, and $p$ can be raised, keeping it below $v$. When $p \geq v$, the expected payoff is $(1 + v - p)p$, with the derivative with respect to $p$ equal to $1 + v - 2p$. If $p = v$, then this derivative is nonpositive when $v \geq 1$. Thus for $v \geq 1$, the best response is $p = v$. If $v < 1$, then the derivative $1 + v - 2p$ is positive at $p = v$, so expected profit rises with a slight increase in $p$. Expected profit is maximized at $p = (v + 1)/2$, and in this case (with $v < 1$) this $p$ is above $v$. These are the only possibilities in SPE.

d. Is there a Nash equilibrium of the game that is not a SPE? If not, show why not. If so, find an NE and compare the players’ payoffs to what they get in the SPE of part c.

Yes. If the buyer uses the strategy example in part b, then a best response for the seller is to offer $p = 0$, and the buyer’s strategy is a best response to that. Then the buyer is rejecting $0 < p < v$, which cannot happen in SPE.

Suppose for the remainder of the problem that the agents’ interaction is the same as above except that before the seller announces an offered price, the buyer can choose its type $I$ from the interval [0, 1]. As before, the seller does not know the buyer’s type when it announces its price offer.

e. Explain what a pure strategy is for the buyer in this new game and give an example of such a strategy.

A pure strategy for the buyer is a choice of $I$ along with a strategy from part b above. This means that there is a rule for reacting to $(I, p)$ even if the buyer did not choose this $I$. An example is $I = 1$ and accept every $p \leq v$ and reject every other $p$ no matter what $I$ is.

f. Does a pure strategy SPE exist for this game? To find out, consider the players’ best responses at each stage of the game. Plot the best response choices of $I$ and $p$ on a single graph, assuming that the seller accepts offers whenever accepting gives at least as high a payoff as rejecting.
No pure strategy SPE exists. In SPE every $p < v + I$ is accepted. In order for the seller to have a pure best response to $I$, $p = v + I$ must also be accepted. Then the seller’s best $p = v + I$. The best response to $p$ is $I = 1$ for all $p < v + (1/2)$, and when this last inequality is replaced by equality, $I = 1$ is also best. But if $p > v + (1/2)$ then $I = 0$ is the only best response. When $I = 0$, the best response is $p = v$, so $I = 1$ is best for the buyer. When $I = 1$, the best response is $p = v + 1$, so $I = 0$ is best for the buyer. There is no pure SPE.