Instructions: Answer any three of the following four questions. Whenever possible, justify your answers. Write your answer to each question in a separate bluebook. Write the number of the question on the cover of the bluebook. DO NOT WRITE YOUR NAME OR STUDENT ID NUMBER on the bluebooks. The exam is 4 hours.

1. Consider an individual who is endowed with one unit of time which can be allocated to leisure (ℓ) or to either of two types of labor. Generally, the person uses its labor income to purchase consumption goods c at the price p. The first type of labor, L₁, pays a lower wage (w₁) but is easier to perform, whereas the second type, L₂, pays a higher wage (w₂ > w₁) but is more difficult. The person’s preferences are given by the strictly concave utility function \( u(c, ℓ, L₁, L₂) \), where the partial derivatives are \( u_c > 0 \), \( u_ℓ > 0 \), \( u_{L₁} < 0 \) and \( u_{L₂} < 0 \).
   a. In terms of \( u \), propose an appropriate way to formalize the expressions “L₁ is easier than L₂” or “L₂ is more difficult than L₁.”
   b. Does the utility function \( u(c, ℓ, L₁, L₂) = c^{\frac{1}{2}} + ℓ^{\frac{1}{3}} + (1 - \frac{L₁}{2} - L₂^2) \) satisfy the criterion you proposed in part (a)?
   c. Assume the person can do L₁ or L₂ but not both, and that it is only possible to choose \( L_i \in \{0, \frac{1}{3}\} \); that is, labor can only be supplied in the discrete amounts 0 or \( \frac{1}{3} \). Explain how the individual would choose which job, if either, to perform.
   d. Now assume that labor can be supplied continuously but that again it is not possible to do both jobs. In this case, explain how the individual would choose which job to accept. Discuss the relevant factors in determining which of the jobs it would accept. Describe circumstances under which it would choose to supply L₁ and others under which it would choose L₂. You may use either the general utility specification \( u(c, ℓ, L₁, L₂) \) or that in part (b).
   e. Next, assume it is possible to supply both types of labor. Formulate the individual’s decision problem and characterize the solution in terms of the appropriate first order conditions.
   f. Prove that in order for the agent to supply positive quantities of both types of labor, it is necessary that L₂ should be more difficult than L₁ according to your definition in part (a).
   g. Finally, suppose that while L₁ is easier than L₂ (i.e., entails less exertion) it is also more tedious or boring. As presently formulated, is the above model sufficient to allow these two factors – exertion and tedium – to be taken into consideration? If so, explain. Otherwise, discuss how the model might be extended to incorporate them.

2. Consider an economy with a single agent, who both produces and consumes, and three commodities: a primary resource (x), a produced good (y), and time. The person is initially endowed with three units of x and one unit of time. The agent supplies labor (L) to produce y. However, the productivity of L is affected by the agent’s health, h. The technology for producing y is given by \( y = f(L, h) = \sqrt{Lh} \). Health is produced from the primary resource and from exercise according to the production function \( h = g(x_h, e) = x_h e \), where \( x_h \) denotes the quantity of x allocated to health and e is time devoted to exercise. The agent’s preferences are described by \( u(x_c, y, ℓ) \), where \( x_c \) is the amount of x consumed directly and ℓ denotes leisure.
   a. What is an allocation in this economy? What quantities must be determined? Identify the feasible allocations. Be precise.
   b. Set up the choice problem facing the agent.
c. For the case in which \( u(x, y, \ell) = xy\ell \), solve the agent’s choice problem.
d. Argue that the solution in (c) is Pareto efficient.
e. If the individual were to behave independently in its consumption and production activities, without regard to its behavior in the other role, would it be possible to achieve the outcome in part (c) as a result of competitive trading among its different “selves,” where each acts as a separate price-taking agent? Explain.

Next, suppose health is a binary variable taking on the value 0 if the agent is in poor health and 1 if it is in good health. Also, suppose the effect of \( x_h \) and \( e \) on health is stochastic rather than deterministic; no longer is \( h = g(x_h, e) \), but now \( p(h = 1|x_h, e) = \frac{x_h e}{3} \), for \( 0 \leq x_h \leq 3 \) and \( 0 \leq e \leq 1 \).

f. Set up the agent’s choice problem in this case.
g. Again assuming \( u(x, y, \ell) = xy\ell \), solve the problem in part (f) and compare the level of exercise and the amount of \( x \) devoted to health here versus in part (c).
h. Do you think the qualitative comparison in part (g) is general or do you think it is peculiar to this particular specification? Discuss the issues involved. What role, if any, does risk aversion play?

3. An economy consists of agents whose ability \( \theta \) takes two possible values, \( \theta_H \) and \( \theta_L \), with \( 0 < \theta_L < \theta_H < 2\theta_L \). The private cost to an agent with ability \( \theta \) of earning \( y \) is \( y^2/2\theta \). The agents’ abilities are private information. When an agent of type \( \theta_i \) earns \( y \) and pays tax \( t \), his utility is \( u_i = y - y^2/2\theta - t \), \( i = h, l \). Suppose that the proportion of each type in the (large) population is \( \frac{1}{2} \). The government can tax or subsidize. Assume that the government tax/subsidy is purely for transferring income. Participation in the tax/subsidy system is mandatory.

a. Formulate the government’s problem of choosing a tax/subsidy scheme that maximizes the social welfare function \( u_H + ru_L \), where \( r \geq 1 \), subject to agents’ incentive constraints and the government’s budget constraint.
b. Ignore the incentive constraint of the low ability person (you will be asked to justify this later) and show that, when \( r > 1 \), the incentive constraint of the high ability person binds at a solution.
c. Characterize the set of the government’s optimal schemes when \( r = 1 \). Assume that tax for the high ability type is non-negative.
d. Describe the government’s optimal scheme when \( r > 1 \) and \( \theta_h = \frac{3}{2}, \theta_l = 1 \).
e. Show that the incentive constraint for the low ability is satisfied at the solutions in (c) and (d).
f. Now, vary \( r \geq 1 \) and trace out (qualitatively) the utility possibility frontier of the economy.
g. Now suppose the government can change agents’ ‘productivity’ by some action (such as opening the market to foreign competition). Give possible examples of the utility possibility frontiers of the economy before and after the governmental action that permit a Pareto improvement among the agents when the government combines its action with a suitable income transfer system. Give an example of such utility possibility frontiers where a Pareto improvement is not possible.

4. A seller offers a potential buyer an item at a price \( p \geq 0 \). The seller’s payoff is \( p \) if the buyer accepts, and 0 otherwise. A type \( I \) buyer \((0 \leq I \leq 1)\) gets payoff \( v + I - (I^2/2) - p \) if it accepts the seller’s offer and \(-I^2/2\) if it rejects the offer. Both agents know \( v \). The buyer knows its type before the seller makes the offer, but the seller does not. The seller’s initial
belief about the buyer’s type \( I \) is represented by a uniform probability distribution on \([0, 1]\).
Both agents are risk neutral.

a. Draw a game tree that can represent the interaction described above.

b. Explain what a pure strategy is for the buyer and give an example of such a strategy.

c. Find a subgame perfect equilibrium (SPE) of the game. Is it possible that there is a SPE in which the seller offers a price \( p < v \)?

d. Is there a Nash equilibrium of the game that is not a SPE? If not, show why not. If so, find an NE and compare the players’ payoffs to what they get in the SPE of part c.

Suppose for the remainder of the problem that the agents’ interaction is the same as above except that before the seller announces an offered price, the buyer can choose its type \( I \) from the interval \([0, 1]\). As before, the seller does not know the buyer’s type when it announces its price offer.

e. Explain what a pure strategy is for the buyer in this new game and give an example of such a strategy.

f. Does a pure strategy SPE exist for this game? To find out, consider the players’ best responses at each stage of the game. Plot the best response choices of \( I \) and \( p \) on a single graph, assuming that the seller accepts offers whenever accepting gives at least as high a payoff as rejecting.