1. A risk neutral, profit-seeking owner of a firm wishes to hire a particular worker. It is publicly known that if the worker spends $t$ hours working for the firm, the firm’s gross profit (prior to any payment to the worker) will be $\pi(t)$, where $\pi' > 0$ and $\pi'' < 0$, and $t$ is publicly observed. It is also publicly known that a worker of type $j = H, L$ working $t$ hours for the firm at wage $w$ will get utility $v(w - g(t, \theta_j))$, where $\theta_H > \theta_L > 0$, $\pi' > 0$, $\pi'' \leq 0$, $g(0, \theta) = 0, \forall \theta$, $g_1 > 0$, $g_1 > 0$, and $g_2(t, \theta) < 0$ and $g_{12}(t, \theta) < 0$ for $t > 0$, with subscripts denoting first and second order partial derivatives.

   a. Suppose that the owner knows the worker’s type. Characterize a contracting offer that is optimal for the owner, being as specific as you can. Interpret your characterization.

   In all the remaining parts of the problem, assume that before making the worker a contracting offer, the owner believes that the worker is of type $j$ with probability $\lambda_j > 0$, where $\lambda_H + \lambda_L = 1$.

   Consider the following possible assumptions about the worker’s information and preferences:

   [1] The worker knows its type before deciding whether to accept the owner’s offer; [2] the worker decides whether or not to accept the owner’s offer, then learns its type, then chooses an offered wage and effort combination or rejects all combinations offered. In case [2], the worker could be an expected utility maximizer, expecting its type to be $j$ with probability $\lambda_j$ (call this case [2e]) or, alternatively, the worker could be pessimistic and care only about its utility in the worst possible outcome (call this case [2p]).

   Consider the optimization problem

   

   \[ \max \sum_j \lambda_j \left( \pi(t_j) - w_j \right) \text{ with respect to } t_H, t_L, w_H, w_L, \text{ subject to} \]

   \[ (P_j): \quad v(w_j - g(t_j, \theta_j)) \geq \bar{u} \text{ for } j = H, L, \text{ and} \]

   \[ (I_j): \quad v(w_j - g(t_j, \theta_j)) \geq v(w_k - g(t_k, \theta_j)) \text{ for } j, k = H, L, k \neq j. \]

   a. Interpret each of the constraints $(P_j)$ and $(I_j)$ under assumption [1].

   b. In which of the cases [1], [2e], [2p], above, is a solution to the optimization problem $(\ast)$ a set of optimal contracts for the owner to offer? Justify your answer. For any remaining cases, write other optimization problem(s) the solutions of which determine contracts that are optimal for the owner to offer.

   d. In at least one of the cases above, problem $(\ast)$ can be used to characterize contract offers that are optimal for the owner. In that case, could a modified version of problem $(\ast)$ be used to characterize the contract offers that are not necessarily optimal for the owner, but are constrained efficient, yielding outcomes that are Pareto efficient among those attainable by a planner who has no more information than the owner? If so, what modification of $(\ast)$ could be used? If not, why not?

   e. Show that the constraints $(P_H)$ and $(I_L)$ in problem $(\ast)$ are redundant since they are implied by the other constraints in the problem.

   f. Suppose $\{(w_j, t_j)\}_{j=H,L}$ solves problem $(\ast)$ with $t_L > 0$. Write first order conditions necessarily satisfied by that solution and use them to compare $t_H$ and $t_L$ to the working hours of types $H$ and $L$ in part a, where the owner knows the worker’s type.
g. In this problem, the only variable that affects the firm’s gross profit is \( t \), and it is publicly observed. Does it matter for the owner’s welfare whether or not the owner knows the worker’s type? Why or why not?

2. Consider the following auction game. A single object is being auctioned off to \( n \) bidders. Each bidder \( i = 1, \ldots, n \) values the object at \( v_i > 0 \). The indices are chosen in a way so that \( 0 < v_n < \cdots < v_1 \). Each bidder can submit any bid \( b_i \geq 0 \).

Consider the second-price format. The bidder whose bid is the highest wins the object; if there are multiple highest bids, then the winner is randomly chosen from bidders with the highest bid and each of them has equal chance to win. The winner, say bidder \( i \), gets a payoff \( v_i - \max b_{-i} \), where \( \max b_{-i} \) is the highest bid made by bidders other than bidder \( i \). All non-winning players receive zero payoff. In this problem consider only pure strategies.

a. Formalize the auction as a normal form game.

b. Show that \( b_i = v_i \) is a weakly dominant strategy for player \( i \).

c. Let the total number of bidders be \( n = 10 \). Find all Nash equilibria of the second-price auction in which player \( i = 8 \) is the only winner of the object. Justify your answer.

Next consider the “semi-second-price” auction format. Everything is the same as in the second-price auction above, except that the winner pays the average of the highest and the second highest bid (if there is a tie at the highest bid then a winner is randomly chosen among highest bidders with equal chance and the winner pays the highest bid). Players who do not win receive zero payoff.

d. Is there a weakly dominant strategy for any player? Justify your answer.

e. Can you find a pure-strategy Nash equilibrium? Explain your answer carefully.

f. Is there any advantage for the seller to use the semi-second-price rather than the second-price auction?

3. Consider an economy in which outputs 1 and 2 are produced under constant returns to scale by identical competitive firms using labor and land as inputs. Both land and labor are essential, i.e., both inputs are needed to produce a positive amount of either output. The unit cost of output \( k \) is \( c_k(w, r) \), where \( c_k \) is continuously differentiable, \( w \) is the price of labor and \( r \) the price of land. Each consumer \( i, i = 1, \ldots, I, I \geq 2 \), has the endowment vector \( (E, L) \gg 0 \) of labor and land and a strictly quasiconcave, continuously differentiable utility function \( u(x_{1i}, x_{2i}, q) \), with strictly positive partial derivatives, where \( x_ki \) is the amount of output \( k \) consumed by \( i \) and where \( q \) is the total quantity of land that is not used as an input in production. Consumers can keep part of their land endowment and not sell all of it to the firms.

a. The assumption that both inputs are essential implies what restriction on the unit cost function \( c_k \)?

b. In a competitive (Walrasian) equilibrium with no disposal for this economy, each agent must be optimizing subject to constraints, given the equilibrium choices of the other agents. Let \( \mathcal{E} \) be the set of competitive equilibria in which every consumer consumes both outputs and every firm uses both inputs. Use the notation above to write a system of equations and/or inequalities that characterize \( \mathcal{E} \). (The equations and inequalities are satisfied by the equilibria in \( \mathcal{E} \) and only by those equilibria.)

c. Is it possible that there is an equilibrium in the set \( \mathcal{E} \) in which not all the land is used in production? Explain your answer.

d. What can be said about the efficiency of the allocations in equilibria in \( \mathcal{E} \)? Try to be as complete as possible. Explain your answer.
e. Are there conditions under which the efficiency of a competitive equilibrium allocation could be improved by taxing the firms a fixed amount per unit of land they use, with the tax revenue paid to the consumers in equal amounts? Explain.

f. If the consumers' preferences changed so that the equilibrium price of output 1 rose slightly, with the price of output 2 fixed, what could be said about the effect on the equilibrium prices of labor and land, assuming that all firms produce before and after the change?

4. Consider a farmer who grows corn each year from seed acquired the year before. Initially (in period 1), the farmer is endowed with the quantity of seed $s_0$ which he devotes entirely to corn production according to the production function $c_t = \theta s_{t-1}$, $\theta \in (0, 1]$. However, subsequently (beginning in period 1), the farmer must decide how much of his annual crop to sell on the market and how much to set aside for seed for next year's production. When making this decision in period $t$, the farmer knows the market price of corn at $t$, denoted $p_t$, but does not know the future prices. However, he does know that in each period prices are distributed according to a stationary distribution with mean $\mu$. Assume the farmer seeks to smooth current (undiscounted) income/profits over time.

Initially, assume the farmer cannot acquire seed from outside sources, but must grow his own.

a. For the case in which there are only two periods, formulate and solve the decision problem facing the farmer.

b. For the case in which $p_1 = \mu$, discuss the behavior of the solution with respect to changes in $s_0$ and $\theta$. How does the solution differ if $p_1 > \mu$ or $p_1 < \mu$?

c. Next, assuming $p_t = \mu$ for all $t$, generalize the problem and the solution to $n$ periods.

(d) Explain why the solution does not involve a constant amount of seed set aside each year.

e. Discuss the limit behavior of the optimal values of $(c_t, s_t)$ as $n$ increases.

Now, assume the farmer is able to buy or sell seed (from period $t - 1$) in period $t$ at the price $r_t$. As in the case of corn, assume the farmer knows $r_t$ when deciding how to allocate output in period $t$ but does not know future seed prices. These are distributed according to a stationary distribution with mean $\beta$.

f. Analyze the farmer's optimal decisions in the event there are only two periods.

g. Contrasting parts (a) and (e), discuss the relative welfare/income level of the farmer in the two cases. Under what conditions will the farmer be better off by participating in the seed market versus operating autonomously, outside of the market?