1. Consider a Robinson Crusoe economy with two production processes. Coconuts, $x$, are produced according to the technology $x = \sqrt{L_x}$, where $L_x$ denotes the quantity of labor devoted to $x$ production. However, the production of bananas, $y$, is stochastic. If $L_y$ units of labor are devoted to $y$ production, then with probability $\theta$ the quantity of $y$ produced will be $\sqrt{L_y}$, and with probability $1-\theta$ output will be 0. Robinson Crusoe (RC) is endowed with one unit of time and has the von Neumann-Morgenstern utility function $u(x, y) = x + y$.

a. How should RC allocate his time between $x$ and $y$ production? How does the optimal allocation vary with $\theta$?

b. Express the optimal utility as a function of $\theta$, and discuss the properties of the indirect utility function so obtained.

c. Suppose $x$ and $y$ production is carried out independently by two profit-maximizing firms who employ labor and distribute any profit to RC, while RC in turn supplies labor and purchases $x$ and $y$ in competitive markets. Argue that the allocation described in part (a) can be decentralized (supported) as a competitive equilibrium and find supporting prices. Explain the interpretation of prices in this context.

d. Suppose it were possible for RC to expend resources (units of $x$) in order to reduce the uncertainty involved in $y$ production. Specifically, if he were to expend $i$ units of $x$, then the resulting probability would be

$$\theta(i) = \theta^\circ + \frac{i}{1+i}(1-\theta^\circ),$$

where $\theta^\circ$ denotes the initial level of $\theta$.

Set up the problem RC would solve in order to determine the optimal levels of $i$, $x$ and $y$. Derive and interpret the first order conditions for a solution. (You do not have to solve for an explicit solution.) Is there necessarily an interior solution to the problem? Explain.

e. Discuss the relationship between this problem and that of determining an optimal portfolio in allocating a fixed budget between a risky versus a safe asset.

2. Consider a pure exchange economy with two consumers and two goods. Each consumer’s consumption set is $\mathbb{R}^2_+$. Consumer 1 prefers consumption vector $x_1 = (x_{11}, x_{21})$ to $z_1 = (z_{11}, z_{21})$ if $x_{11} > z_{11}$, or if both $x_{11} = z_{11}$ and $x_{21} > z_{21}$ hold. Consumer 2 has the utility function $u_2(x_{12}, x_{22}) = x_{12}x_{22}$. Consumer 1 has the initial endowment vector $(1, 0)$, and consumer 2 has the initial endowment $(0, 1)$.

a. Show that at a strictly positive price vector $p = (p_1, p_2) \gg 0$ and a wealth level $w \geq 0$, the ordinary competitive (Walrasian) demand vector of consumer 1 is the same as if consumer 1 had the utility function $u_1(x_{11}, x_{21}) = x_{11}$.

b. Are the preferences of consumer 1 locally nonsatiated? Justify your answer.

c. Are the preferences of consumer 1 convex? Explain.
d. Use the notation above to give a formal definition of a Pareto efficient (Pareto optimal) allocation for this economy.

e. Find all the Pareto efficient allocations for this economy and draw them in an Edgeworth box diagram. Is the initial endowment Pareto efficient? Justify your answers.

f. Use the notation above to give a formal definition of a competitive (Walrasian) equilibrium for this economy. Note that in a competitive equilibrium, each consumer must be choosing an optimal consumption vector in the consumer’s budget set.

g. For a general pure exchange economy with private ownership, state conditions (as weak as you can give) on the fundamentals of the economy that are sufficient to ensure that a competitive equilibrium exists.

h. Show that no competitive equilibrium exists for the particular economy specified above. Which of the sufficient conditions from your answer to part g is (or are) violated in the particular economy?

i. Consider a modification of the particular economy above in which the initial endowment of consumer 1 is \((1, 1)\) and the initial endowment of consumer 2 is \((0, 0)\). The consumers’ preferences remain as specified above. Show that this initial endowment is Pareto efficient (or refer to your answer to part e). Is there a competitive equilibrium with this initial endowment as the equilibrium allocation? Justify your answer. How is this related to the second fundamental welfare theorem?

3. Two firms’ have developed similar computer programs and are competing to sell them to a government. The government will buy from the firm whose program wins a contest for speed (or buys from each firm with probability 1/2 if the program speeds are equal). But before the contest, each firm must choose closed (C) or open (O) architecture for its program, which determines the program’s compatibility with existing software. The net gains to the winner and the loser of the speed contest are shown in the table below, with the winner’s net gain listed first in each payoff pair. The firms’ behavior can be represented as a game in which the firms’ program speeds are chosen by nature as independent draws from a uniform distribution on \([0, 1]\). Each firm observes its own program speed \(v_i\), but not the speed of its rival. Then the firms independently choose open or closed architecture (without communicating). These choices do not affect the speeds of their programs. Finally, the government tests the programs and buys one. All of the above is common knowledge.

<table>
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<th>C</th>
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<tr>
<td>Winner</td>
<td>(2, -2)</td>
<td>((4, -1))</td>
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<tr>
<td>Loser</td>
<td>((1, -2))</td>
<td>((0, 0))</td>
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a. Define what a pure strategy for one of the firms is. Give an example of a pure strategy for a firm.

b. Suppose that firm 2 uses a strategy in which it chooses C if its program speed is greater than some number \(V\) and chooses O otherwise. Suppose that the speed of firm 1 is \(v_1 > V\). Find the expected net gain for firm 1 if it chooses C, given firm 1’s information when it has to make its architecture choice. Explain very carefully why the expected profit is what you wrote. NOTE that whether firm 1 picks a row or a column in the table depends on whether it has a higher or lower speed than firm 2.

c. Under the assumptions of part b, write the expected net gain for firm 1 if it chooses
O, given its information when it chooses its architecture. Compare this expected net gain to what you found in part b.

d. Show that there is a number \( V \) and a Nash equilibrium for the above game in which each firm chooses \( C \) (closed architecture) if and only if its program speed is greater than \( V \). You must consider the firm’s choice of architecture when its speed is above \( V \) and when its speed is below \( V \).

e. Find the expected net gain of each firm in the Nash equilibrium of part d.

f. Now, suppose that before the firms choose their architecture, each firm \( i \) can either publicly reveal its program’s correct speed \( v_i \) (which can be verified) or it can keep the speed a secret until after the architectures are chosen. The firms’ decisions to reveal or not are made simultaneously. Then after whatever revelation is made, the firms simultaneously choose their architectures and the game continues as before. Give a definition of a pure strategy for a firm in this new game, being as specific as possible.

g. Define a pure strategy sequential equilibrium for the game of part f.

h. For the game of part f, consider the joint strategy in which neither firm ever reveals its speed and the firms play the Nash equilibrium of part d. Explain why this joint strategy is NOT part of a sequential equilibrium of the game of part f.

i. Show that there is a sequential equilibrium for the game in part f in which each firm reveals its speed whenever the speed is positive. Note that you need to specify the equilibrium beliefs in information sets that are not reached in equilibrium.

4. A consumer will either suffer a loss of \( L \) dollars or else will suffer no loss during a particular period. The loss will occur with probability \( \pi \in (0, 1) \) if the consumer takes care to avoid it and will occur with probability \( \pi' > \pi \) if the consumer does not take care. The consumer seeks to maximize expected utility, and obtains utility \( u(w) - 1 \) from sure wealth \( w \) if the consumer takes care, and obtains utility \( u(w) \) from sure wealth \( w \) if the consumer does not take care. A monopoly insurance company offers the consumer a contract under which the consumer pays the insurer \( p \) whether or not the loss occurs and the insurer pays the consumer a benefit \( B \geq 0 \) if and only if the loss does occur. The consumer either accepts or rejects the contract. The insurance company knows all the above information and can tell whether or not the loss \( L \) occurs, but cannot observe whether the consumer takes care to avoid the loss. The insurance company seeks to maximize its expected net revenue from the insurance contract. The above information determines a game played by the insurer and the consumer. In all parts of the question below, assume:

(A) There is a Nash equilibrium in which the consumer accepts the offered contract and takes care to avoid the loss.

(B) The function \( u \) is differentiable with \( u' > 0 \) and \( u'' \leq 0 \).

a. Give an economic interpretation for the restriction \( u'' \leq 0 \).

b. Specify a maximization problem that determines a contract offered in a Nash equilibrium of the game described above. Use the first order conditions to characterize such a contract, being as specific as possible.

c. For this part of the problem only, suppose that the insurer can observe whether the consumer takes care and can make the benefit it pays in case a loss occurs depend also on whether care is taken. Specify a maximization problem that determines a contract that is optimal for the insurer in this case. Compare the coverage under this contract to the coverage under the contract of part b. Compare the expected net revenue of the
insurer under this contract to the expected net revenue in part b. Treat the cases of a risk neutral and a strictly risk averse consumer separately.

d. Now return to the situation described above part (a) of this problem, but suppose that the loss described above might also occur in a second period. The interaction between the consumer and the insurer is repeated in the second period. The consumer seeks to maximize the sum of its expected utilities in the two periods, and the insurer seeks to maximize the sum of its expected net revenues. The consumer can choose not to buy insurance in either period. At the beginning of the first period, the insurer can offer a contract (and commit to its terms) promising benefits to be paid in either or both periods, depending on the losses incurred before or during the period of the benefit payment. Specify a maximization problem that determines a contract of this type that is optimal for the insurer.

e. Use the first order conditions to characterize a solution to the maximization problem in part d. Can a solution be different from a repetition of contracts that are optimal for the insurer in each period separately? Explain.

f. State conditions on the fundamentals under which there is an optimal two-period contract for the insurer (as in part d) that consists of a repetition of contracts that are optimal in a single period. This optimal two-period contract must be such that the second period insurance benefit depends only on whether a loss occurs in the second period.