1. Consider an island economy with a single agent who is endowed with one unit of time all or part of which can be supplied as labor \((L)\) to produce food \((y)\) or consumed as leisure \((\ell)\). Food is produced according to the technology \(y = f(L)\). However, production also generates pollution as a byproduct which adversely affects the welfare of the agent. Overall, the agent’s utility is given by \(u(y, \ell) = y\ell - c(y)\), where \(c(y)\) denotes the welfare cost of producing \(y\).

a. What is an allocation in this economy? What quantities must be determined? Identify the feasible allocations. Be precise.

b. Set up the choice problem facing the agent, assuming that the agent can choose any feasible input and output combination.

For the remainder of the problem, suppose \(f(L) = L\) and \(c(y) = \frac{1}{3}y\).

c. Solve the agent’s choice problem in this case.

d. Argue that the solution in part c is Pareto efficient.

e. Explain the meaning of the Second Welfare Theorem in this context. Does the theorem ”hold” here? That is, does the conclusion follow under these circumstances?

Next, suppose instead there are two identical individuals each with the tastes and endowment as described above. Denote agent \(i\)’s quantities as \(L_i, \ell_i\) and \(y_i\), for \(i = 1, 2\). Then total output is \(y = f(L_1 + L_2) = y_1 + y_2\), total (pollution) costs are \(c(y)\), and \(u_i(y_i, \ell_i) = y_i\ell_i - c(y)\).

f. Set up the Social Planner’s Problem for determining the Pareto efficient allocations.

g. If each individual were to contribute the same amount of labor and receive the same amount of food as in part c, would the outcome be Pareto efficient? Explain.

h. Instead, suppose that each agent decides how to allocate its time separately, taking the other agent’s decision as given, and assuming it will consume what its labor produces. Argue that in this case, the outcome would be the same as in part c and explain why.

i. Discuss the proposal to partition the island into two halves and have each of the individuals engage in production separately (incurring only the welfare cost associated with their own production), rather than produce jointly. Evaluate this proposal.

2. Consider two duopolists, 1 and 2, who produce a homogeneous product. Their cost functions are given, respectively, by \(c_1(y_1) = y_1^2\) and \(c_2(y_2) = y_2^2 + ky_2\), for some \(k > 0\). The firms act as Cournot competitors facing uncertain (future) market demand described by the inverse demand function \(p(y, \alpha)\), where \(y\) is aggregate output and \(\alpha\) is a random variable distributed on the interval \([\alpha, \bar{\alpha}]\) with known density \(f(\alpha)\). The firms must choose their outputs prior to the realization of market demand \((\alpha)\).
a. In light of the above, set up the decision problem facing firm $i$ assuming each firm seeks to maximize its expected profit taking the other firm’s output as given. (Do not solve the problem.)
b. For the special case in which $p(y, \alpha) = -y + \alpha$, solve for the Cournot equilibrium as a function of the expected value of $\alpha$, denoted $\hat{\alpha}$.
c. Argue that if $\hat{\alpha} > 3k$, then firm 1’s revenue will be greater and it’s costs less than firm 2’s in Cournot equilibrium, and hence 1’s profits will be greater.
d. In light of part c, suppose that the government decides to subsidize firm 2 in the event of a loss in order to ensure that there remain at least two competitors in the industry. This works as follows. As before, the firms first engage in Cournot competition, choosing their output levels to maximize their expected profits. Market demand is then realized. If the market price is below firm 2’s break-even price, denoted $p(y_2)$, the government would raise the price for firm 2 (only) so that it would be able to break even. Assuming firm 2 knows it will be subsidized in the event of a loss, explain how the government intervention would affect its decision problem.
e. Discuss the effects of $\hat{\alpha}$ and $k$ on the likelihood of the need for government subsidization.
f. Next, suppose that even when subsidies are not necessary (i.e., when firm 2 would be profitable otherwise), the government decides to provide a price subsidy to firm 2 since firm 1 has a cost advantage. In this case, firm 1 would receive the market price $p(y, \alpha)$, but firm 2 would receive $p(y, \alpha) + \varepsilon$ per unit. Prove that this would always have the desired effect of increasing firm 2’s market share relative to firm 1’s.

3. Apple’s market research shows that there are two types of consumers for its iMac computer and bMac computer (baby iMac). The bMac is physically the same as the iMac but has some functions disabled. Both iMac and bMac cost 300 dollars to produce per unit. Both consumer types, $h$ and $l$, get utility $m$ if they have $m$ dollars and no computer: $u_h(0, m) = u_l(0, m) = m$. High type consumers with $m$ dollars (after paying for a computer) get utility $u_h(i, m) = 1500 + m$ and $u_h(b, m) = 800 + m$, where $i$ denotes having one iMac and $b$ having one bMac. Low types get utility $u_l(i, m) = 600 + m$ and $u_l(b, m) = 500 + m$. Apple knows that there are one million consumers of high type and 2 million consumers of low type. Once Apple announces a price, it must serve all consumers who demand its product at that price. The consumers start with enough money to buy a computer if it raises their utility. They demand at most one computer during the period considered.

a. Suppose Apple sells iMacs at 1500/unit and bMacs at 500/unit. Find the maximum sales and profits Apple could get.
b. Suppose Apple offers only iMacs. Find profit maximizing price, sales and profit.
c. Suppose Apple offers only bMacs. Find profit maximizing price, sales and profit.
d. Suppose Apple offers iMacs at price $p_i$ and bMacs at price $p_l$. Show that no prices are such that high types buy bMacs and low types buy iMacs.
e. Find profit maximizing prices $p_i$ and $p_l$ that can induce high types to buy the iMac and low types to buy the bMac. What is the corresponding profit?
f. Do part e assuming instead that the number of low types is 1 million. In this case, find profit maximizing prices and compare the resulting profit to what Apple receives if it offers only the iMac at a profit maximizing price.
4. A worker has to decide whether to finish her degree or not before applying for a job with a particular firm. The firm knows that the worker is of high or low ability and will know if the worker finishes her degree. Finishing the degree costs the worker 1 if she is of high ability and 2 otherwise. If the worker is of high ability, then she will produce 5 units of revenue for the firm if she finishes her degree and 3 units if she does not. If the worker is of low ability she will produce 2 units of revenue for the firm if she finishes her degree and 1 unit if she does not. The firm pays each of its workers with a degree $W$ ($0 < W < 5$) and pays each of its workers without a degree 1. Initially, before the worker decides about her degree, the firm believes with probability $\lambda \in (0, 1)$ that she is of high ability and decides whether to hire her or not. The interaction between the worker and firm is represented, as a game of common knowledge, in the tree in Figure 1, below, where the worker is player [1] with the first listed payoff, and the firm is player [2] with the second listed payoff.

a. Why might it make sense to model a worker’s cost of finishing a degree as depending on her ability even when there is no discrimination among students in pricing of education? 
b. Label the players’ moves in the tree in Figure 1. Refer to your labels in your answers to the remaining parts of this problem.

c. Under what (if any) conditions is there a Nash equilibrium (NE) in which the worker finishes her degree for sure if she has low ability and does not if she has high ability? 
d. Under what (if any) conditions on the exogenous variables $\lambda$ and $W$ is there a NE in which for sure the worker does not finish her degree if she has low ability and for sure she finishes it if she has high ability? Interpret the conditions if there are any.

e. Find and interpret all conditions on the exogenous variables under which there is a weak perfect Bayesian equilibrium (WPBE, i.e., weak sequential equilibrium) where, for sure, the worker finishes her degree no matter what her ability is.

f. Find and interpret all conditions on the exogenous variables under which there is a weak perfect Bayesian equilibrium (WPBE, i.e., weak sequential equilibrium) where, for sure, the worker does not finish her degree no matter what her ability is.

g. What can be said about the efficiency of the outcomes in the equilibria in parts c, d, and e? Define an appropriate notion of efficiency for the economy consisting of the worker and firm. How does your answer depend on the values of $\lambda$ and $W$?

Figure 1
Answers: 1.a. $((L, y), (y, l))$. Feasible: $L, l \geq 0, L + l \leq 1, y \leq f(L)$

b. $\max_{L, l, y} \ell - c(y) \text{ s.t. } L, l \geq 0, L + l \leq 1, y \leq f(L)$; or $\max_{\ell} f(1 - \ell) \ell - c(f(1 - \ell))$

s.t. $0 \leq \ell \leq 1$

c. $\max(1 - \ell)(\ell - \frac{1}{4}) \text{ s.t. } 0 \leq \ell \leq 1 \text{ yields } \ell = \frac{5}{8} \text{ (therefore, } L = y = \frac{3}{8}) \text{ and } u = \frac{9}{64}. \text{ (check SOC)}$

d. Social welfare is individual welfare.

e. Can the (efficient) outcome in c be supported as a competitive equilibrium? Note: preferences are quasiconcave.

f. $\max_{L_1, \ell_1, y_1, \ell_2, y_2} (y_1 \ell_1 - c(y_1 + y_2)) \text{ s.t. } (y_2 \ell_2 - c(y_1 + y_2)) \geq \overline{w}_2; L_i, \ell_i \geq 0, L_i + \ell_i \leq 1,$

for $i = 1, 2$; and $y_1 + y_2 \leq f(L_1 + L_2)$;

g. $\ell_i = \frac{5}{8}; L_i = y_i = \frac{3}{8}$ and $u_i = \frac{3}{16} \text{ with } \frac{3}{8} \text{ vs. max } 2[\frac{1}{2} f(2(1 - \ell_i)) - c(f(1 - \ell_i)) + \ell - c(y)] \geq \ell_i - c(y) \ell_i$

h. Since $f$ and $c$ are linear, the choices of one agent affect the objective function of the other simply by adding a constant to it. (The marginal product $f'$ and marginal cost $c'$ for a single agent are unaffected by the other agent’s choices.) Therefore, the solution to a single agent’s optimization problem is unaffected by the other agent’s choices.

i. This means reverting to part c, with $\ell_i = \frac{3}{4}, L_i = y_i = \frac{1}{4} \text{ and } u_i = \frac{9}{64}. \text{ Versus here max symmetric utility is } u_i = \frac{1}{16}.$

2. a. $\max_{y_2} \int_\alpha (p(y_1 + y_2, \alpha) y_1 - c(y_1)) f(\alpha) d\alpha.$

b. $\hat{y}_1 = \frac{3\alpha + k}{15}, \hat{y}_2 = \frac{3\alpha - 4k}{15}, \text{ where } \hat{\alpha} \text{ is the expected value of } \alpha.$

c. $p(y, \alpha)\hat{y}_1 > p(y, \alpha)\hat{y}_2$ if and only if $\hat{\alpha} > 3k, c(\hat{y}_1) < c(\hat{y}_2)$

d. For each $y_2$, the break-even price is given by $p(y_2) = c_2(y_2) = 0$. For $c_2(y_2) = y^2 + ky, p(y_2) = y_2 + k$. Let $p_+(y, \alpha) = \max\{p(y, \alpha), p(y_2)\}$ = $\max\{-(y_1 + y_2) + \alpha, y_2 + k\}$. Then 2 would now solve $\max_{y_2} \int_\alpha (p_+(y_1 + y_2, \alpha) y_2 - c_2(y_2)) f(\alpha) d\alpha.$

e. Cournot price is $p^c = \frac{-2\hat{\alpha} + k}{5} + \alpha$ versus $\hat{p}(\hat{y}_2) = \frac{3\hat{\alpha} + 11k}{15}.$ Hence, $p^c < \hat{p}(\hat{y}_2)$ iff $\alpha < \frac{9\hat{\alpha} + 8k}{15}$. From the latter, the RHS is increasing in both $\hat{\alpha}$ and $k$. Therefore, for given $\alpha$, the greater the cost premium $k$, the more likely the need for subsidization. Moreover, from part b, both firms will expand production in response to an increase in $\hat{\alpha}$. This will decrease the equilibrium price and possibly lead to further subsidization.

f. Firm 1’s reaction function would be the same, namely, $y_1 = \frac{\hat{\alpha} - y_2}{4}$. However, firm 2’s is now $y_2 = \frac{\hat{\alpha} + y_2}{4} - k$. The new equilibrium is then $y_1^* = \frac{3\hat{\alpha} - e + k}{15}, y_2^* = \frac{3\hat{\alpha} + 4e - 4k}{15}$. Hence, $y_2^* > y_2$ and $y_1^* < y_1^c$.

3.a. Both types will buy bMac. Sales: 3 million bMacs. Total profit $= 200$ million $= 600$ million.

b. If priced only for high type, the optimal price for iMac ($p_h$) is 1500, with 1 million iMacs sold. Profit is 1200$-1$ million $= 1200$ million. To sell to both types, optimal price is 600, with 3 million iMacs sold. Profit is 300$-3$ million $= 900$ million. So, optimal $p_h$ is 1500, with 1 million iMacs sold and 1200 million profit.

c. It is optimal to serve only the high type: optimal price of bMac ($p_l$) is $800$; sales: 1 million at profit of $500$ million. Serving both types: $p_l = 500$; sales: 3 million at profit of $600$ million. So, optimal price is $p_l = 500$.
d. If a high type buys a bMac, $800 - p_l \geq 1500 - p_h$. If a low type buys a bMac, $600 - p_h \geq 500 - p_l$. These inequalities imply $100 \geq p_h - p_l \geq 700$. Thus, they cannot both hold.

e. If high types buy iMac and low types buy bMac, $1500 - p_h \geq 800 - p_l$, $500 - p_l \geq 600 - p_h$, $1500 \geq p_h$, $500 \geq p_l$. So, $700 \geq p_h - p_l \geq 100$, $1500 \geq p_h$, $500 \geq p_l$. Apple maximizes $(p_h - 300) \cdot 100$ million $+(p_l - 300) \cdot 200$ million subject to $700 \geq p_h - p_l \geq 100$, $1500 \geq p_h$, $500 \geq p_l$. Thus, $p_h = 1200$, $p_l = 500$. Total profit is $1300$ million.

f. To accommodate both types, Apple maximizes $(p_h - 300) \cdot 100$ million $+(p_l - 300) \cdot 100$ million subject to $700 \geq p_h - p_l \geq 100$, $1500 \geq p_h$, $500 \geq p_l$. Total profit is $1100$ million. In this case, it is better for Apple to ignore low types and offer iMac only at $1500$ since this will result in the profit of $1200$ million.

4a. A worker with lower ability might have to spend more time on school work in order to finish the degree (hence give up time for earning money or other pursuits) or might need to take some courses over and have to pay for them.

b. In the tree below, F and S [respectively, f and s] represent finishing the degree and stopping before finishing when the worker is of high ability [resp., low ability]. H and N represent hiring and not hiring a worker with a degree and h and n represent hiring and not hiring worker without a degree.

c. Let $\sigma_H$ and $\sigma_h$ be the probabilities of hiring a worker with and without a degree, respectively (in the firm’s strategy). If, in NE, the worker finishes her degree for sure when she has low ability, then the worker’s expected payoff from not getting the degree, $\sigma_h$, is no greater than $\sigma_H W - 2 < \sigma_H W - 1$, which is the worker’s expected payoff from getting the degree with high ability. So she gets the degree in such NE for sure.

d. NE best response requires that the firm hires a worker with a degree for sure. F and s are best only if $W - 2 \leq \sigma_h \leq W - 1$ for some $\sigma_h \in [0, 1]$. Such a $\sigma_h$ exists if $1 \leq W \leq 3$. The wage paid to a worker with a degree cannot be too high or too low.

e. If the worker finishes her degree no matter what, then the firm believes with probability $\lambda$ that she is of high ability if she has the degree and it hires her with positive probability in that case (otherwise s is best). This implies $0 \leq \lambda(5-W)+(1-\lambda)(2-W) = 2 + 3\lambda - W$. For f to be best, we need $\sigma_H W - 2 \geq \sigma_h$. Since WPBE places no restriction on the belief when the worker has no degree, WPBE can include $\sigma_h = 0$ and $\sigma_H = 1$, so the restrictions on the exogenous variables are $2 \leq W \leq 2 + 3\lambda$. Again, there is a limited range of possible wages $W$. The lower bound on $W$ is higher than in part d, but the upper bound could be higher or lower than in d.

f. In WPBE, the firm believes a worker without a degree is of high ability with probability $\lambda > 0$, so chooses $\sigma_h = 0$. For S to be best, we need $\sigma_h = 1 \geq \sigma_H W - 1$, which holds if $\sigma_H = 0$. This is best for the firm if $W \geq 2$ and the firm believes a worker with a degree is of low ability (a belief compatible with WPBE). If $W < 2$, then $\sigma_H = 1$ in WPBE and we need $1 \geq W - 1$ or $W \leq 2$. Thus, for any values of $\lambda$ and $W$ there are WPBE in which, for sure, the worker does not finish the degree.

g. A natural measure of efficiency is the total surplus, which equals total revenue minus cost of finishing the degree. The wage payment is simply a transfer from firm to worker and does not affect the total surplus, though it might affect the equilibrium allocation. It is inefficient for the high ability type of worker not to finish her degree, since the cost is less than the gain in revenue. For a low ability worker, the cost is the same as the gain in revenue, so it does not matter if that type worker finishes the degree. We conclude
that the equilibria in parts d and e are efficient and those in f are not. Efficient equilibria require \( W \in [1, \max \{3, 2 + 3\lambda\}] \).