Instructions: Answer any three of the four numbered problems below. Justify your answers whenever possible. Write your answers to separate problems in separate collated answer sheets. Write the number of the problem you are answering AND NOTHING ELSE on the cover sheet. You may not use any electronic devices. The exam lasts 4 hours.

1. Consider an island economy with a single agent who is endowed with one unit of time all or part of which can be supplied as labor (L) to produce food (y) or consumed as leisure (ℓ). Food is produced according to the technology \( y = f(L) \). However, production also generates pollution as a byproduct which adversely affects the welfare of the agent. Overall, the agent’s utility is given by \( u(y, \ell) = y\ell - c(y) \), where \( c(y) \) denotes the welfare cost of producing \( y \).

   a. What is an allocation in this economy? What quantities must be determined? Identify the feasible allocations. Be precise.
   
   b. Set up the choice problem facing the agent, assuming that the agent can choose any feasible input and output combination.

   For the remainder of the problem, suppose \( f(L) = L \) and \( c(y) = \frac{1}{4}y \).

   c. Solve the agent’s choice problem in this case.
   
   d. Argue that the solution in part c is Pareto efficient.
   
   e. Explain the meaning of the Second Welfare Theorem in this context. Does the theorem "hold" here? That is, does the conclusion follow under these circumstances?

   Next, suppose instead there are two identical individuals each with the tastes and endowment as described above. Denote agent \( i \)'s quantities as \( L_i, \ell_i \) and \( y_i \), for \( i = 1, 2 \). Then total output is \( y = f(L_1 + L_2) = y_1 + y_2 \), total (pollution) costs are \( c(y) \), and \( u_i(y_i, \ell_i) = y_i\ell_i - c(y) \).

   f. Set up the Social Planner’s Problem for determining the Pareto efficient allocations.
   
   g. If each individual were to contribute the same amount of labor and receive the same amount of food as in part c, would the outcome be Pareto efficient? Explain.
   
   h. Instead, suppose that each agent decides how to allocate its time separately, taking the other agent’s decision as given, and assuming it will consume what its labor produces. Argue that in this case, the outcome would be the same as in part c and explain why.
   
   i. Discuss the proposal to partition the island into two halves and have each of the individuals engage in production separately (incurring only the welfare cost associated with their own production), rather than produce jointly. Evaluate this proposal.

2. Consider two duopolists, 1 and 2, who produce a homogeneous product. Their cost functions are given, respectively, by \( c_1(y_1) = y_1^2 \) and \( c_2(y_2) = y_2^2 + ky_2 \), for some \( k > 0 \). The firms act as Cournot competitors facing uncertain (future) market demand described by the inverse demand function \( p(y, \alpha) \), where \( y \) is aggregate output and \( \alpha \) is a random variable distributed on the interval \([\alpha_l, \alpha_u]\) with known density \( f(\alpha) \). The firms must choose their outputs prior to the realization of market demand (\( \alpha \)).
a. In light of the above, set up the decision problem facing firm \( i \) assuming each firm seeks to maximize its expected profit taking the other firm’s output as given. (Do not solve the problem.)

b. For the special case in which \( p(y, \alpha) = -y + \alpha \), solve for the Cournot equilibrium as a function of the expected value of \( \alpha \), denoted \( \tilde{\alpha} \).

c. Argue that if \( \tilde{\alpha} > 3k \), then firm 1’s revenue will be greater and it’s costs less than firm 2’s in Cournot equilibrium, and hence 1’s profits will be greater.

d. In light of part c, suppose that the government decides to subsidize firm 2 in the event of a loss in order to ensure that there remain at least two competitors in the industry. This works as follows. As before, the firms first engage in Cournot competition, choosing their output levels to maximize their expected profits. Market demand is then realized. If the market price is below firm 2’s break-even price, denoted \( p(y_2) \), the government would raise the price for firm 2 (only) so that it would be able to break even. Assuming firm 2 knows it will be subsidized in the event of a loss, explain how the government intervention would affect its decision problem.

e. Discuss the effects of \( \tilde{\alpha} \) and \( k \) on the likelihood of the need for government subsidization.

f. Next, suppose that even when subsidies are not necessary (i.e., when firm 2 would be profitable otherwise), the government decides to provide a price subsidy to firm 2 since firm 1 has a cost advantage. In this case, firm 1 would receive the market price \( p(y, \alpha) \), but firm 2 would receive \( p(y, \alpha) + \varepsilon \) per unit. Prove that this would always have the desired effect of increasing firm 2’s market share relative to firm 1’s.

3. Apple’s market research shows that there are two types of consumers for its iMac computer and bMac computer (baby iMac). The bMac is physically the same as the iMac but has some functions disabled. Both iMac and bMac cost 300 dollars to produce per unit. Both consumer types, \( h \) and \( l \), get utility \( m \) if they have \( m \) dollars and no computer: \( u_h(0, m) = u_l(0, m) = m \). High type consumers with \( m \) dollars (after paying for a computer) get utility \( u_h(i, m) = 1500 + m \) and \( u_h(b, m) = 800 + m \), where \( i \) denotes having one iMac and \( b \) having one bMac. Low types get utility \( u_l(i, m) = 600 + m \) and \( u_l(b, m) = 500 + m \). Apple knows that there are one million consumers of high type and 2 million consumers of low type. Once Apple announces a price, it must serve all consumers who demand its product at that price. The consumers start with enough money to buy a computer if it raises their utility. They demand at most one computer during the period considered.

a. Suppose Apple sells iMacs at 1500/unit and bMacs at 500/unit. Find the maximum sales and profits Apple could get.

b. Suppose Apple offers only iMacs. Find profit maximizing price, sales and profit.

c. Suppose Apple offers only bMacs. Find profit maximizing price, sales and profit.

d. Suppose Apple offers iMacs at price \( p_h \) and bMacs at price \( p_l \). Show that no prices are such that high types buy bMacs and low types buy iMacs.

e. Find profit maximizing prices \( p_h \) and \( p_l \) that can induce high types to buy the iMac and low types to buy the bMac. What is the corresponding profit?

f. Do part e assuming instead that the number of low types is 1 million. In this case, find profit maximizing prices and compare the resulting profit to what Apple receives if it offers only the iMac at a profit maximizing price.
A worker has to decide whether to finish her degree or not before applying for a job with a particular firm. The firm knows that the worker is of high or low ability and will know if the worker finishes her degree. Finishing the degree costs the worker $1$ if she is of high ability and $2$ otherwise. If the worker is of high ability, then she will produce $5$ units of revenue for the firm if she finishes her degree and $3$ units if she does not. If the worker is of low ability she will produce $2$ units of revenue for the firm if she finishes her degree and $1$ unit if she does not. The firm pays each of its workers with a degree $W$ ($0 < W < 5$) and pays each of its workers without a degree $1$. Initially, before the worker decides about her degree, the firm believes with probability $\lambda \in (0, 1)$ that she is of high ability and decides whether to hire her or not. The interaction between the worker and firm is represented, as a game of common knowledge, in the tree in Figure 1, below, where the worker is player $[1]$ with the first listed payoff, and the firm is player $[2]$ with the second listed payoff.

a. Why might it make sense to model a worker’s cost of finishing a degree as depending on her ability even when there is no discrimination among students in pricing of education?

b. Label the players’ moves in the tree in Figure 1. Refer to your labels in your answers to the remaining parts of this problem.

c. Under what (if any) conditions is there a Nash equilibrium (NE) in which the worker finishes her degree for sure if she has low ability and does not if she has high ability?

d. Under what (if any) conditions on the exogenous variables $\lambda$ and $W$ is there a NE in which for sure the worker does not finish her degree if she has low ability and for sure she finishes it if she has high ability? Interpret the conditions if there are any.

e. Find and interpret all conditions on the exogenous variables under which there is a weak perfect Bayesian equilibrium (WPBE, i.e., weak sequential equilibrium) where, for sure, the worker finishes her degree no matter what her ability is.

f. Find and interpret all conditions on the exogenous variables under which there is a weak perfect Bayesian equilibrium (WPBE, i.e., weak sequential equilibrium) where, for sure, the worker does not finish her degree no matter what her ability is.

g. What can be said about the efficiency of the outcomes in the equilibria in parts c, d, and e? Define an appropriate notion of efficiency for the economy consisting of the worker and firm. How does your answer depend on the values of $\lambda$ and $W$?