1. In practice, firms often price their products by “marking up” a fixed percentage over (average) cost. To investigate the consequences of markup pricing, consider a single firm that faces the demand \( Q = 90 - P \), for \( P \leq 90 \). The firm’s **TOTAL** cost function is \( C(Q) = 20Q \).

a. If the firm marks its prices up 50% over average cost, how much would it produce? What price would it charge? And what would be its profits?

b. In contrast, determine the firm’s profit maximizing price and output decisions and its maximal profits.

c. Given that the firm can make more money by behaving as in part b rather than as in part a, give one reason why it could choose a markup price.

d. In this example, would a 50\% markup lead to a more or less efficient outcome than the profit maximizing rule in part b? How do you define efficiency? Explain.

e. Is there a general relationship between markup pricing and market efficiency when production exhibits constant returns to scale?

f. Is there a general relationship between markup pricing and market efficiency under decreasing returns to scale?

g. For the firm in part a, derive conditions under which an increase in average cost would increase profits.

h. For a profit maximizing firm with the cost curve \( C(Q) = aQ^2 \), is it possible that an increase in \( a \) (hence, marginal cost) would increase profits? Explain.

i. Is it possible that either a profit maximizing firm or a firm using markup pricing would choose not to operate in this market?

2. First, consider an economy with one consumer and two commodities, labor \((L)\) and food \((y)\). Food can be produced from labor according to the production function \( y = L(4 - L) \). The consumer is endowed with 4 units of labor and has preferences represented by \( u(y, L) = y(4 - L) \).

a. Characterize the set of Pareto efficient allocations for this economy.

b. Suppose the consumer owns the technology but it is operated independently as a profit maximizing enterprise. Compute a competitive equilibrium, taking food as numeraire.

Next, suppose there are two consumers, \( A \) and \( B \), each of whom is endowed with 4 units of labor and has the utility function \( u_i(y_i, L_i) = y_i(4 - L_i) \), where \( y_i \) and \( L_i \) denote the quantities of food and labor of agent \( i \).

c. Characterize the interior Pareto efficient allocations in this case. How do they differ from each other?

d. Is it possible that at an efficient allocation, an agent might work \((L_i > 0)\) but consume no food \((y_i = 0)\)? Explain.

e. Suppose that here, too, resources were allocated via the Walrasian mechanism with each consumer owning half of the firm. Prove that the equilibrium wage would be lower than in the equilibrium in part b. Explain why it is.

f. Under these circumstances, could there be an asymmetric Walrasian equilibrium in which the consumers get different consumption bundles? Explain.

g. Explain what is meant by the statement, “the decreasing portion of the production function is economically irrelevant.”

h. Suppose that instead of \( L = L_1 + L_2 \) it were the case that \( L = L_1 + \alpha L_2 \), for \( 0 < \alpha < 1 \).
Discuss as thoroughly as possible how this would affect the analysis and properties of a competitive equilibrium.

3. A single monopolistic seller sells products to a continuum of buyers of size 1. Buyers are of one of two types $i = 1, 2$. Every type 2 buyer receives utility equal to

$$16q - \frac{q^2}{2} - t$$

if he purchases $q$ units of the good for a total payment of $t$ dollars. Every type 1 buyer receives utility equal to

$$12q - \frac{q^2}{2} - t$$

if he purchases $q$ units of the good for a total payment of $t$ dollars. It costs the seller a constant average cost of $4$ per unit to produce the good. Let $n_i$ be the fraction of buyers of type $i$, so $n_1 + n_2 = 1$. Each buyer receives a reservation utility equal to zero if the buyer does not purchase anything from the monopolistic seller.

a. Suppose the seller can observe each buyer’s type and can force each type $i$ to choose between a contract $(q_i, t_i)$ or else buying nothing. What is the seller’s profit-maximizing (i.e. optimal) contract $(q_i, t_i)$ for each type $i$?

For the rest of the problem, suppose that buyer types are not observable to the monopolist. The monopolist can offer a menu of two contracts $\{(q_1, t_1), (q_2, t_2)\}$ to buyers. If a buyer selects contract $(q_i, t_i)$ then the buyer is entitled to receive $q_i$ units of the good by paying the amount $t_i$ to the seller regardless of the buyer’s true type. For parts b through d, assume $n_1 = n_2 = 1/2$.

b. Formulate the monopolist’s profit-maximizing problem: write down the objective function and all the constraints including incentive constraints (ICs) and individual rationality constraints (IRs) for both types. Call this the original problem (OP).

c. Consider the seller’s problem but without the type 2 buyer’s IR constraint (IR2) and without the type 1 buyer’s IC constraint (IC1). Call this the relaxed problem (RP). Find the optimal set of contracts $(q_i, t_i)$ that maximize the seller’s profits in this relaxed problem.

d. Show that the solution in part c also satisfies the additional constraints, IR2, IC1, in the original problem. Argue that based on this result the optimal contracts found in RP in fact are also optimal for the original problem OP.

e. Explain how the distribution of the two types $(n_1, n_2)$ affects the optimal contracts.

4. Two producers can grow food for a consumer who cares only about food and money. Producer $j$ ($j = 1, 2$) can plant $q_j > 0$ units of seed at a cost of $(1/3) + q_j$ (in units of money) and then produce $q_j$ units of food from the seed at no additional cost. If a producer plants no seed, it has no cost. The agents’ interaction can be described as a game in which the producers independently plant $q_j \geq 0$ units and pay any cost of planting. Next, they independently choose food prices $p_j \geq 0$ (in terms of money) and then the consumer chooses to buy $c_j \in [0, q_j]$ units of food from each producer $j$. The payoff of producer $j$ is its revenue $p_j c_j$ minus its cost. (It can borrow to pay for seed at no interest and repay its loan from its revenue.) The consumer’s payoff is $3c - (1/2)c^2 - p_1 c_1 - p_2 c_2$, where $c = c_1 + c_2$. All this information is common knowledge.
a. How many pure strategies does the consumer have? Give an example of one.
b. Show that in the consumer’s best response to \((q_1, q_2, p_1, p_2)\), if \(p_1 < p_2\) and \(q_1 + p_1 \leq 3\), then \(c_1 = q_1\).

In parts c and d, consider only subgames that follow the producers’ choices of amounts to plant: \(q_1 > 0\) and \(q_2 > 0\).

c. Show that there is no pure subgame perfect equilibrium (SPE) of this subgame in which \(p_1 < p_2\) and \(c_2 > 0\). Hint: Use part b and consider deviation by producer 1.
d. Use parts b and c to show that this subgame has no pure SPE in which \(0 < p_1 < p_2\). Hint: Consider deviation by producer 2.

In the remaining parts of the problem, consider the whole game.

e. The whole game has a pure subgame perfect equilibrium (SPE) in which the quantities \(q_j\) are quantities the producers would choose if they acted as Cournot duopolists selling to the consumer. Find the outcome prices and levels of \(q_j\) and \(c_j\) and explain why they can arise in SPE.
f. Find a SPE of the whole game in which \(q_2 = 0\). Explain why it is SPE and compare the total surplus (the sum of the three agents’ payoffs) to that in the SPE of part e.

**Answers:**

1a. \(P_{MU} = 30, Q_{MU} = 60, \pi_{MU} = 600\).

b. Profit maximizing price, output and profit:

Setting marginal revenue equal to marginal cost, \(90 - 2Q = 20\) yields \(Q_{\text{max}} = 35\), \(P_{\text{max}} = 55\), \(\pi_{\text{max}} = 1225\)

c. If the firm did not know the demand it was facing, then it could not compute the profit maximizing supply.

d. If we consider the welfare of the buyers, the markup yields a more efficient outcome than profit maximization. The efficient quantity occurs at \(P = MC\) or \(P_{eff} = 20\) and \(Q_{eff} = 70\).

e. Raising the markup slightly above 0, raises the profit of the sellers, but reduces the social efficiency of the allocation (measured by total surplus, including the buyers’ surplus). Raising the markup above the profit maximizing level (in the example of part b, so that \(P = MU \cdot 20 > 55\), or \(MU > 2.75\)) reduces both profit and total surplus.

f. There is no general relationship between markup pricing and market efficiency under decreasing returns to scale. If the markup is sufficiently high, markup pricing will be less efficient. If it is lower, it will be more efficient than profit maximization. However, it is possible that for a very low markup (significantly below \(MC - AC\)), the outcome is less efficient than with profit maximization.

g. In part a, an increase in average cost could increase profits. Writing profits as a function of average cost (\(AC\)), where \(P = 1.5AC\), yields:

\[\pi = 1.5AC(90 - 1.5AC) - AC(90 - 1.5AC) = 45AC - 0.75AC^2.\]

Therefore, \(\frac{d\pi}{dAC} > 0\) iff \(AC < 30\).

h. For a profit maximizing firm with the cost curve \(C(Q) = aQ^2\), an increase in \(a\) cannot increase profit. Equating \(MR = MC\) yields \(Q = \frac{45}{1+a}\) and hence \(P = \frac{45(1+2a)}{1+a}\). Writing \(\pi\) as a function of \(a\), \(\pi = \frac{2025}{1+a}\). Hence, profits are decreasing in \(a\).

i. If the costs are prohibitively high, neither a profit maximizing firm nor a firm using markup pricing would choose to operate in this market. But as long as \(P > AC\) at some output level, profits are positive for both kinds of behavior.
2. a. A Pareto efficient (PE) allocation for this economy solves
\[ \max_{y,L} u(y, L) = y(4 - L) \text{ subject to } y = L(4 - L), \text{ so } L_e = \frac{4}{3}, y_e = \frac{32}{9}. \]
b. In a competitive equilibrium, with food as numeraire, the allocation is as in part a (by the first welfare theorem). The equilibrium wage is \( w = 4/3 \), as required by the foc for profit maximization at the equilibrium allocation.
c. If there are two consumers, as described, PE allocations solve
\[ \max_{y_1, L_1, y_2, L_2} y_1(4 - L_1) \text{ s.t. } y_2(4 - L_2) \geq \bar{u}_2 \text{ and } y_1 + y_2 \leq (L_1 + L_2)(4 - L_1 - L_2). \]
At an interior solution, \( y_2(4 - L_2) = \bar{u}_2 \),
\[ y_1 = (4 - L_1)(4 - 2L_1 - 2L_2) \]
\[ y_2 = (4 - L_2)(4 - 2L_1 - 2L_2) \]
\[ y_1 + y_2 = (L_1 + L_2)(4 - L_1 - L_2). \]
The different PE allocations differ from each other in the distribution of utilities of the two consumers. When \( \bar{u}_2 \) is higher, consumer 2 gets higher utility and consumer 1 lower utility. Dividing the equation for \( y_1 \) by the one for \( y_2 \) and using \( \bar{u}_i = y_i(4 - L_i) \), we see that when \( \bar{u}_2 \) is higher, the ratios \( (y_2/y_1)^2 \) and \( y_2/y_1 \) are higher.
d. It is possible that at a Pareto efficient allocation, an agent might work (\( L_i > 0 \)) but consume no food (\( y_i = 0 \)). For example, agent \( j \)'s most preferred feasible allocation would be \( y_j = 4, y_i = 0, L_j = 0, L_i = 2 \). Any deviation from this will make \( j \) worse off.
e. The consumers are identical, so in (Walrasian) competitive equilibrium, they have equal wealth. This implies that their utilities are equal. Since their utility functions are strictly quasiconcave, their demand functions are single valued, so their consumption vectors are equal. Since the allocation is PE, the foc's in part c imply \( L \equiv L_1 = L_2, y \equiv y_1 = y_2 = 4(4 - L)(1 - L), y = 2L(2 - L) \), so \( L \) solves the quadratic equation
\[ 2(4 - L)(1 - L) = L(2 - L), \]
which implies that \( L < 1 \) and that 0 equals \( 3L^2 - 12L + 8 \). Since this last expression is positive at \( L = 2/3 \) and negative at \( L = 1 \), the solution is in \((2/3, 1)\) and total labor supply is \( 2L > 4/3 \). Thus the wage is less than in part a. The consumers get less disutility of labor, each supplying less than \( 4/3 \) units, therefore the equilibrium wage is lower and demand for labor is higher.
f. There cannot be an asymmetric Walrasian equilibrium, as explained in the answer to e.
g. The decreasing portion of the production function is economically irrelevant unless the allocation of resources is very inefficient. As long as the firm seeks profit and the price of its input is nonnegative, it will not operate where the marginal product of its input is negative.
h. If \( L = L_1 + \alpha L_2 \), for \( 0 < \alpha < 1 \), then the different consumers' labor inputs have different productivity. In competitive equilibrium with perfect information about the productivities, the consumers would receive different wages. However, because the inputs are perfectly substitutable in \( \alpha \) to 1 ratio, the equilibrium wages must be such that \( w_2 = \alpha w_1 \) in any equilibrium in which both consumers supply labor.

3.a. If the seller can observe each buyer’s type and can force each type \( i \) to choose between a contract \((q_i, t_i)\) or no contract, then in the profit maximizing contract for
type 2, $q^e_2$ solves

$$\max_q 16q - \frac{q^2}{2} - 4q$$

Result: $q^e_2 = 12$ and then $t^e_2 = 16q^e_2 - \frac{(q^e_2)^2}{2} = 120$. In the optimal contract for type 1, $q^e_1$ solves

$$\max_q 12q - \frac{q^2}{2} - 4q$$

Result: $q^e_1 = 8$ and then $t^e_1 = 12q^e_1 - \frac{(q^e_1)^2}{2} = 64$.

b. When the monopolist cannot observe the buyer’s type and $n_1 = n_2 = 1/2$, optimal contracts solve

$$\max_{(q_i, t_i)} 0.5(t_2 - 4q_2) + 0.5(t_1 - 4q_1)$$

subject to:

$$16q_2 - 0.5(q_2)^2 - t_2 \geq 16q_1 - 0.5(q_1)^2 - t_1 \quad (IC_2)$$

$$12q_1 - 0.5(q_1)^2 - t_1 \geq 12q_2 - 0.5(q_2)^2 - t_2 \quad (IC_1)$$

$$16q_2 - 0.5(q_2)^2 - t_2 \geq 0 \quad (IR_2)$$

$$12q_1 - 0.5(q_1)^2 - t_1 \geq 0 \quad (IR_1) \quad \square$$

where $(IR_1)$ and $(IR_2)$ are individual rationality or participation constraints.

c. The relaxed problem $(RP)$ is

$$(RP): \max_{(q_i, t_i)} 0.5(t_2 - 4q_2) + 0.5(t_1 - 4q_1)$$

subject to:

$$16q_2 - 0.5(q_2)^2 - t_2 \geq 16q_1 - 0.5(q_1)^2 - t_1 \quad (IC_2)$$

$$12q_1 - 0.5(q_1)^2 - t_1 \geq 0 \quad (IR_1)$$

Both constraints are binding: If IR1 is not binding then reduce $t_1$ should increase profits; if IC2 is not binding then reduce $t_2$ should increase profits.

Use the two binding constraints to get $t_1$ and $t_2$:

$$t_1 = 12q_1 - 0.5(q_1)^2$$

$$t_2 = 16q_2 - 0.5(q_2)^2 - 4q_1$$

and plug them in the objective function to get

$$\max_{q_1, q_2} 0.5\{12q_2 - 0.5(q_2)^2 + 4q_1 - 0.5(q_1)^2\} \quad (0.1)$$
The answer is $q_1 = 4, q_2 = 12$. Then $t_1 = 40, t_2 = 104$. \(\square\)

d. The solution in part c satisfies the additional constraints, IR2, IC1, in the original problem.
IR2: $16q_2 - 0.5(q_2)^2 - t_2 = 120 - 104 = 16 > 0$, satisfied.
IC1: $12q_1 - 0.5(q_1)^2 - t_1 \geq 12q_2 - 0.5(q_2)^2 - t_2$
\[\iff 0 \geq 12 \times 12 - 0.5(12)^2 - 104 = -32, \text{ satisfied.}\]

Since every solution to the problem in part b must satisfy the constraints in part c, the solution to (RP) in part c is a solution to the original problem in part b.

e. If $(n_1, n_2)$ changes, there is no change in the solution to (RP) until we reach Eq. (0.1). We keep $n_1, n_2$ before we plug in $t_1, t_2$ to get:

\[
\max_{q_1, q_2} n_2 \{16q_2 - 0.5(q_2)^2 - 4q_1 - 4q_2\} + n_1 \{12q_1 - 0.5(q_1)^2 - 4q_1\}
\]
\[
= n_2 \{16q_2 - 0.5(q_2)^2 - 4q_2\} + (8n_1 - 4n_2)q_1 - 0.5n_1(q_1)^2
\]
\[
= n_2 \{16q_2 - 0.5(q_2)^2 - 4q_2\} + (12n_1 - 4)q_1 - 0.5n_1(q_1)^2
\]

Thus, $q_2 = 12$ remains unchanged. As for $q_1$, it is positive only if $12n_1 - 4 > 0$ or $n_1 > 1/3$, in which case $q_1 = (12n_1 - 4)/n_1$. One can see that $q_1$ increases as $n_1$ increases: the seller offers more quantity to type 1 when it is a bigger fraction of the population. Otherwise, $q_1 = 0$. Payments $t_1, t_2$ are calculated using the previous formulas.

4a. A pure strategy for the consumer is a function that assigns $c = (c_1, c_2) \leq q = (q_1, q_2)$ to each $(q, p) = (q_1, q_2, p_1, p_2) \geq 0$. There are infinitely many such functions, so the consumer has infinitely many pure strategies. An example is $c = q$ for each $(q, p) \geq 0$.

b. In a best response, the consumer maximizes $3(c_1 + c_2) - (1/2)(c_1 + c_2)^2 - p_1c_1 - p_2c_2$ subject to $0 \leq c_j \leq q_j$. Let $\lambda_2$ be the Lagrange multiplier of the constraint $c_2 \leq q_2$. Suppose that $p_1 < p_2$ and that there is a solution with $c_1 < q_1$. The first order conditions are $3 - (c_1 + c_2) - p_1 \leq 0$ and $3 - (c_1 + c_2) - p_2 - \lambda_2 \leq 0$, with equality if $c_2 > 0$. Therefore $3 - c_1 - c_2 \leq p_1 < p_2 + \lambda_2$, so $c_2 = 0$ and $3 \leq c_1 + p_1 < q_1 + p_1$. Thus, if $q_1 + p_1 \leq 3$, then $c_1 = q_1$.

c. The answer to b shows that if in pure SPE $p_1 < p_2$ and $c_2 > 0$, then $c_1 = q_1$. If producer 1 raises $p_1$ slightly, the consumers’ best response remains $c_2 > 0$ and $c_1 = q_1$, so producer 1 raises its revenue with no change in cost.

d. By c, $c_2 = 0$ in pure SPE with $0 < p_1 < p_2$. If $c_1 = 0$, then, by the answer to b, $p_1 \geq 3$ and reducing $p_1$ raises producer 1’s revenue. If $c_1 > 0$, then producer 2 can charge $p_2 \in (0, p_1)$ and raise its revenue above 0.

e. Parts c and d rule out SPE with $q_1 > 0, q_2 > 0$ and unequal prices. Let $p_1 = p_2 = p$ and consider the Cournot duopoly outcome. The consumer demands a total of $3 - p$ units if that amount is produced. There is no wasted output in SPE. Otherwise a producer could raise its payoff by producing slightly less. (The SPE prices and $c_1$ and $c_2$ stay the same.) Therefore, in SPE, $p = 3 - q_1 - q_2$. Producer 1 maximizes $(3 - q_1 - q_2)q_1 - q_1$. The first order condition is $2 - q_2 - 2q_1 = 0$ and, since the producers are identical, $q_1 = q_2 = 2/3$, $p = 3 - (4/3) = 5/3$. Each producer’s payoff is $[(5/3) - 1](2/3) - (1/3) = 1/9$, so they do choose to produce. Note that this is the Cournot duopoly outcome.

This is a SPE outcome in the subgame after the $q_j$’s are chosen. [Proof: lowering a price reduces revenue when outputs are fixed. Raising a price, say $p_1$, yields revenue $p_1(3 - q_2 - p_1)$, with derivative $3 - q_2 - 2p_1 \leq 3 - (2/3) - (10/3) < 0$.] In every SPE of
a subgame following choices of positive $q_j$’s, the producers choose the same price $p$ and sell a total of $3 - p$ units when $p \leq 3$, so the duopoly solution characterizes SPE choices of $q_j$’s in the whole game.

f. If $q_2 = 0$ in SPE, then producer 1 chooses $q_1$ to maximize $(3 - q_1)q_1 - q_1$ at $q_1 = 1$, with $p_1 = 2$ and $c_1 = 3 - p_1 = 1$ and positive payoff $1 - (1/3) = 2/3$. This is SPE in the subgame following choices of $q_j$. In the whole game, if producer 2 chooses $q_2 > 0$, the price in the remaining subgame is $3 - 1 - q_2$ and the producer 2 gets payoff $(1 - q_2)q_2 - (1/3)$, which is negative at its maximizer $q_2 = 1/2$. So producer 2 has no profitable deviation. Total surplus is $(2/3) + 3 - (1/2) - 2 = 7/6$ in the monopoly case of part f, higher than the duopoly surplus of $(2/9) + 3 \cdot (4/3) - (1/2)(16/9) - (5/3)(4/3) = 1/9$. 