Instructions: Answer any three of the four numbered problems. Justify your answers whenever possible. Write your answer to each question in a separate bluebook. Write the number of the question AND NOTHING ELSE on the cover of the bluebook. No electronic devices may be used. The exam lasts 4 hours.

1. Consider a large conglomerate or multinational firm with \( n \) different divisions. Each division produces a different output using two inputs, one which is specific to that division and another, called “administrative services” (AS), which is used in every division. The technology of division \( i \) is given by \( y_i = \ln x_i + \ln a_i \), where \( x_i \) is the idiosyncratic input and \( a_i \) is AS. Division \( i \) purchases \( x_i \) in a competitive (input) market at the price \( w_i \) and sells \( y_i \) in a competitive output market at the price \( p_i \). The firm is deciding between two organizational structures: (1) it can produce AS “in house,” i.e., within the conglomerate, or (2) it can “outsource” AS and purchase it from outside vendors. In the event of the former, the marginal cost of providing the aggregate quantity \( a = \sum_{i=1}^{n} a_i \) within the firm is given by \( MC(a) = a \). Alternatively, the price of purchasing AS from an outside vendor is \( \tau \).

A problem with producing AS internally is to determine how much each division should receive. One option is to create an internal market in which the central administration announces a transfer price, \( t \), and promises to supply any amount of AS at this price. Each division can then purchase as much AS as it wishes at the price \( t \).

a. Under this scheme, determine the price the central administration should charge for AS and the quantity each division will purchase.

b. How is the optimal transfer price affected by a change in the price of \( y_i \)? Why? How would this affect the profits of division \( i \)? Division \( j \)? Overall profits, which consist of the sum of the profits of the separate divisions along with any profit or loss associated with the production and distribution of AS?

c. Argue that under this scheme, allowing each division to behave independently, maximizing its own profits, would also lead to maximum overall profits.

d. Evaluate the following statement, i.e., state whether it is True, False or Uncertain and explain your answer.

If provision of AS is internal to the firm, then any amount paid by a production division is received by the central administrative office. Since this is simply a transfer, it will not affect overall profits. Therefore, it does not matter what price the central office charges.

e. Suppose the outside vendor price \( \tau \) is less than \( t \). Referring to your answer to part b, explain why it would be beneficial for the conglomerate to outsource AS, that is, to buy it from the outside vendor rather than to produce it internally.

f. From the above, explain when the conglomerate should produce AS internally and when should it outsource.

g. Next, suppose that instead of producing different products which are sold separately, each division produces a component of a single commodity. The components are then assembled at a central production facility according to the production function \( Y = F(y_1, ..., y_n) \), and \( Y \) is sold at the market determined price \( P \). As in the previous
case, suppose each division manager behaves autonomously and can “sell” its output (internally) to the central facility at marginal cost. Will this lead to maximum profits overall? Explain.

h. Returning to the case in which the divisions produce and sell different products, suppose instead that AS is nonrivalrous, that is, the use by division \( i \) does not lessen the amount available for division \( j \). Therefore, all divisions use the entire quantity of \( a \) available. Again, assuming that \( a \) is produced internally, set up the conglomerate firm’s decision problem for determining the quantity of \( a \) to produce and characterize a solution.

i. In this case, would it suffice to decentralize the production decisions to the divisions in order to maximize overall firm profits, that is, to allow the divisions to behave independently, each purchasing (some) units of \( a \)? Explain.

2. An auctioneer auctions off one indivisible object to more than one (potential) bidders. The bidders’ valuations are distributed independently as follows. Each bidder has valuation 1 with probability \( \frac{1}{2} \) and valuation 2 with probability \( \frac{1}{2} \). Each bidder knows his valuation only, but the prior distribution of valuations is common knowledge. The auctioneer uses either a first price sealed bid auction or a second price sealed bid auction but restricts the allowable bids to \( \{b_1, b_2\} \). In either type of auction, the highest bidder wins the object if there is a single highest bidder. If there are more than one highest bidders, each highest bidder has an equal probability of winning the object. In a first price auction, the winner pays the highest bid. In a second price auction, the winner pays the highest bid among the rest of bids after excluding the winner’s bid. Those who do not win neither pay nor receive anything.

a. Suppose there are two bidders. Find the set of nonnegative \( \{b_1, b_2\} \) that form a symmetric and strictly increasing Bayesian Nash equilibrium in which both bidders participate regardless of their types (valuations). At a symmetric and strictly increasing Bayesian Nash equilibrium, a type 1 bidder (a bidder with valuation 1) bids \( b_1 \) and a type 2 bidder bids \( b_2 \), where \( b_2 > b_1 \). A bidder (after knowing his valuation) participates in the auction if and only if his expected payoff is nonnegative.

b. In part a, what would be the expected revenue maximizing choice of \( b_1 \) and \( b_2 \) for the auctioneer? Set up the problem and solve it. What is the maximum expected revenue?

c. Assume there are two bidders and suppose the auctioneer conducts a second price auction. What is the set of nonnegative \( \{b_1, b_2\} \) that form a symmetric and strictly increasing dominant strategy equilibrium under this auction and induce participation of both bidders? Assume that a bidder participates if and only if he gets a nonnegative payoff regardless of the other’s bid when he follows an equilibrium strategy.

d. In part c, what is the expected revenue maximizing choice of \( b_1 \) and \( b_2 \) for the auctioneer? Set up the problem and solve it. What is the maximum expected revenue?

e. Suppose in parts a and b that there are three bidders with independent and identical prior distributions of types. The prior distributions are the same as the one given above. Solve parts a and b with this change.

f. Suppose in parts c and d there are three bidders with the given independent and identical prior distributions of types. Solve parts c and d with this change.

g. Compare your answers to the different parts above and interpret the results.
3. Consider the class $\mathcal{E}$ of private ownership economies with $L$ goods, where each firm $j = 1, \ldots, J$, has a compact production set $Y_j \subset \mathbb{R}^L$ containing $0 \in \mathbb{R}^L$ and each consumer $i = 1, \ldots, I$, has a continuous, quasiconcave utility function $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ and an endowment vector $e_i \in \mathbb{R}^L_+$, $e_i \neq 0$, and owns the share $\theta_{ij} \geq 0$ of firm $j$, with $\sum_i \theta_{ij} = 1, \forall j$. In economies in $\mathcal{E}$, an allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is feasible if and only if $x_i \in \mathbb{R}^L_+$, $y_j \in Y_j$, $\forall i, j$, and $\sum_i (x_i - e_i) = \sum_j y_j$.

a. Use the notation above to give a formal definition of a competitive (Walrasian) equilibrium of an economy in $\mathcal{E}$.

b. Prove that in a competitive equilibrium of an economy in $\mathcal{E}$, every consumer $i$ spends its entire wealth.

c. Define formally additional restrictions on the exogenously given fundamentals of an economy in $\mathcal{E}$ (i.e., on the $Y_j$, $u_i$, $e_i$, $\theta_{ij}$) that imply that every competitive equilibrium allocation of the economy is Pareto optimal (Pareto efficient). Try to make the restrictions you state as weak as possible.

d. Define formally additional restrictions on the exogenously given fundamentals of an economy in $\mathcal{E}$ that imply that a competitive equilibrium exists. Try to make the restrictions as weak as possible.

e. Let $\hat{a} = (\hat{x}_1, \ldots, \hat{x}_I, \hat{y}_1, \ldots, \hat{y}_J)$ be a Pareto optimal (Pareto efficient) allocation for an economy in $\mathcal{E}$. Suppose that there is a price equilibrium with transfers with price vector $p$ in which each firm $j$ chooses production vector $\hat{y}_j$ and each consumer $i$ chooses consumption vector $\bar{x}_i$ satisfying $p\bar{x}_i = p\hat{x}_i$. (In such a price equilibrium with transfers, each firm $j$'s choice $\hat{y}_j$ maximizes its profit at the price vector $p$, each consumer $i$'s choice $\bar{x}_i$ maximizes its utility over a budget set $\{x_i \in \mathbb{R}^L_+ : px_i \leq w_i\}$, and the resulting allocation is feasible.) Note that $(\bar{x}_1, \ldots, \bar{x}_I, \bar{y}_1, \ldots, \bar{y}_J)$ need not equal $\hat{a}$. Without assuming the additional restrictions in parts c and d, prove that the economy has a price equilibrium with transfers with price vector $p$ and allocation $\hat{a}$.

In parts f and g, allow prices to be unrestricted in competitive equilibrium. Each price can be positive, negative or 0.

f. Consumer $i$ is called completely satiated at $\bar{x}_i \in \mathbb{R}^L_+$ if the inequalities $x_i \geq \bar{x}_i$ and $x_i \neq \bar{x}_i$ imply $u_i(x_i) < u_i(\bar{x}_i)$. Describe a pure exchange economy in $\mathcal{E}$ (an economy with no firms) in which consumer 1 has monotone utility $(x_1 \geq z_1 \geq 0 \Rightarrow u_1(x_1) > u_1(z_1))$, consumer 2 is completely satiated at some $\bar{x}_2$, and there is no competitive equilibrium. Try to make your example as simple as possible. Show that one of the restrictions you listed in part d is violated in your example. Show that there is no competitive equilibrium in your example and explain briefly why that is the case.

g. If some consumer is completely satiated at some consumption vector in an economy in $\mathcal{E}$, does the economy necessarily have no competitive equilibrium? Justify your answer.
4. A monopolistic seller of a single product serves a market with a continuum of buyers of size one. Buyers are of two types. Ex ante it is known that a fraction \( \pi_i \) of the buyers is of type \( i = 1, 2 \). Type \( i \) buyer's utility function is given by \( u_i(x) = \theta_i \sqrt{x} - t \) where \( x \) is the quantity of the good consumed, \( t \) is the payment to the seller, and the parameters \( \theta_i \) satisfy \( 0 < \theta_1 < \theta_2 \). The seller's marginal cost of production is constant at \( c > 0 \), and fixed cost is zero. Type 2 buyer has a reservation utility equal to zero while type 1 has a reservation utility equal to \( v \). Except in part e, assume \( v = 0 \).

a. If the seller can observe each buyer's type and can price the product according to buyer's type, what is the seller's profit-maximizing (i.e. optimal) pricing strategy (i.e. identify the quantity to sell to each type \( i \) and the payment to charge for it)?

For the rest of the problem, suppose that buyer types are not observable to the monopolist. The monopolist can offer a menu of two contracts \( \{(x_1, t_1), (x_2, t_2)\} \) to buyers. If a buyer selects contract \( (x_i, t_i) \), then the buyer is entitled to receive \( x_i \) units of the good by paying the amount \( t_i \) to the seller.

b. Formulate the monopolist's profit-maximizing problem: write down the objective function and all the constraints, including incentive constraints (ICs) and individual rationality constraints (IRs) for both types. Call this the original problem.

c. Solve the seller's problem without type 2 buyer's IR constraint and type 1 buyer's IC constraint. Call this the relaxed problem.

d. Solve the original problem by arguing that the solution to the relaxed problem in c is also a solution to the original problem in b. This can be accomplished by showing that the solution in c satisfies the additional constraints in the original problem.

e. Suppose \( v > 0 \). Discuss how the monopolist's solution might be different from the case when \( v = 0 \).