Instructions: Answer any three of the four numbered problems. Justify your answers whenever possible. Write your answer to each question in a separate bluebook. Write the number of the question AND NOTHING ELSE on the cover of the bluebook. No electronic devices may be used. The exam lasts 4 hours.

1. a. Consider a firm that produces output $q$ and faces the demand function $q^d(p) = 100 - p$. The cost of producing $q$ is given by $c(q) = q^2$. Assuming the firm is able to determine both the price and quantity, determine its optimal decisions.

b. Now assume that the firm is able to advertise its product. If it advertises at the level $a$, it incurs the cost $\$a^2$ and demand for its product increases to $\tilde{q}^d(p) = 100 - p + 2a$. Determine its optimal price, quantity and advertising decisions.

c. Next, assume the firm can advertise as in part b. However, the effect of advertising is stochastic: with probability $\pi$ it will increase demand to $\tilde{q}^d(p) = 100 - p + 2a$ and with probability $(1 - \pi)$ demand will be unaffected, i.e., the same as in part a. Assume the firm must determine its output and advertising levels before knowing which demand function will occur, but it can adjust its price instantaneously, after its demand function is realized. Assuming the firm is risk neutral, determine its optimal output and advertising decisions conditional on $\pi$. Would it always choose to advertise for all $\pi \geq 0$? Show that its expected profits are increasing in $\pi$. Also, explain the complication that would arise if it had to commit to a price at the same time it determined its output.

d. One can interpret part c as follows: If consumers are exposed to the firm’s advertising, then they will increase their demand from $q^d$ to $\tilde{q}^d$. However, there is an exogenous probability of $\pi$ that they will be exposed. Now, assume that the more the firm advertises, the greater the likelihood its ads will be seen. Hence, the likelihood demand will increase is endogenous. Suppose, in particular, the likelihood is $\pi(a) = 1 - \frac{1}{1+a}$. In this case, set up the decision problem facing the firm -- you do not have to solve it.

e. Returning to the deterministic case, suppose now that the firm is one of two Cournot competitors which together face the demand in part a. Also, each has the cost function specified there. Determine the Cournot equilibrium output and price.

f. Again suppose that firms can advertise; if firm $i$ advertises at the level $a_i$, it incurs the cost $\$a_i^2$ and overall demand increases to $\tilde{Q}^d = 100 - p + 2(a_1 + a_2)$. Define a strategy for firm $i$ and determine a Cournot-Nash equilibrium. Compare the outcome here to that in part e.

g. Suppose the firms were able to coordinate their output and advertising decisions in order to maximize joint profits. Discuss the consequence and contrast this to your answer in part b.

h. Evaluate the following statement pertaining to the Cournot model with advertising:

Since advertising by your competitor increases demand for your product but does not entail a cost, there is an incentive for each firm to “under advertise”.

2. A cooperative farm has \( I \geq 2 \) workers. During a time period one unit long, each worker \( i = 1, \ldots, I \) gets utility \( m_i + \phi(x_i) \) when working for \( L_i = 1 - m_i \) units of time and consuming \( x_i \geq 0 \) units of corn, where \( 0 \leq L_i \leq 1, \phi' > 0, \) and \( \phi'' < 0 \). The workers produce a total of \( x \geq 0 \) units of corn during the period if and only if the sum of their working times is \( c(x) \geq 0 \), where \( c(0) = 0, c' > 0, c'' > 0, c'(0) < \phi'(0) \), and \( \phi'(x/I) < c'(x) \) when \( c(x) \geq I \). The workers have no source of corn other than what they produce.

a. In the two-good economy consisting of the farm and its workers, an allocation can be represented by the list of consumption and labor amounts \((x_i, L_i)\) for the workers and total demand for labor and supply of output by the farm. Explain why a Pareto optimal (Pareto efficient) allocation with consumption and labor levels \((\bar{x}_i, \bar{L}_i)\) must maximize \( \phi(x_1) - L_1 \) over all allocations, subject to \( \phi(x_i) - L_i \geq \phi(\bar{x}_i) - \bar{L}_i \) for \( i = 2, \ldots, I \), and \( 0 \leq L_i \leq 1 \) for \( i = 1, \ldots, I \), and other feasibility constraints.

b. Use first order conditions from the optimization in part a to characterize an arbitrary Pareto optimal allocation for this economy. Be as specific as possible, considering possible corner solutions. In particular, show that a positive amount of corn is produced.

c. Show that there is a Pareto optimal allocation in which worker 1 consumes no corn, whereas every other worker works and consumes corn. How much does worker 1 work in such an allocation? What can be said about the fairness of such an allocation?

d. Suppose that the workers treat their farm as a competitive firm that hires them and sells them corn in competitive markets. They all receive equal shares of the firm’s profits. Characterize competitive equilibrium in this private ownership economy. Show that in equilibrium, every worker works and consumes corn.

e. In part d, the allocation is different from the Pareto optimal allocation in part c. Does this mean that the Second Fundamental Welfare Theorem does not apply to this economy?

f. Suppose instead that the workers receive equal shares of the output of the farm and that they choose their working times independently. Describe the corresponding noncooperative game played by the workers, whose strategies are their working times. Characterize each pure strategy Nash equilibrium of this game. Compare the resulting total output to what it is in a Pareto optimal allocation. How do the workers’ utility levels in Nash equilibrium compare to what they are in the competitive equilibrium in part d?

3. An auctioneer auctions off one indivisible object to two potential bidders in a sealed bid auction. The bidders’ valuations are distributed independently as follows: Bidder \( L \) has valuation 1 with probability 1/2 and valuation 2 with probability 1/2. Bidder \( H \) has valuation 1 with probability 1/4 and valuation 2 with probability 3/4. Each bidder knows his valuation only but the prior distribution of valuations is common knowledge.

a. Suppose that the auctioneer conducts a first price sealed bid auction. If there is a tie in the bids, each bidder wins with probability 1/2 and the winner pays the (highest) bid. A bidder (after knowing his valuation) participates in the auction if and only if his expected surplus is non-negative. Find the set of non-negative \( \{b_1, b_2\} \) that form a strictly increasing, symmetric Bayesian Nash equilibrium in which both bidders participate regardless of their valuations. Draw the set of feasible \( \{b_1, b_2\} \). At a strictly increasing, symmetric Bayesian Nash equilibrium, a type 1 bidder (any bidder with valuation 1) bids \( b_1 \) and a type 2 bidder bids \( b_2 > b_1 \) regardless of whether the bidder is \( H \) or \( L \).
b. In part a, suppose the auctioneer chooses a non-negative pair \( \{b_1, b_2\} \), \( b_2 > b_1 \) that induces participation of all bidders regardless of their valuations and that forms a strictly increasing, symmetric Bayesian Nash equilibrium. What would be the expected revenue maximizing choice of \( b_1 \) and \( b_2 \) for the auctioneer? What is the maximum expected revenue?

c. Suppose that the auctioneer conducts the first price auction as in part a. But now suppose that a bidder (of any type) participates in the auction if and only if he can guarantee himself a nonnegative surplus regardless of the other bidder’s bid. Find all values of \( \{b_1, b_2\} \) that make \( b_1 \) a dominant strategy for type 1 (any bidder with valuation 1) and \( b_2 (> b_1) \) a dominant strategy for type 2 (any bidder with valuation 2) and that induce all bidders of all types to participate. Draw the set of such \( \{b_1, b_2\} \).

d. In part c, suppose that the auctioneer chooses non-negative \( \{b_1, b_2\} \) that induces participation of all bidders of all types and that satisfies all the other conditions listed. Derive the expected revenue maximizing choice of \( \{b_1, b_2\} \) by the auctioneer and the expected maximum revenue.

e. One can compare the maxima obtained in parts b and d without computing them. Explain why.

4. Consider \( N \)-player normal form games in which each player \( i \) has a (non-empty, finite) pure strategy set \( S_i \) and a utility (payoff) function \( u_i \).

a. Define a **strictly dominated (pure) strategy**.

b. In the following two person game, the row player has strategies \{T, M, B\} and the column player has strategies \{L, R\}. Solve the game by applying the iterated elimination of strictly dominated strategies (IDSDS). Justify each step carefully.

\[
\begin{array}{cc}
L & R \\
T & (2, 0) & (0, 1) \\
M & (0, 0) & (3, 1) \\
B & (1, 1) & (1, 0) \\
\end{array}
\]

c. Show that a pure strategy played with positive probability in a mixed strategy Nash equilibrium survives any number of rounds of iterated deletion of strictly dominated (pure) strategies.

d. Show that if IDSDS produces a unique strategy profile then it is a Nash equilibrium.

e. Consider a Cournot duopoly model with linear demand \( P = a - Q \) and \( P = 0 \) if \( Q > a \), where \( Q = q_1 + q_2 \) and \( a > 0 \). Both firms have zero fixed cost and constant marginal cost \( c \) and \( 0 \leq c < a \). Each firm \( i \)'s output \( q_i \) belongs to the set \( S_i = [0, \infty) \). Solve the game by applying IDSDS (in spirit) and taking the limit of the outcomes as the number of rounds of iteration goes to infinity.

f. Similar to part e, but assuming there are *three* firms, what is the result of IDSDS?
Answers

a. \( q = 25, p = 75, \text{profit} = 1250 \)

b. \( q = 50, a = 50, p = 150, \text{profit} = 2500 \)

c. \( q = \frac{50}{2-\pi^2}a = \frac{50\pi}{2-\pi^2}, \bar{p} = \frac{50(3+2\pi-2\pi^2)}{2-\pi^2}, p = \frac{50(3-2\pi^2)}{2-\pi^2}, E(\text{profit}) = \frac{2500}{2-\pi^2} \). \( a = 0 \) at \( \pi = 0 \); otherwise, \( a > 0 \). \( \frac{\partial E(\text{profit})}{\partial \pi} > 0 \). If the firm were to choose a single \( p \) and \( q \) to maximize \( E(\text{profit}) \), the market would not clear in either state.

d. \( \max_{q,a} (1 - \frac{1}{1+a}) \left((100 - q + 2a)q - q^2 - a^2\right) + \left(\frac{1}{1+a}\right) \left((100 - q)q - q^2\right) \)

e. \( q_i = 20, p = 60, \text{profit}_i = 800, \text{total profit} = 1600 \)

f. A strategy is a pair \((q_i, a_i)\). \( q_i = 100, a_i = 100, p = 300, \text{profit}_i = 10,000, \text{total profit} = 20,000 \)

g. The firms could increase profit without bound. In part (b) a single firm would incur the entire quadratic cost associated with advertising. Whereas here, for the same level of advertising, the cost could be spread over the two firms, decreasing the overall amount.

h. Each firm will advertise less in part (f) than in part (g).

Answers: 2a. If the allocation \( \bar{a} \) with worker \( i \) getting \((\bar{x}_i, \bar{L}_i)\) does not solve the maximization problem, then another feasible allocation gives worker \( 1 \) higher utility \( 1 - L_i + \phi(x_i) \) and at least as much utility to every other worker. This contradicts the assumption that \( \bar{a} \) is Pareto optimal.

b. Since \( \phi' > 0, \forall i \), it is inefficient for total output to be more than the sum of the workers' corn consumption. So the output constraint can be written as \( \sum L_i = c(\sum x_i) \) and the Lagrange function for the optimization in part a can be written as \( \lambda_1(\phi(x_1) - L_1) + \sum_{i>1} \lambda_i(\phi(x_i) - L_i + \bar{L}_i - \phi(\bar{x}_i)) + \gamma[\sum L_i - c(\sum x_i)] + \sum \mu_i(1 - L_i) \), where the nonnegativity constraints for \( L_i \) and \( x_i \) are omitted. The first order conditions are \( \lambda_i\phi'(x_i) - \gamma c'(\sum x_i) \leq 0 \), with equality if \( x_i > 0 \), and \( \gamma - \lambda_i - \mu_i \leq 0 \), with equality if \( L_i > 0 \) at a solution, where \( \lambda_i \geq 0, \mu_i \geq 0, \mu_i(1 - L_i) = 0 \), and some \( \lambda_i > 0 \). The last condition implies \( \gamma > 0 \). For each \( i \) with \( x_i > 0 \) and \( L_i < 1 \) we have \( \lambda_i = \gamma > 0 \) and \( \phi'(x_i) = c'(x) \), where \( x \) is total output.

If no corn is produced, then \( L_i = x_i = \mu_i = 0, \forall i \). Therefore, \( \gamma \leq \lambda_i \), and \( \gamma \phi'(0) \leq \lambda_i \phi'(0) \leq \gamma c'(0), \forall i \), contradicting the assumption \( \phi'(0) > c'(0) \).

c. If \( x_i > 0 \) and \( L_i > 0 \) for some \( i > 1 \) in a Pareto optimal allocation with \( x_1 = 0 \), then the first order conditions imply \( \lambda_i\phi'(x_i) = \gamma c'(\sum x_i) \geq \lambda_1\phi'(x_1) = \lambda_1\phi'(0) \). Since \( \phi'(0) > \phi'(x_i) \), we have \( \lambda_1 < \lambda_i = \gamma - \mu_i \leq \gamma \), which implies \( \mu_1 > 0 \) and \( L_1 = 1 \), so worker \( 1 \) works the maximum amount in any Pareto optimal allocation in which \( x_i > 0 \) and \( L_i > 0 \) for \( i > 1 \). Such a Pareto optimal allocation can be obtained by maximizing the Lagrange function in the answer to part b with respect to the allocation, letting \( \lambda_1 = 0 \) and \( \lambda_i = 1, \forall i > 1 \). In the solution, \( L_1 = 1, x_1 = 0, L_i > 0, \) and \( x_i > 0, \forall i > 1 \).

d. The utility functions are locally nonsatiated, so an equilibrium allocation is Pareto optimal and a positive amount of corn is produced. The price of corn is positive; otherwise the firm does not produce. Also the price of labor time is positive; otherwise the firm has no profit-maximizing output level. Letting labor time be numeraire, the firm maximizes its profit \( \pi = px - c(x) \) at output level \( x \) with \( p = c'(x) \). Each worker \( i \) maximizes \( 1 - L_i + \phi(x_i) \) subject to the budget constraint \( px_i + (1 - L - i) \leq 1 + \pi \). The
first order condition implies \( \phi'(x_i) \leq p = c'(x) \), with equality if \( x_i > 0 \), and \( L_i = px_i - \pi \).
Since \( x > 0 \), some \( x_i \) is positive, so \( \phi'(x_i) = c'(x) \) for that \( i \). If \( x_i = 0 \) for some \( h \), then
\[
\phi'(x_h) = \phi'(0) > \phi'(x_i) = c'(x),
\]
contradicting the first order condition. Therefore \( x_i > 0 \) for all \( i \) and \( x_i = x_i^* \) and \( \phi'(x_i^*) = c'(x) \). There is an \( x > 0 \) with labor requirement \( c(x) < I \) since \( \phi'(0) > c'(0) \) and \( \phi'(x/I) < c'(x) \) when \( c(x) \geq I \). Therefore each \( L_i < 1 \).

e. The answer to d shows that \( x_i > 0, \forall i \) in competitive equilibrium, whereas \( x_i = 0 \) in the Pareto optimal allocation in part c. The second welfare theorem is not contradicted. It says that under the convexity in this economy, each Pareto optimal allocation is part of a price quasi-equilibrium with transfers. Transfers are needed to make the allocation in part c a price quasi-equilibrium.

f. Each \( i \) takes \( L_{-i} = \sum_{h \neq i} L_h \) as given and maximizes \( \phi(x/I) - L_i \) with respect to \( x \) and \( L_i \), subject to \( \sum_{h=1}^{H} L_h = c(x) \). The solution is the same for each \( i \). If \( x > 0 \) at a solution, then the first order conditions imply \( \phi'(x/I) = Ic'(x) \). If \( \phi'(0) < Ic'(0) \), then each \( L_i = 0 \) and \( x = 0 \). In both cases, per capita consumption and labor supply are less than in any Pareto optimal allocation and each worker gets lower utility than in the Pareto optimal competitive allocation of part d.

Answers: 3a. Payoff matrices are:

\[
\begin{bmatrix}
\text{bidder bids } b_1 & \frac{1}{2} (1 - b_1) & 0 \\
\text{bidder bids } b_2 & 1 - b_2 & \frac{1}{2} (1 - b_2)
\end{bmatrix}
\]

for bidder \( L \) type 1, the expected surplus is maximized at \( b_1 \) if \((1 - b_1) \frac{1}{2} + (1 - b_2) \frac{1}{2} \geq (1 - b_2) \frac{1}{2} + (1 - b_2) \frac{1}{2} \geq 0 \). Bidder \( L \) type 2 participates if \((2 - b_2) \frac{1}{4} + (2 - b_2) \frac{1}{2} \geq 0 \). Similarly, a bidder \( H \) type 1 maximizes the expected surplus by choosing \( b_1 \) if \((1 - b_1) \frac{1}{2} + (1 - b_2) \frac{1}{2} \geq (1 - b_2) \frac{1}{2} + (1 - b_2) \frac{1}{2} \). Bidder \( H \) type 2 maximizes expected payoff at \( b_2 \) if \((2 - b_2) \frac{1}{2} + (2 - b_2) \frac{1}{2} \geq (2 - b_1) \frac{1}{2} \). Bidder \( H \) type 1 participates if \((1 - b_1) \frac{1}{2} + (2 - b_2) \frac{1}{2} \geq 0 \). Bidder \( H \) type 2 participates if \((2 - b_2) \frac{1}{2} + (2 - b_2) \frac{1}{2} \geq 0 \).

These inequalities simplify to: \( 8 \geq 5b_2 - b_1 \geq 4, 4 \geq 3b_2 - b_1 > 2, 1 \geq b_1 \geq 0, 2 \geq b_2 \). To satisfy \( b_2 > b_1 \), we need to remove \( \{b_1 = 1, b_2 = 1\} \) to obtain the feasible set.

b. Winning bid is \( b_1 \) precisely when both bidders are type 1. This happens with probability \( \frac{1}{8} \).\( \frac{7}{8} > \frac{1}{8} \) subject to \( \{b_1, b_2\} \in F \) gives \( b_1 = 1, b_2 = \frac{5}{3} \) with the maximum \( \frac{19}{12} \approx 1.5833 \).

c. Incentive constraints: \( \frac{1}{2} (1 - b_1) \geq 1 - b_0, 0 \geq \frac{1}{2} (1 - b_2), 2 - b_2 \geq \frac{1}{2} (2 - b_1), \frac{1}{2} (2 - b_2) \geq 0 \). Participation constraints: \( \frac{1}{2} (1 - b_1) \geq 0, 2 - b_2 \geq 0 \). These simplify to: \( b_1 \leq 1, 1 \leq b_2 \leq 2, 1 \leq 2b_2 - b_1 \leq 2 \). To satisfy \( b_2 > b_1 \), we need to remove \( \{b_1 = 1, b_2 = 1\} \) to obtain the feasible set \( F \).

d. \( \{b_1 = 1, b_2 = \frac{3}{2}\} \). maximum \( \frac{23}{16} \approx 1.4375 \).

e. The feasible set described in c. is contained in the feasible set described in a. Thus, the maximum in b. is greater than or equal to the maximum in d.

Answer: (a)
(b) Let $\sigma$ be a mixed strategy Nash and let $\tilde{S}_i$ be the support of strategy of $\sigma_i$. If $s_i \in \tilde{S}_i$, $s_i$ is a best response for player $i$ to $\sigma_{-i}$, the others’ mixed strategy profile. Then, $s_i$ is not strictly dominated. Then none of elements in $\tilde{S}_i$ for any player $i$ can be deleted in the first round. Since all $\{\tilde{S}_i\}$ survived the first round and the set of surviving strategies of player $i$ is monotonically decreasing, any $s_i \in \tilde{S}_i$ remains a best response to $\sigma_{-i}$ in the second round and thus is not strictly dominated in the second round. Applying the argument repeatedly, $\tilde{S}_i$ survives all future rounds.

(d) Let $s_i \in S_i \setminus \{s_i^*\}$ be such that $u_i(s_i, s_{-i}^*) \geq u_i(s_i^*, s_{-i}^*)$ for all $s_i^* \in S_i \setminus \{s_i^*\}$. But $s_i$ is strictly dominated in some round. This means in particular, there is $s_i'' \in S_i$ such that $u_i(s_i'', s_{-i}^*) > u_i(s_i, s_{-i}^*)$. This means $s_i'' = s_i^*$. Therefore $s_i^*$ is the best response to $s_{-i}^*$.

(e) Player $i$’s best response function is given by $r_i(q_j) = \frac{a-c-q_j}{2}$, $r_i(q_j) = 0$ if $q_j > a-c$. The first-round of deletion produces the strategy sets $S^1_i = [x_1 \equiv 0, y_1 \equiv \frac{a-c}{2}]$ as any $q_i > y_1$ is strictly dominated by $y_1$ given that the payoff function $u_i(q_i, q_j) = (a-c-q_i-q_j)q_i$ is strictly concave in $q_i$. Now for $q_j \in S^1_j$, firm $i$’s best responses $r_i(q_j)$ range from $x_2 \equiv r_i(y_1)$ to $y_2 \equiv r_i(x_1)$. By strict concavity of $u_i$, any $q_i < x_2$ is strictly dominated by $x_2$ and any $q_i > y_2$ is strictly dominated by $y_2$. Therefore the second round of deletion results in $S^2_i = [x_2, y_2]$. Similar arguments show that the result after $n > 1$ rounds of deletion is $S^n_i = [x_n, y_n]$ where

$$x_n = r_i(y_{n-1}) = \frac{a - c - y_{n-1}}{2} \quad \text{and} \quad y_n = r_i(x_{n-1}) = \frac{a - c - x_{n-1}}{2}.$$ 

with initial conditions $x_1 = 0, y_1 = (a-c)/2; x_2 = (a-c)/4, y_2 = y_1 = (a-c)/2$.

Then

$$x_n = \frac{a - c - y_{n-1}}{2} = \frac{a - c - \frac{a-c-x_{n-2}}{2}}{2} = \frac{a - c}{4} + \frac{1}{4}x_{n-2}$$

This implies as $n \to \infty$, $\lim x_n = \frac{a-c}{4}(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots) = \frac{a-c}{3}$. Similarly, $\lim y_n = \frac{a-c}{3}$. Thus in the limit, IDSDS results in the Cournot equilibrium $(q_1^* = q_2^* = \frac{a-c}{3})$.

(f) The first-round of deletion produces the strategy sets $S^1_i = [0, \frac{a-c}{2}]$. But then any $q_1 \in S^1_i$ is a best response to some profile $(q_2, q_3) \in [0, \frac{a-c}{2}] \times [0, \frac{a-c}{2}]$. Therefore we can go no further.