Instructions: Answer any three of the four numbered problems. Justify your answers whenever possible. Write your answer to each question in a separate bluebook. Write the number of the question AND NOTHING ELSE on the cover of the bluebook. In problem 4 you must also write on the exam sheet and turn it in with your bluebooks. No electronic devices may be used. The exam lasts 4 hours.

1. Consider an individual with preferences, $\succ$, defined over two commodities, money, $m$, and TVs, $x$. While $m$ can be consumed in any quantity, $x$ is binary: she can either consume 0 or 1 unit. Her preferences are strictly monotonic in both commodities. Assume that for any bundle $(1, m)$ there is an equivalent amount of money, $\mu(m)$, such that $(1, m) \sim (0, \mu(m))$, where $\sim$ represents indifference. The consumer is initially endowed with $M$ units of money and 0 units of $x$. The price of a TV at her present location is $p(0) > 0$.

a. Explain why $\mu(m) > m$.
b. Under what conditions would she choose to purchase a TV at her present location and when would she choose not to do so?
c. Generally, suppose the price of a TV at location $d$ is $p(d)$, where $d$ denotes the distance from 0. In addition, the cost of traveling distance $d$ is $cd$, where $c > 0$. Suppose the only options are (i) purchase a TV at 0, (ii) purchase a TV at $d$, or (iii) do not purchase a TV. Under what conditions would she choose to purchase a TV at $d$?
d. Next, suppose $p(0)$ is known, but in order to determine the price at location $d$ it would be necessary to travel to that location and thus incur the cost $cd$. Suppose the person’s beliefs about $p(d)$ are that it is distributed on $[p, \bar{p}]$ with density $f(p)$. Also assume she has the von Neumann - Morgenstern utility function $u(x, m) = 1000x + m$. Under what condition would it be worth traveling to $d$ in search of a lower price rather than purchasing a TV at location 0?
e. Again consider the case where the agent must travel to $d$ to discover $p(d)$ as in part d. Now suppose $d$ is an integer and that stores are located at each integer distance up to $d$. Assume the consumer is presently at $(d - 1)$ having traveled from 0 and having learned the price at each store along the way including $(d - 1)$. Her options at this point are to purchase a TV at any of the stores she has visited so far (without incurring any additional transportation cost) or to continue searching and go on to $d$. Determine the criterion for continuing her search. (Hint: would she continue if $p(d - 1) = \bar{p}$? If $p(d - 1) = p$?)
f. How would your answer to part e change if she would have to incur an additional cost of $c(d - d')$ in order to return to the store at location $d'$?

**Answers:**

1b. Purchase when $\mu(M - p(0)) > M$.
c. $\mu(M - p(d) - cd) > \max\{\mu(M - p(0)), M\}$
d. $\hat{p} + cd < p(0)$
e. Let $p_s$ satisfy $p_s + c(d - 1)) = \hat{p} + cd$ so that $p_s = \hat{p} + c$. The optimal decision would be to purchase a TV at the first location at which $p(d) \leq p_s$ and to continue to search otherwise.
f. It wouldn’t. Having already decided to continue at each store she visited up to $(d - 1)$, raising the cost of accepting an offer at $d' < d - 1$ would not make it more attractive and thus more likely to be accepted.
2. Consider an economy with $I \geq 3$ consumers, $I$ firms, 2 private goods, and no technology for disposing of goods. Each consumer $i$ ($i = 1, \ldots, I$) owns firm $i$ and initially owns $c_i > 0$ units of money. Consumer $i$ gets utility $m_i + \phi_i(x_i)$ from consuming $m_i$ units of money and $x_i \geq 0$ units of food, where $\phi' > 0$ and $\phi'' < 0$. Consumers can consume negative or positive amounts of money. Each firm $i$ produces 0 units of food when it uses no input and produces $q_i > 0$ units of food when it uses $K_i + c(q_i)$ units of money as input, where $c(0) = 0$, $c' > 0$, $c'' > 0$, and $0 \leq K_1 < K_2 < \cdots < K_I$.

a. Use the notation above to specify the set of feasible allocations in this economy.

b. Characterize the set of Pareto efficient (PE) allocations, being as specific as possible. Compare the output levels of the different firms in a given PE allocation. Explain why it is possible that some firms do not produce in a PE allocation. What can be said about which firms they are?

c. Compare the levels of food consumption by different consumers in a given PE allocation. Compare the levels of food consumption that any given consumer gets in different PE allocations for the same economy. Explain how the different PE allocations differ from each other.

d. Define a competitive (Walrasian) equilibrium (CE) for this economy. Characterize a CE in which money is numeraire (assuming that such a CE exists). Be as specific as possible. For each firm $i$ that produces in a CE, find an inequality that relates $K_i$ to the output level and the marginal and total cost of firm $i$ in equilibrium.

e. Is a CE allocation necessarily Pareto efficient in this economy?

f. Show that a CE allocation does not necessarily exist in this economy. Explain why CE might not exist. Use the notation above to specify an allocation that might plausibly arise if no CE exists. What can be said about the efficiency of such an allocation?

Answers: a. An allocation $(m_i, x_i, -z_i, q_i)_{i=1}^I$ is feasible if \( \sum_{i=1}^I (m_i + z_i - c_i) = 0 \) and \( \sum_{i=1}^I (x_i - q_i) = 0 \), where, for all $i$, $x_i \geq 0$, $q_i \geq 0$, and $z_i = K_i + c(q_i)$ if $q_i > 0$ and $z_i = 0$ otherwise.

b. If firm $i$ produces in a PE allocation, then each firm $j < i$ produces. Otherwise, switching the $q_i$ units of output from firm $i$ to firm $j$ reduces total input use without reducing output. This can raise all consumers’ utilities. So in a PE allocation, the first $J$ firms produce for some $J \leq I$.

If one feasible allocation is Pareto superior to another, then the first allocation has higher sum of consumers’ utilities and therefore higher total surplus \( \sum_{i=1}^I \phi_i(x_i) - \sum_{i=1}^J [K_i + c(q_i)] \). (This follows from replacing $\sum m_i$ in the sum of utilities by $\sum (c_i - z_i)$ and using the equality for $z_i$.) Thus a PE allocation maximizes total surplus subject to the feasibility condition \( \sum_{i=1}^I x_i = \sum_{i=1}^J q_i \). The first order conditions for this maximization imply $c'(q_i) = \gamma, \forall i \leq J$ and $\gamma \geq i \phi'(x_i), \forall i$, with equality if $x_i > 0$, where $\gamma$ is the Lagrange multiplier of the feasibility constraint. Therefore $q_i = q_1, \forall i \leq J$. The first $J$ firms produce the same amount and the other firms do not produce. If $K_i$ is very big, then having firm $i$ produce reduces total surplus. So if some firms have very big fixed (but not sunk) cost $K_i$ they do not produce in a PE allocation.

c. Since $\phi'' < 0$, the first order condition in the answer to b implies that in a PE allocation in which at least one firm produces, a consumer $i$ with $\phi'(0) > c'(q_1)/i$ consumes $x_i$ units of food with $\phi'(x_i) = c'(q_1)/i$. Therefore, $i$ consumes more food than every consumer $h < i$. There is a unique number of firms producing and a unique $q_1$ output level of firm 1 in a PE allocation. This follows from the fact that raising $q_1$ raises $c'(q_1)$ and reduces every $x_i > 0$
satisfying the first order conditions for Pareto efficiency. So there is a unique \( q_i \) with the first order conditions satisfied along with feasibility: \( \sum_{i=1}^{I} x_i = \sum_{i=1}^{J} q_i = Jq_1 \).

If the number \( J \) of producing firms increases and the output of each producing firm does not fall, then the first order conditions imply that \( \sum x_i \) falls or remains constant, so demand is less than supply of food. Therefore, if \( J \) increases and the first order conditions for efficiency continue to hold, every producer produces less, every consumer who consumes consumes more, and total output rises. The marginal gain in total surplus decreases at the total output rises, but the cost of an additional firm \( K_i \) rises, so there is a unique PE number \( J \) of producing firms and a unique PE level of output for each firm and food consumption for each consumer. The only way that PE allocations can differ from each other is in the consumers’ consumption of money.

d. A CE with money as numeraire is a feasible allocation and a price \( p \) of food such that each consumer \( i \) chooses \( m_i \) and \( x_i \) to maximize \( m_i + i\phi(x_i) \) subject to \( m_i + px_i = e_i + \pi_i \), where \( \pi_i \) is the equilibrium profit of firm \( i \). Firm \( i \) maximizes its profit, which is 0 if it does not produce and is \( pq_i - c(q_i) - K_i \) if it produces \( q_i > 0 \). Every firm that produces produces \( q_i \) such that \( p = c'(q_i) \) and \( K_i \leq pq_i - c(q_i) = q_i c'(q_i) - c(q_i) \). The consumer \( i \)'s first order condition can be written as \( io'(x_i) \leq p \), with equality if \( x_i > 0 \).

e. A CE allocation is PE. This is a private ownership economy with locally nonsatiated consumers so the first welfare theorem applies.

f. A CE allocation is PE, but Pareto efficiency uniquely determines the output levels of all the firms. At these output levels it is possible that some firm \( i \) that produces in a PE allocation has \( K_i > q_i c'(q_i) - c(q_i) \). This happens, for example if \( c' \) is very small and nearly constant. Then \( c(q_i) \approx kq_i \) for some small constant \( k \) and \( q_i c'(q_i) - c(q_i) \approx kq_i - kq_i = 0 < K_i \). In that case, firms that produce in a PE allocation cannot make a profit in CE. The firms might use market power instead. One possibility is that they act as Cournot oligopolists so that the allocation is a Nash equilibrium with the firms’ output levels as their pure strategies. The game is well-defined because the consumers have demand functions that depend only on the price of food, not on the consumers’ wealth levels, which depend on the firms’ profits. In Cournot equilibrium, the output price is higher than marginal cost, so the allocation is Pareto inefficient. The equilibrium output level is too low.

3. Consider the pricing problem faced by a monopolistic seller. There is a continuum of potential buyers of size 1. Each buyer demands at most one unit of the good. Buyers are of three types, \( i = 0,1,2 \). Each type \( i \) buyer values one unit of the good at \( V_i = \theta_i \sqrt{x} \), where \( x \geq 0 \) is the quality of the good and \( \theta_i \) is a parameter that describes the buyer’s taste for quality. Assume \( 0 < \theta_0 < \theta_1 < \theta_2 \). Let \( n_i \) be the fraction of buyers of type \( i \). It costs the monopolist \( C'(x) = c \cdot x \), where \( c > 0 \) is constant, to produce one unit of the good of quality \( x \). The monopolist’s payoff is its expected profit. The buyers get payoffs equal to the value to them of what they buy minus what they pay.

a. Suppose the monopolist can directly observe buyer type and can offer contracts contingent on type. Characterize the profit maximizing set of contracts for the monopolist.

For the rest of the problem, suppose that types are not observable to the monopolist. The monopolist offers a menu of contracts of the form \((p_i, x_i)\) where a type \( i \) contract is meant for type \( i \) buyers.

b. Formulate the monopolist’s pricing problem with incentive and participation constraints, assuming each buyer has a reservation payoff equal to zero.

c. Consider the relaxed monopoly pricing problem (RP) in which only the following downward adjacent incentive constraints (DAIC) and a participation constraint (P0) for type
\[ i = 0 \text{ are imposed.} \]
\[ \theta_2 \sqrt{x_2} - p_2 \geq \theta_2 \sqrt{x_1} - p_1, \quad (\text{DAIC2}) \]
\[ \theta_1 \sqrt{x_1} - p_1 \geq \theta_1 \sqrt{x_0} - p_0, \quad (\text{DAIC1}) \]
\[ \theta_0 \sqrt{x_0} - p_0 \geq 0 \quad (\text{P0}) \]

Show that all these constraints bind in a solution to this relaxed problem.

d. Show that if the solution to the relaxed problem (RP) satisfies monotonicity, i.e. \( x_2^* \geq x_1^* \geq x_0^* \), then all of the incentive constraints and participation constraints in the original problem are satisfied and the solution to the relaxed problem is also a solution to the original problem. Interpret this monotonicity condition.

e. Solve the relaxed problem (RP). Compare the optimal quality levels \( (x_2^*, x_1^*, x_0^*) \) to the quality levels the monopolist would choose in part a. Discuss and interpret any differences.

Answers: 3a. Offer \( (x_i^E, p_i^E) \) to each type \( i \) buyer, where each
\[ x_i^E = \arg \max_x \theta_i \sqrt{x} - c \cdot x \]
and \( p_i^E = \theta_i \sqrt{x_i^E} \).

b. \[ \max_{x_i, p_i} \Pi = \sum_{i=0}^{2} n_i(p_i - C(x_i)) \text{ s.t.} \]
\[ \theta_i \sqrt{x_i} - p_i \geq \theta_i \sqrt{x_j} - p_j, \quad \forall i, j \]
\[ \theta_i \sqrt{x_i} - p_i \geq 0, \quad \forall i. \]

c. If (P0) does not bind, increase \( p_1 \) slightly; If (DAICi) is not binding for \( i = 1, 2 \), then increase \( p_i \) slightly. In each case, the profit will be increased while the constraints are intact.

d. By (DAIC) we have \( \theta_2(\sqrt{x_2} - \sqrt{x_1}) \geq (p_2 - p_1) \) and \( \theta_1(\sqrt{x_1} - \sqrt{x_0}) \geq p_1 - p_0 \). Hence:
\[ \theta_2(\sqrt{x_2} - \sqrt{x_0}) \geq \theta_2(\sqrt{x_2} - \sqrt{x_1} + \sqrt{x_1} - \sqrt{x_0}) \]
\[ \geq \theta_2(\sqrt{x_2} - \sqrt{x_1}) + \theta_2(\sqrt{x_1} - \sqrt{x_0}) \]
\[ \geq \theta_2(\sqrt{x_2} - \sqrt{x_1}) + \theta_1(\sqrt{x_1} - \sqrt{x_0}) \]
\[ \geq p_2 - p_1 + p_1 - p_0 = p_2 - p_0 \]

which says type 2 does not prefer the contract \( (x_0, p_0) \) designed for type 0.

Moreover, given that DAICs are binding in optimum and \( x_1 - x_2 \leq 0 \) and \( x_0 - x_1 \leq 0 \) we have
\[ \theta_1(\sqrt{x_1} - \sqrt{x_2}) \geq \theta_2(\sqrt{x_1} - \sqrt{x_2}) = p_1 - p_2 \]
\[ \theta_0(\sqrt{x_0} - \sqrt{x_1}) \geq \theta_1(\sqrt{x_0} - \sqrt{x_1}) = p_0 - p_1 \]
\[ \theta_0(\sqrt{x_0} - \sqrt{x_2}) \geq \theta_1(\sqrt{x_0} - \sqrt{x_1}) + \theta_2(\sqrt{x_1} - \sqrt{x_2}) \geq p_0 - p_1 + p_1 - p_2 = p_0 - p_2 \]
which are the three “upward” incentive constraints.

For the participation constraints, note that for \( i = 1, 2 \), the incentive constraints imply
\[ \theta_i \sqrt{x_i} - p_i \geq \theta_i \sqrt{x_0} - p_0 \geq \theta_0 \sqrt{x_0} - p_0 \geq 0. \]
e. Using the binding constraints $\text{DAIC2}$, $\text{DAIC1}$ and $\text{P0}$ to get
\[
p_0 = \theta_0 \sqrt{x_0}
\]
\[
p_1 = \theta_1 \sqrt{x_1} - (\theta_1 - \theta_0) \sqrt{x_0}
\]
\[
p_2 = \theta_2 \sqrt{x_2} - (\theta_2 - \theta_1) \sqrt{x_1} - (\theta_1 - \theta_0) \sqrt{x_0}
\]
and then substitute for prices in the objective function:
\[
\Pi = n_2 (\theta_2 \sqrt{x_2} - cx_2) + n_1 \left\{ \left[ \theta_1 - \frac{n_2}{n_1} (\theta_2 - \theta_1) \right] \sqrt{x_1} - cx_1 \right\} + n_0 \left\{ \left[ \theta_0 - \frac{n_1}{n_0} (\theta_1 - \theta_0) - \frac{n_2}{n_0} (\theta_1 - \theta_0) \right] \sqrt{x_0} - cx_0 \right\}
\]

Then first-order conditions on $x_i$’s lead to:
\[
\theta_2 \cdot \frac{1}{2 \sqrt{x_2}} = c
\]
\[
\left( \theta_1 - \frac{n_2}{n_1} (\theta_2 - \theta_1) \right) \cdot \frac{1}{2 \sqrt{x_1}} = c
\]
\[
\left( \theta_0 - \frac{n_1 + n_2}{n_0} (\theta_1 - \theta_0) \right) \cdot \frac{1}{2 \sqrt{x_0}} = c,
\]
and if there is no $x \geq 0$ that satisfies $i$-th equation then $x_i^* = 0$, otherwise $x_i^*$ is given by the $i$-th equation above.

Note that except for type $i = 2$, other buyers get a quality less than the efficient level $x_i^E = \theta_i$.

f. It is sufficient if $x_2^* \geq x_1^* \geq x_0^*$. It is clear $x_2^*$ is greater than both $x_1^*$ and $x_0^*$, so only need $x_1^* \geq x_0^*$. Using answers from (e), we can find one sufficient condition as follows:
\[
\theta_1 - \frac{n_2}{n_1} (\theta_2 - \theta_1) \geq \theta_0 - \frac{n_1 + n_2}{n_0} (\theta_1 - \theta_0)
\]
or the simpler (note: $n_0 = 1 - n_1 - n_2$):
\[
\frac{1}{n_0} (\theta_1 - \theta_0) \geq \frac{n_2}{n_1} (\theta_2 - \theta_1).
\]

One interpretation is that given distribution $(n_0, n_1, n_2)$, the valuation difference between type 0 and type 1 should be sufficiently large compared to that between types 1 and 2.

4. A monopoly firm $I$ knows that firm $E$ is considering entering its market and knows that the product design team at firm $E$ is either good (G) or bad (B). Firm $I$ initially believes that the design team is more likely good than bad. Firm $E$ knows the quality of its design team. If $E$ decides to enter the market, it can do so with either a high or low investment. If the investment is high and the design team is good, then $E$’s product quality is high. If the investment is low and the design team is bad, the product quality is low. The product quality is medium if either the investment is high and the design team is bad or else if the investment is low and the design team is good. Firm $I$ can see the quality of $E$’s product (high, medium, or low), but does not directly know the quality of $E$’s design team before deciding whether to fight or accommodate $E$ in the market.

If $E$ stays out of the market, its payoff is 0 and $I$’s is 5. The two tables below show the payoffs to $E$ and $I$, depending on their decisions, when $E$ enters the market. The left table shows the payoffs when $E$’s design team is good and right table shows the payoffs if the team is bad. The structure of this interaction is common knowledge to $E$ and $I$. 

a. The interaction between $E$ and $I$ can be represented by a game with the tree above. Writing on this sheet, label the moves, the players that move them, and the payoffs in the tree above. (Turn in this sheet with your bluebook.) Explain briefly how you can tell which player moves in which information set.

b. Explain how you can tell that the decisions represented in the tables above are not the players’ pure strategies. How many pure strategies does $E$ have in the game? Give an example of one of them.

c. How many pure strategies does $I$ have in the game? List two of them.

d. Does $E$ have any weakly dominated strategies in the game? If so find one.

e. Find every sequential equilibrium (SE) in which $I$ plays a pure strategy. Justify your answer. Explain what the SE outcomes are and discuss whether they are plausible. Is there an SE in which $I$ is sure about whether or not $E$’s design team is good?

f. Is there a Nash equilibrium (NE) in which $E$ chooses not to enter the market no matter what? If so, discuss the plausibility of this outcome. If not, explain why not.

Answers: a. Nature moves first, determining whether $E$’s design team is good or bad. This move represents $I$’s uncertainty and since $I$ initially believes that the team is more likely good, Nature chooses “good” with probability $p > 1/2$. $I$ observes the quality of $E$’s product if $E$ enters, so $E$ moves second and $I$ moves last.
b. A pure strategy for $E$ is a function assigning a move to every information set where $E$ moves. $E$ can either have a good or bad team and in each information set it has 3 moves: out, high or low investment. There are three moves when the team is bad for every one of three moves when it is good, so $3^2 = 9$ pure strategies. An example is playing out whether the team is good or bad.
c. $I$ has three information sets determined by the quality of $E$’s product. In each information set, $I$ has two moves, so $I$ has $2^3 = 8$ pure strategies. One is fight no matter what the quality of $E$’s product is. Another is accommodate no matter what the quality is.
d. Every pure strategy in which $E$ makes high investment when its team is bad is weakly dominated. For example, choosing high investment no matter what is weakly dominated by choosing high investment when the team is good and low investment when the team is
bad. $E$’s payoff is strictly higher if the team is bad, no matter what $I$’s strategy is.

e. Sequential rationality requires that $I$ accommodates if $E$’s product quality is high and fights if it is low. As part of a best reply, $E$ chooses high investment if its design team is good. Let $I$ believe with probability $\mu$ that the design team is good when the product quality is medium. Sequential rationality of fighting in that case requires $2 \geq \mu + 3(1 - \mu)$ or $\mu \geq 1/2$. If $I$ fights, then $E$’s best reply is high if the team is good and out if the team is bad. Thus, medium quality is reached with 0 probability. Every belief $\mu \geq 1/2$ is consistent since there is a sequence of totally mixed strategies in which $E$ chooses with probability $\gamma/k$ low investment when the team is good and chooses with probability $\beta/k$ high investment when the team is bad. As $k \to \infty$, these probabilities approach the equilibrium probabilities and the conditional probability of the node after low investment is $p\gamma/[p\gamma + (1 - p)\beta]$, which equals $\mu$ with appropriate choices of $\gamma$ and $\beta$.

Next, suppose $I$ accommodates when $E$’s product quality is medium. Then $\mu \leq 1/2$. Any such $\mu$ can occur in SE by the same reasoning as above if $E$ stays out for sure when its design team is bad. There are also SE in which $E$ chooses high investment with positive probability when its design team is bad. Then rationality of beliefs requires $\mu = 0$. This shows that there are three types of SE in which $I$ plays a pure strategy. $I$ might fight for sure when the quality is medium. Then $\mu \geq 1/2$ and $E$ plays out for sure when its design team is bad. In another type of SE, $\mu \leq 1/2$, $I$ accommodates for sure if $E$’s product quality is medium and $E$ stays out if its team is bad. Alternatively, in which $E$ enters with high investment with positive probability when its team is bad. Then $\mu = 0$ and $I$ accommodates for sure if $E$’s product quality is medium. In all three types of SE, it is possible that $I$ is sure either that the design team is good ($\mu = 1$) or bad ($\mu = 0$).

An equilibrium dominance argument raises doubts about the plausibility of SE with $\mu \geq 1/2$. If the design team is good and $E$ enters with low investment, $E$’s payoff is less than the equilibrium payoff no matter how $I$ responds. This is not true if the design team is bad. Then entering with high investment gives $E$ at least the equilibrium payoff if $I$ accommodates when the product quality is medium.

f. There is a NE in which $I$ fights for sure in every information set and $E$ stays out no matter what. For this to be $E$’s best reply, $I$ must threaten to fight when $E$’s product quality is high, an incredible threat. This strategy is not sequentially rational for $I$. 