Instructions: Answer any three of the four problems below. Write your answer to each problem in a separate bluebook. Write the number of the problem on the cover of the bluebook. DO NOT WRITE YOUR NAME OR STUDENT ID NUMBER on the bluebooks. The exam lasts 4 hours.

1. Consider an $L$-good private ownership economy in which every consumer has a strictly positive endowment vector $e_i \gg 0$ in $\mathbb{R}^L$ and a utility function $u_i : \mathbb{R}_+^L \to \mathbb{R}$ with the following properties:
   (a) $u_i$ is strictly quasiconcave; and
   (b) the preferences determined by $u_i$ are locally nonsatiated.
In addition, every consumer entirely owns its own firm and every firm has exactly one owner. The firms have the same convex production set $Y \in \mathbb{R}^L$, which has the following property:
   (c) $y \in Y \Rightarrow ty \in Y$ for all $t \in \mathbb{R}_+$.
A feasible allocation in this economy is one in which each consumer $i$ gets a consumption vector $x_i \in \mathbb{R}_+^L$ and its firm chooses production vector $y_i \in Y$ such that $\sum_i (x_i - e_i - y_i) \leq 0$.

   a. Give a mathematical definition of strict quasiconcavity for a function $u_i$.
   b. Use the notation above to define a competitive (Walrasian) equilibrium (without disposal) for this economy.
   c. Interpret property (c).
   d. What can be said about the profit of firm $i$ (owned by consumer $i$) in a competitive equilibrium of this economy? Be as specific as possible and show that your answer is correct.
   e. We call a competitive equilibrium in this economy self-sufficient if each consumer $i$ gets a consumption vector $\bar{x}_i$ with $\bar{x}_i - e_i \in Y$. Explain why the term “self-sufficient” applies to such an equilibrium.
   f. Is a self-sufficient competitive equilibrium allocation in this economy necessarily Pareto efficient?

Suppose now that this economy has a self-sufficient competitive equilibrium in which each consumer $i$ gets consumption vector $\bar{x}_i$. Consider another competitive equilibrium with price vector $p$, consumption vector $x_i$ and production vector $y_i$ for each consumer $i$.

   g. Explain why $p\bar{x}_i \leq pe_i$ and $u_i(x_i) \geq u_i(\bar{x}_i)$ for each consumer $i$.
   h. Show that there is a feasible allocation in which each consumer $i$ gets the consumption vector $(1/2)(x_i + \bar{x}_i)$. Is it possible that $x_i \neq \bar{x}_i$ for some consumer $i$? Interpret your answer and show that it is correct. It may help if you draw a graph.

2. There are two types of workers of equal numbers. The total number of workers is normalized to be 1. A worker is either a high productivity type ($\theta = 2$) or a low productivity type ($\theta = 1$). Each worker chooses a task of difficulty $e$, $0 \leq e \leq 2$. The government cannot observe the type of worker or the difficulty of the task a worker performs but it can observe the income of a worker. A worker of type $\theta$ working in a type $e$ task receives income $w = \theta(1 + e)$. The government needs to raise a tax revenue of 1 to meet its fiscal needs but seeks to raise it in a welfare maximizing manner. The utility function of a type $\theta$ who chooses a task of difficulty $e$ and pays income tax $t$ is $u_\theta(e, w, t) = w - \frac{e^2}{t} - t$, where $w = \theta(1 + e)$. The government maximizes $W = 2u_l + u_h$ subject to constraints.
a. Suppose that the government is restricted in its choice of income tax schedule to an affine income tax schedule $t = a + bw$ where $0 < b < 1$. Which task $e$ would each type choose? Formulate the problems and solve them.

b. Given the results in part a and the welfare function of the government, formulate the government’s problem of choosing optimal affine income tax schedule to raise the revenue of 1. Find the optimal marginal tax rate $b$.

c. Now, suppose instead that the government offers a menu for the two types to choose from: \{\{w_h, t_h\}, \{w_l, t_l\}\}. Those with income $w_\theta$ are taxed $t_\theta$. Here, \{\{w_\theta, t_\theta\}\} is intended for type $\theta$ by the government. If a type $\theta$ worker is indifferent between \{\{w_h, t_h\}\} and \{\{w_l, t_l\}\}, assume that the worker chooses \{\{w_\theta, t_\theta\}\}. The tax revenue raised is 1. Formulate the government problem of maximizing the welfare criterion $W$ given all relevant constraints. (Hint: since $w_h = \theta h (1 + e_h)$, type $h$ chooses $w_h$ by choosing $e_h$. Similarly, for type $l$.)

d. Show that the optimum solution to part c gives a social welfare level at least as great as that obtained in b. (Hint: you do not need to solve the problem in c to show this.)

3. a. Consider a consumer with income $M > 0$ who purchases commodities $x$ and $y$ at market prices $p_x > 0$ and $p_y > 0$ respectively. However, in addition to incurring the direct cost associated with such expenditures, the consumer is also required to pay a transaction fee of 0.5% of the value of its purchases.

   (i) Write the budget equation for the consumer.

   (ii) How does this differ from the standard budget equation?

b. In contrast, suppose the consumer was endowed with the bundle of goods $\omega = (\omega_x, \omega_y) \gg 0$ rather than with income. In this case, the transaction cost is incurred as a result of changes from $\omega$ and is paid in units of $x$, the numeraire. (That is, the transaction cost is incurred whether the consumer is a net buyer or a net seller of $x$.) Such costs are increasing in the size of the transaction.

   (i) Graphically depict the consumer’s budget constraint which takes into consideration the transaction cost. Also, depict a typical consumer’s equilibrium.

   (ii) Is it possible that a consumer would be a net buyer of $x$ without the transaction cost, but would choose to be a net seller of $x$ with the cost?

   (iii) Suppose the transaction cost is 0.5% of the amount of $x$ traded. Write an expression for the consumer’s budget constraint and draw the consumer’s budget set.

c. Discuss the difference between the formulations in parts a and b.

d. Finally, consider an exchange economy with two consumers, 1 and 2, both of whom are as described in part b. Let $\omega^i = (\omega^i_x, \omega^i_y)$ denote the endowment of consumer $i$.

   (i) Assume that the consumers incur the cost of their own transactions. Set up the social planner’s problem for determining the Pareto efficient allocations.

   (ii) Derive and interpret/explain the (necessary) first order conditions for efficiency at a solution to the planner’s problem where consumer 1 is a net buyer of $x$. 

4. The tree below represents a game in which an entrepreneurial scientist (player 1) has an idea for a new project. If she does not develop it enough to show to a potential collaborator (player 2), then both she and player 2 get 0 payoff. If she does develop it and show it to player 2, then player 2 can accept or refuse to collaborate. If player 2 refuses, then player 2 gets 0 payoff. If player 2 accepts, then the two work together. Player 1 can complete the research and publish it with player 2, giving both of them a payoff of 2. Alternatively, player 1 can switch to exploiting a commercial application of the research and leave the results unpublished. In that case, player 2, gets payoff −2. All the payoffs are listed in the tree below, with the payoff of player 1 listed first. One fourth of entrepreneurial scientists are committed to publishing research while the others are not. The committed ones get payoffs listed in the tree branches following C. The uncommitted ones get payoffs in branches following U. At the beginning of their interaction, player 2 thinks of player 1 as a random selection from the population of entrepreneurial scientists. The players are rational expected payoff maximizers and have common knowledge of this and of the structure of their interaction, as represented in the game tree.

a. Does player 1 know whether she is committed to publishing when she decides whether to develop her idea? How can you tell?
b. How many pure strategies does player 1 have? Give an example of one.
c. How many pure strategies does player 2 have? Give an example of one.
d. Find all the subgames of the game.
e. Does this game have a subgame perfect Nash equilibrium (SPE) in which players 1 and 2 choose pure strategies? If so, find one. Show all your work and interpret your result. Be careful to consider best responses by player 2 to the entire pure strategy of player 1.
f. Find a sequential equilibrium (SE) in which the research is completed and published with positive probability. Does player 2 get a positive expected payoff? Show that your answers are correct.