Instructions: Answer any three of the following four questions. Whenever possible, justify your answers. Write your answer to each question in a separate bluebook. Write the number of the question on the cover of the bluebook. DO NOT WRITE YOUR NAME OR STUDENT ID NUMBER on the bluebooks. The exam lasts 3 hours.

1. Consider a population of workers consisting of equal numbers of two types, \( t = l, h \). The marginal productivity of a worker with education level \( e \in [0, 16] \) is \( e \) for \( t = l \) and \( 2e \) for \( t = h \). Workers know their types but the firms cannot identify them directly. The firms are risk neutral. They base their wage offers on \( e; \bar{e} \) which is observable, and compete in a Bertrand manner: they make wage offers to a worker that is equal to their expected marginal productivity of the worker. Utility functions \( u_t(w, e) \) of workers \( t = h, l \) are strictly concave and satisfy \( u_t(w, e) > 0; u_t(e) < 0 \) (subscripts indicate partial derivatives), and the single crossing property: 
\[
\frac{u_t(w, e)}{u_t(w, \bar{e})} > \frac{u_h(w, e)}{u_h(w, \bar{e})}
\]
at each \((w, e)\) . The wage expectation of a worker with education \( e \) is denoted by \( w(e) \). A worker of type \( t \) chooses \( e = e_t \) that maximizes \( u_t(w(e), e) \). For simplicity, we assume that all maximizers are unique in this problem. Let \( \mu(e) \) be the belief of firms that a worker is type \( h \) given their education level of \( e \).

a. Show why the single crossing assumption implies that an indifference curve of type \( h \) worker meets an indifference curve of type \( l \) worker at most once.

b. Draw the game tree of the above signaling game.

c. What is a (pure) strategy of a worker? What is a (pure) strategy of a firm?

d. What would be an equilibrium if employers could directly identify workers’ types? Draw a graph to illustrate your answer.

e. It can be shown that in a sequential equilibrium, firms’ beliefs on the type of workers are the same and thus they offer the same \( w(e) \) after observing \( e \). Show that in a sequential equilibrium, competing firms offer a wage \( w(e) \) between \( e \) and \( 2e \) to a worker with education level \( e \).

f. Under what assumption on \( \mu(e) \), would the wage offer function \( w(e) \) be increasing in \( e \) in a sequential equilibrium?

g. Let \( \bar{e} = \arg \max_e u_l(e, e) \). We assume that \( \bar{e} < 16 \). Show that type \( l \) obtains at least the utility level \( u_l(\bar{e}, \bar{e}) \) at any (separating or pooling) sequential equilibrium.

h. Consider an education level \( e \) satisfying \( u_l(2e, e) < u_l(\bar{e}, \bar{e}) \). Explain why a type \( l \) would not choose such an education level.

i. Let \( \hat{f}_1 \) be a level of education satisfying \( u_l(2\hat{f}_1, \hat{f}_1) = u_l(\bar{e}, \bar{e}) \) and let \( \hat{f}_2 = \arg \max_{\hat{f}_1 \leq e \leq 16} u_h(2e, e) \). That is, \( e_h = \hat{f}_2 \) maximizes \( u_h(2e, e) \) for any education level \( e \) at least as large as \( \hat{f}_1 \). Assume in the rest of the problem 1. \( \hat{f}_1 < 16 \), 2. \( \hat{f}_2 > \hat{f}_1 \) and 3. type \( h \) indifference curve passing through \((32, 16)\) does not meet \((e, e)\). Explain why a type \( h \) worker can guarantee himself the
utility level of \( u^h(2\hat{f}_2, \hat{f}_2) \) in any sequential equilibrium.

j. Explain why at a separating equilibrium, \( e_h = \hat{f}_2 \). Illustrate such a separating equilibrium with a graph.

k. Under the assumption that \( \hat{f}_2 > \hat{f}_1 \), there cannot be a pooling equilibrium. Explain.

2. A society is composed of three agents, \( A, B \) and \( C \) with von Neumann and Morgenstern (Bernoulli) utilities for money \( u_A(m) = \sqrt{m}, \ u_B(m) = m \) and \( u_C(m) = m^2 \), respectively. In the absence of redistribution, the agents’ gross incomes would be \( m_A = 46, m_B = 92 \) and \( m_C = 138 \), respectively.

Now, imagine that the agents were behind a (thin) veil of ignorance in which they know their preferences (as above), but not which of the three income levels they will be assigned. That is, with equal probability, one of the agents will receive gross income 46, one will receive 92 and one will receive 138, but it is not yet known which.

a. Devise a social welfare system, or a redistribution scheme, to be implemented after incomes are realized, so as to maximize the “outcome” of the “worst off” individual (i.e., the lowest ranked agent according to that outcome measure) when the outcome is measured by:

i. income

ii. expected utility (tax conditional solely on income)

iii. ex post utility (tax can be conditional on identity)

b. Such a Rawlsian social objective is often taken to rationalize the provision of social insurance among risk averse agents. For example, if agent \( A \) were to vote over redistributive tax policies, its preferences would accord with those of a Rawlsian social planner. Is this social objective function appropriate here? And why?

c. Contrast your answers to part (a) with the case of a Utilitarian social planner who seeks to maximize the aggregate “outcome”. Again consider each of the three outcome measures.

Next, consider a thick veil of ignorance behind which the agents do not even know their preferences (although it is known they are expected utility maximizers.) Now assume there is an equal chance of each agent being assigned to each utility function and receiving any of the three income levels.

d. How does this affect your answers to part (a)?

3. Several trucking firms buy diesel oil from refineries at a port and deliver it to a large city. The oil is used for home heating and as fuel for trucks. Let \( q_j \) be the amount of diesel oil delivered to the city by trucking firm \( j \). The inverse demand function for diesel oil in the city is \( p = a - bq \), where \( p \) is the price of diesel oil sold in the city and \( q \) is the total quantity delivered by the trucking firms \( (a > 0, b > 0) \). The refineries charge \( r \) per gallon for diesel oil. Each trucking firm fuels its own trucks using \( \gamma < 1 \) gallons of diesel oil per gallon it delivers to the city. It also has other variable costs of \( w \) per gallon delivered and fixed costs.

a. Consider the game played by the trucking firms, in which they independently choose the quantities of diesel oil they deliver, and their payoffs are their profits from the sale of oil in the
city. Find a pure strategy Nash equilibrium of the game in terms of the exogenous variables of the problem. Is there only one pure strategy Nash equilibrium?

b. If the price \( r \) were slightly higher what would be the effect on the equilibrium profit of a typical trucking firm and on the price \( p \) of diesel oil in the city in the Nash equilibrium you found in part a? Is the change in equilibrium \( p \) as large as the change in \( r \)? Interpret your answer.

c. How would the equilibrium profit of trucking firm 1 differ if \( \gamma \) were slightly smaller? Be as specific as possible.

d. Carefully formulate a system of equations characterizing Nash equilibrium if the trucks of firm 1 used their fuel more efficiently, while the fuel efficiency of all the other trucks (along with all other exogenous variables) remained as before. You do not need to find the new equilibrium. Compare the effect of this improvement in efficiency to the effect of a small reduction in \( \gamma \) (the effect found in part c). Explain the comparison. Can the envelope theorem be applied to compute the effect of a slight improvement in the fuel efficiency of the trucks of firm 1 on that firm’s profit? Explain why or why not.

e. Starting from the original Nash equilibrium in part a, suppose that without bearing any additional costs firm 1 could save a small fixed amount of oil by improving the fuel efficiency of its trucks. What would be the value of this efficiency improvement to firm 1 per unit of oil saved? Is it the price \( r \) that the firm pays for oil at the refinery, or the price \( p \) at which it sells oil in the city, or something in between? Explain.

4. Consider the following strategic-form game:

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(a) Find a Nash equilibrium.
(b) Show that the equilibrium is unique.

Next consider the game with all payoffs multiplied by \(-1\):

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(c) Show that there is a unique Nash equilibrium for this game.