1. In this question, we will use the framework of the real business cycle model to study consumption in the face of anticipated productivity shocks. Welfare of the representative agent is given by

\[
\sum_{t=0}^{\infty} \beta^t E_0 [\ln C_t + \psi \ln (1 - \ell_t)] , \quad 0 < \beta < 1 ,
\]

where \(C_t\) denotes consumption, and \(\ell_t\) denotes labor supply. Assuming the household is endowed with one unit of time, then \((1 - \ell_t)\) represents leisure. The household’s budget constraint is given by

\[
C_t + K_{t+1} = (1 + r_t) K_t + w_t \ell_t,
\]

with \(K_t\) denoting capital, \(C_t\) consumption, \(r_t\) the real interest rate, and \(w_t\) the wage rate. Note that we have simplified by setting capital depreciation to zero. The firm chooses capital and labor to maximize profits given by

\[
\text{profits} = Y_t - w_t \ell_t - r_t K_t , \quad \text{where} \quad Y_t = A_t K_t^\alpha (1 - \ell_t)^{1-\alpha}
\]

where \(Y_t\) is output, and \(A_t\) is productivity. We assume that at some unknown future date, there will be an adverse productivity shock. Specifically, there is a fixed probability \(\rho\) that \(A_{t+1} < A_t\) and a probability \(1 - \rho\) that \(A_{t+1} = A_t\). Assume initially that the adverse shock lasts for a single period and then reverts to its constant prior value. This possible future adverse productivity shock is the only source of uncertainty in the economy.

(a) Set up the Bellman equation and derive first order conditions for households. Use recursive notation, dropping time subscripts, and denoting future values with primes. Solve for the agent’s Euler equation in consumption and for the equilibrium relationship between consumption and leisure.

\[
V(A, K) = \max_{C, K'} \left\{ \ln C + \psi \ln (1 - \ell) + \beta E (K', A') + \lambda \left[ (1 + r) K + w\ell - C - K' \right] \right\}
\]

\[
\frac{1}{C} - \lambda = 0 \quad \text{envelope}
\]

\[
\psi \frac{1}{1 - \ell} = \lambda w
\]

\[
\beta E \frac{\partial V(K', A')}{\partial K'} = \lambda
\]

implying that

\[
\frac{\partial V(K, A)}{\partial K} = \lambda (1 + r)
\]

\[
\frac{\partial V'(K', A')}{\partial K'} = \frac{1}{C'} (1 + r')
\]
Euler equation and relationship between consumption and leisure become.

\[
\frac{1}{C} = \beta E \frac{1}{C'} (1 + r') \tag{EE}
\]

\[
\psi \frac{1}{1 - \ell} = w \frac{1}{C} \tag{LL}
\]

(b) Assume that the firm picks the capital stock and labor to maximize profits and solve for first order conditions on both capital and labor.

\[
(1 - \alpha) \frac{Y_t}{\ell_t} = w_t \tag{w}
\]

\[
\alpha \frac{Y}{K} = r \tag{r}
\]

(c) List the equations of the model determining the non-stochastic steady-state values of the variables with \(A\) fixed at its initial value, and solve for steady state values of \(C, Y, K, \ell, r, w\).

The five equations necessary for steady state are labeled.

\[
1 = \beta (1 + r) \tag{EE}
\]

\[
1 + r = \beta^{-1}
\]

\[
C + K = (1 + r) K + w \ell = K + Y \tag{FBC}
\]

\[
C = Y
\]

\[
\alpha \frac{Y}{K} = r = \beta^{-1} = \frac{1 - \beta}{\beta} \tag{r}
\]

\[
K = \frac{\alpha \beta Y}{1 - \beta}
\]

\[
(1 - \alpha) \frac{Y}{\ell} = w \tag{w}
\]

\[
(1 - \alpha) \frac{1}{\ell} = \frac{w}{C} = \psi \frac{1}{1 - \ell} \tag{ll}
\]

\[
\ell = \frac{1 - \alpha}{\psi + 1 - \alpha}
\]

\[
Y = AK^\alpha \ell^{1 - \alpha} = A \left( \frac{\alpha \beta Y}{1 - \beta} \right)^\alpha \left( \frac{1 - \alpha}{\psi + 1 - \alpha} \right)^{1 - \alpha} \tag{1}
\]

\[
Y = A \left( \frac{\alpha \beta}{1 - \beta} \right)^\frac{\alpha}{1 - \alpha} \left( \frac{1 - \alpha}{\psi + 1 - \alpha} \right)
\]

\[
K = A \left( \frac{\alpha \beta}{1 - \beta} \right)^\frac{1}{1 - \alpha} \left( \frac{1 - \alpha}{\psi + 1 - \alpha} \right)
\]

\[
w = A (1 - \alpha) \left( \frac{\alpha \beta}{1 - \beta} \right)^\frac{\alpha}{1 - \alpha}
\]
(d) Linearize these equations about this steady state for which \( A_t = A \), letting lower-case letters with hats denote percent deviations from this steady state, for all variables except \( \hat{r} \), which denotes the level deviation, not the percent deviation.

The following contains all linearizations, but you are only asked for the first two.

\[
- \ln C_t = \ln \beta - \ln C' + \ln (1 + r') \\
\dot{c} = c' - \dot{r} \\
\ln \psi - \ln (1 - \ell) = \ln w - \ln C \\
\frac{\ell}{1 - \ell} = \hat{\psi} - \dot{c} \\
\ln (1 - \alpha) + \ln Y - \ln \ell = \ln w \\
\dot{y} = \dot{\ell} + \dot{w} \\
\ln \alpha + \ln y - \ln k = \ln (1 + r) \approx r \\
\dot{y} = \dot{k} + \dot{r} \\
\exp(\ln C) + \exp(\ln K') = \exp(\ln K) + \exp(\ln Y) \quad \text{(FBC)} \\
\frac{1}{C} \dot{c} + \frac{1}{K} \dot{k}' = \frac{1}{Y} \dot{y} - \frac{1}{K} \dot{k} \quad \text{(2)} \\
\ln Y = \ln A + \alpha \ln K + (1 - \alpha) \ln \ell \\
\dot{y} = \dot{a} + \alpha \dot{k} + (1 - \alpha) \dot{\ell}
\]

(e) Define permanent income and the permanent income theory of consumption. Compare permanent income with income in periods prior to the adverse shock. Use the permanent income theory of consumption to compare consumption to income in periods prior to the adverse shock and in the period when the shock actually occurs. Explain.

Permanent income is the constant value of income which would yield the expected present value of income plus annuitized initial wealth. The permanent income theory of consumption says that consumption equals permanent income. Prior to the shock, permanent income is less than actual income, implying that consumption is less than income. When the shock occurs, it was unexpected (there was always a constant probability of the shock and in the period in which it is realized, actual income falls more than expected income). Permanent income and consumption both fall, but by much less than actual income, since future productivity reverts immediately back to its initial level.

(f) Use the agent’s flow budget constraint to explain how capital is changing over time. What does the change in capital imply about the growth of current and expected future consumption?

With consumption less than income, agents are accumulating capital, implying that income is growing, raising permanent income and allowing current and expected future consumption to grow.

(g) Does the Euler equation exhibit precautionary savings? Explain. Does the linearized version of the Euler equation exhibit precautionary savings? Explain. In which version would you expect capital to grow faster prior to the shock? Explain.
The Euler equation has precautionary saving since

\[ E_t \frac{1}{C_{t+1}} > \frac{1}{E_t C_{t+1}} \]

The linearized version does not since

\[ E_t \hat{c}_{t+1} = E_t \hat{c}_{t+1} \]

With precautionary saving, consumption is lower and capital accumulation is higher, therefore capital would grow faster in the non-linearized version of the model.

(h) Before the adverse shock, \( E_t \hat{c}_{t+1} \) = \( E_t \hat{c}_{t+1} < 0 \) (since the capital stock is higher than in the steady-state with \( A = 1 \)), Therefore, expected future consumption is lower than current consumption even though consumption is growing over time prior to the shock. Explain this.

Expected future consumption is the probability that the shock does not occur, multiplied by higher consumption plus the probability that the shock does occur, multiplied by lower consumption. Together this can imply that expected future consumption is less than current consumption even though in the absence of the shock, the actual value of future consumption will exceed current consumption.

(i) In periods prior to the adverse shock, explain the wealth effect of this expected shock on the decision to supply labor and relate your answer to the first order equation for leisure.

Since there is a positive probability of this adverse shock, wealth is lower, consumption is lower, and labor supply is higher for each wage.
2. One-sided search with continuing wage adjustment.

**Time**: Discrete, infinite horizon.

**Demography**: Single worker who lives forever.

**Preferences**: The worker is risk-neutral. He discounts the future at the rate $r$.

**Endowments**:

When unemployed, the worker receives the value of leisure, $b$, per period. Also with probability $\alpha$ he gets an offer of employment at a wage $w \sim F(.)$ on $[0, \bar{w}]$ where $\bar{w} > b$.

Whenever employed, the wage (in the same job) is subject to change with probability $\gamma$. New wages are also drawn from $F(.)$. **Note**: (i) the worker does not get a choice to stay on the old wage but can quit to unemployment, (ii) this is not a one-time wage adjustment – as long as the worker remains employed at the firm he is subject to further wage adjustments.

Also when employed with probability $\lambda$ the job is destroyed and the worker moves to unemployment. Assume that the events of getting a wage adjustment and job destruction are mutually exclusive (i.e. not independent) and that $\lambda + \gamma < 1$.

(a) Let $U$ be the value to unemployment and $V(w)$ be the value to employment at current wage, $w$. Write down the flow asset value equations and briefly explain each one.

\[ rU = b + \alpha E_{w} \max \{ V(w) - U, 0 \} \]  \hspace{1cm} (3)

\[ rV(w) = w + \gamma E_{w'} \max \{ V(w') - V(w), U - V(w) \} + \lambda (U - V(w)) \]  \hspace{1cm} (4)

Equation (3) is standard. The second term in the RHS of equation (4) reflects the choice between continuing to work at the new wage, $w'$, and quitting.

(b) Show that the asset value equation for $V(w)$ can be rewritten to make it clear that the reservation wage, $w^*$, above which the worker accepts a job is the same as the one below which he quits. Briefly explain that result.

The reservation wage is defined by $V(w^*) = U$. Subtract $(U - V(w))$ from the second term on the right to get

\[ rV(w) = w + \gamma E_{w'} \max \{ V(w') - U, 0 \} + (\gamma + \lambda) (U - V(w)) \]  \hspace{1cm} (5)

Result follows by comparison of (5) and equation (3). When the worker’s wage is adjusted the decision he faces (to quit or keep the job) is identical to the one he faces when unemployed and receives an offer.

(c) Solve for $U$ in terms of parameters and $w^*$.

Multiply equation (5) by $\alpha$ and equation (3) by $\gamma$. Subtract the latter from the former to get:

\[ \alpha (r + \gamma + \lambda)V(w) = \alpha w - \gamma b + [r \gamma + \alpha (\gamma + \lambda)] U \]  \hspace{1cm} (6)

Evaluation at $w^*$ implies

\[ rU = \frac{\alpha w^* - \gamma b}{\alpha - \gamma} \]  \hspace{1cm} (7)
(d) Use this to obtain the reservation wage equation. (Alternatively you can do it using integration by parts and obtaining $V'(w)$ from the asset value equation for $V(w)$.)

Substitute from equation (7) into (6) to obtain $V(w)$ in terms of $w^*$. Then solve for

$$V(w) - U = \frac{w - w^*}{r + \gamma + \lambda}.$$ 

Into equation (3):

$$\frac{\alpha w^* - \gamma b}{\alpha - \gamma} = b + \frac{\alpha}{r + \gamma + \lambda} \int_{w^*}^{w_0} (w - w^*) dF(w)$$

Simplify to obtain

$$w^* = b + \frac{\alpha - \gamma}{r + \gamma + \lambda} \int_{w^*}^{w_0} (w - w^*) dF(w)$$

(e) What happens to the reservation wage if $\gamma > \alpha$? Briefly interpret this result.

If $\gamma > \alpha$, the reservation wage is below $b$. In the standard model $\gamma = 0$. There, the second term means that $w^* > b$ because there is an option value to remaining unemployed. Now with $\gamma > 0$ there is a also an option value to employment. When $\gamma > \alpha$ the latter option value is larger. People will take a utility cut to get the chance at a higher arrival rate of future wage adjustments.

(f) If this person were one out of a unit mass (continuum) of similar workers, obtain an expression for $u$, the steady state share of the population in unemployment, in terms of parameters and $F(w^*)$.

The aggregate flow from unemployment to employment is

$$\alpha(1 - F(w^*))u.$$ 

The aggregate flow from employment to unemployment is now

$$(\lambda + \gamma F(w^*)) \left(1 - u\right).$$

In steady state these are equal so

$$u = \frac{\lambda + \gamma F(w^*)}{\alpha(1 - F(w^*)) + \lambda + \gamma F(w^*)}.$$
3. Consider the following version of a Lucas tree model. Welfare of the representative agent is given by

\[ E_0 \left( \sum_{t=0}^{\infty} \beta^t u(c_t) \right), \quad 0 < \beta < 1, \quad u(c_t) > 0, \quad u'(c_t) \geq 0, \quad u''(c_t) \leq 0 \]

where \( c_t \) denotes consumption. There are two types of trees in this economy. The first is a safe tree which produces \( d \) dividends each period. The second is a risky tree which produces \( 2d \) dividends with probability 0.5 and no dividends with probability 0.5. The dividends from both trees are not storable. The economy starts off with each household owning one such tree. Let \( p_s^t \) be the price at time \( t \) of a title to all future dividends from a safe tree and \( p_r^t \) the price of all future dividends from a risky tree.

Let \( R_t^{-1} = R_t^{-1}(d_t) \) be the time-\( t \) price of a risk-free discount bond that pays one unit of consumption at time \( t+1 \). Finally, let \( x_t \) denote the consumer’s financial resources, which she allocates between risk-free bonds \( (b_t) \), safe stocks \( (s_t^s) \), risky stocks \( (s_t^r) \) and consumption \((c_t)\).

(a) Write down the consumer’s problem in recursive form (Bellman equation) and find the first order conditions. Write expressions for the bond Euler equation and for the stock Euler equation.

\[
V(x_t, d_t) = \min_{\lambda_t \geq 0, c_t, s_{t+1}^s, s_{t+1}^r, b_{t+1}} \{ u(c_t) + \lambda_t [c_t - p_s^t s_{t+1}^s - p_r^t s_{t+1}^r - R_t^{-1} b_{t+1} - c_t] + \beta E_t V(x_{t+1}, d_{t+1}) \}
\]

\[
x_{t+1} = (d_{t+1} + p_r^t) s_{t+1}^r + (d_{t+1} + p_s^t) s_{t+1}^s + b_{t+1}
\]

First order conditions are:

\[
u'(c_t) = \lambda_t
\]

\[-\lambda_t p_s^t + \beta E_t \left\{ \frac{\partial V(x_{t+1}, d_{t+1})}{\partial x_{t+1}} [d + p_s^t] \right\} = 0
\]

\[-\lambda_t p_r^t + \beta E_t \left\{ \frac{\partial V(x_{t+1}, d_{t+1})}{\partial x_{t+1}} [d_{t+1} + p_r^t] \right\} = 0
\]

\[-\lambda_t R_t^{-1} + \beta E_t \frac{\partial V(x_{t+1}, d_{t+1})}{\partial x_{t+1}} = 0
\]

The envelope condition requires

\[
\frac{\partial V(x_t, d_t)}{\partial x_t} = \lambda_t
\]

Combining this with the first order condition on consumption and taking it forward one period yields

\[
\frac{\partial V(x_{t+1}, d_{t+1})}{\partial x_{t+1}} = \lambda_{t+1} = u'(c_{t+1})
\]

Substituting into the Euler equations for stocks and for bonds yields

\[-u'(c_t) p_s^t + \beta E_t \left\{ u'(c_{t+1}) [d_{t+1} + p_s^t] \right\} = 0
\]

\[-u'(c_t) p_r^t + \beta E_t \left\{ u'(c_{t+1}) [d_{t+1} + p_r^t] \right\} = 0
\]

\[-u'(c_t) R_t^{-1} + \beta E_t u'(c_{t+1}) = 0
\]
(b) What is the equilibrium value for consumption in periods when risky dividends are high? In periods when risky dividends are low? Label each with superscripts $h$ and $l$.

$$c^h = 3d$$
$$c^l = d$$

(c) Use the values for equilibrium consumption in your first order conditions to take the expectations writing out the first order conditions in terms of probabilities of each state and the marginal utility of consumption in those states. Label the future price of each type of stock with an additional superscript, $h, l$ to denote its value when the price is high versus low. Rearrange your first order conditions to express prices of both stocks and the price of bonds ($R_t^{-1}$) on the left-hand-side.

$$p_t^s = \frac{0.5\beta \left\{ u'(3d) \left[ d + p_{t+1}^{s,h} \right] + u'(d) \left[ d + p_{t+1}^{s,l} \right] \right\}}{u'(c_t)}$$
$$p_t^r = \frac{0.5\beta \left\{ u'(3d) \left[ 2d + p_{t+1}^{s,h} \right] + u'(d) \left[ p_{t+1}^{s,l} \right] \right\}}{u'(c_t)}$$
$$R_t^{-1} = \frac{0.5\beta \left[ u'(3d) + u'(d) \right]}{u'(c_t)}$$

(d) Will the interest rate (not the price of bonds) be higher or lower if the current period is one with high dividends? Explain.

If dividends are currently high then $u'(c_t) = u'(3d)$ implying a low marginal utility of consumption, a high bond price, and a low interest rate. Current income is high relative to expected future income, requiring a lower interest rate to convince agents to consume the higher income today.

(e) Now, recognize that the current price of stocks is state-dependent and is not explicitly dependent on time. Therefore, $p_t^{s,h} = p_{t+1}^{s,h}$ and so forth. Drop the time subscripts and solve for the state dependent prices of each stock price, where the state is high or low dividends.

$$p_t^{s,h}u'(3d) = 0.5\beta \left\{ u'(3d) \left[ d + p^{s,h} \right] + u'(d) \left[ d + p^{s,l} \right] \right\}$$
$$p_t^{s,l}u'(d) = 0.5\beta \left\{ u'(3d) \left[ d + p^{s,h} \right] + u'(d) \left[ d + p^{s,l} \right] \right\}$$

Averaging the two stock prices yields

$$0.5 \left[ u'(3d)p^{s,h} + u'(d)p^{s,l} \right] = 0.5\beta \left\{ u'(3d) \left[ d + p^{s,h} \right] + u'(d) \left[ d + p^{s,l} \right] \right\}$$

Solving for the weighted average of prices as a function of dividends yields

$$\left[ u'(3d)p^{s,h} + u'(d)p^{s,l} \right] = \frac{\beta}{1 - \beta}d \left[ u'(3d) + u'(d) \right]$$

Substituting into the first equation yields

$$p^{s,h}u'(3d) = 0.5\beta \left( \frac{\beta}{1 - \beta} + 1 \right) d \left[ u'(3d) + u'(d) \right]$$
$$= 0.5 \frac{\beta}{1 - \beta} d \left[ u'(3d) + u'(d) \right]$$
\[ p^{s,h} = 0.5 \frac{\beta}{1 - \beta} \frac{d[u'(3d) + u'(d)]}{u'(3d)} = R_t^{-1} \frac{1}{1 - \beta} d \]

Similarly \( p^{s,l} \) can be computed as

\[ p^{s,l} = 0.5 \frac{\beta}{1 - \beta} \frac{d[u'(3d) + u'(d)]}{u'(d)} . \]

Using the same steps, we can solve for the state-dependent prices of the risky asset.

\[ p^{r,h} u'(3d) = 0.5 \beta \left\{ \left[ u'(3d) \left[ 2d + p^{r,h} \right] + u'(d) \left[ p^{r,l} \right] \right] \right\} \]

\[ p^{r,l} u'(d) = 0.5 \beta \left\{ \left[ u'(3d) \left[ 2d + p^{r,h} \right] + u'(d) \left[ p^{r,l} \right] \right] \right\} \]

averaging

\[ 0.5 \left[ u'(3d) p^{r,h} + u'(d) p^{r,l} \right] = 0.5 \beta \left\{ \left[ u'(3d) \left[ 2d + p^{r,h} \right] + u'(d) \left[ p^{r,l} \right] \right] \right\} \]

\[ \left[ u'(3d) p^{r,h} + u'(d) p^{r,l} \right] = \frac{\beta}{1 - \beta} u'(3d) \left[ 2d \right] \]

\[ p^{r,h} u'(3d) = 0.5 \beta \left( \frac{\beta}{1 - \beta} + 1 \right) 2d \left[ u'(3d) \right] \]

\[ = 0.5 \frac{\beta}{1 - \beta} 2d \left[ u'(3d) \right] \]

\[ p^{r,h} = 0.5 \frac{\beta}{1 - \beta} \frac{2d [u'(3d)]}{u'(3d)} = \frac{\beta}{1 - \beta} d \]

\[ p^{r,l} = 0.5 \frac{\beta}{1 - \beta} \frac{2d [u'(3d)]}{u'(d)} \]

(f) Compare your solutions for the prices of the risky and safe asset for the high-income state and determine whether the risky asset or the safe asset has the higher price. Explain.

\[ p^{s,h} = 0.5 \frac{\beta}{1 - \beta} \frac{d[u'(3d) + u'(d)]}{u'(3d)} \]

\[ p^{r,h} = 0.5 \frac{\beta}{1 - \beta} \frac{2d [u'(3d)]}{u'(3d)} \]

Since \( u'(d) > u'(3d) \), the price of the risky asset is lower. Agents are risk averse and are willing to pay more for the safe asset.
4. Optimal growth with a payroll tax

**Time:** Discrete, infinite horizon.

**Demography:** Single representative household and representative firm owned by the household. (Both firm and household take prices as given.) Both live forever.

**Preferences:** household preferences are represented by the instantaneous utility function, \( u(c_t, l_t) \) where \( c_t \) is period \( t \) consumption and \( l_t \) is leisure. The function \( u(\cdot, \cdot) \) is bounded above, twice differentiable, increasing in both arguments, and strictly concave with \( \lim_{c \to -\infty} u_1(c, l) = \infty \), and \( \lim_{l \to 0} u_2(c, l) = \infty \) where \( u_i(\cdot, \cdot), \ i = 1, 2 \) is the derivative w.r.t. the \( i \)th argument. The household’s discount factor is \( \beta < 1 \).

**Technology:** The firm has access to the production function, \( f(k, h) \) where \( k \) is its capital input and \( h \) is its labor input. The function \( f(\cdot, \cdot) \) exhibits constant returns to scale, is twice differentiable, concave and increasing in both arguments. Standard Inada conditions apply. Capital depreciates with use at the rate \( \delta \).

**Endowments:** The Household is initially endowed with \( k_0 \) units of capital and each period has one unit of time to distribute between work and leisure.

**Institutions:** There is a government which has to meet expenditures, \( g_t \), each period. Government expenditures are thrown in the ocean. The government also has the power to tax. It will use two taxes, a time varying lump sum tax, \( \tau_t \), and a fixed proportional tax on labor income, \( \phi \). (Both taxes are levied on the household not the firm; \( g_t \) is assumed to be less than the productive capacity of the economy.)

(a) Write down and solve the Planner’s problem for this economy. (The solution should be in the form of a system of three difference equations in \( k_t, c_t \) and \( h_t \).)

The Planner solves:

\[
\max_{\{c_t, l_t, h_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)
\]

subject to:

\[
f(k_t, h_t) + (1 - \delta)k_t = k_{t+1} + c_t + g_t
\]

\[
h_t = 1 - l_t, \quad k_0 \text{ given.}
\]

In recursive form this becomes (they can use Lagrangians or dynamic programming).

\[
V(k_t) = \max_{k_{t+1}, h_t} \{u(f(k_t, h_t) + (1 - \delta)k_t - k_{t+1} - g_t, 1 - h_t) + \beta V(k_{t+1})\}
\]

FOCs:

\[
h_t : \quad -u_2(t) + u_1(t) f_2(t) = 0
\]

\[
k_{t+1} : \quad -u_1(t) + \beta V'(k_{t+1})
\]

where \( u_i(t) = u_i(c_t, 1 - h_t) \) etc.

Envelope condition is:

\[
V(k_t) = u_1(t) [f_1(t) + 1 - \delta]
\]

Into \( k_{t+1} \) equation leads to the system:

\[
-u_2(c_t, 1 - h_t) + u_1(c_t, 1 - h_t) f_2(k_t, h_t) = 0
\]

\[
-u_1(c_t, 1 - h_t) + \beta u_1(c_{t+1}, 1 - h_{t+1}) [f_1(k_{t+1}, h_{t+1}) + 1 - \delta] = 0
\]

\[
f(k_t, h_t) + (1 - \delta)k_t - k_{t+1} - c_t - g_t = 0
\]
(b) Write down and solve the problem faced by the representative household.

Household solves:

\[
\max_{\{c_t, l_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)
\]

Subject to:
\[
c_t = w_t h_t (1 - \phi) + r_t k_t - k_{t+1} + (1 - \delta) k_t - \tau_t
\]
\[
h_t = 1 - l_t, \quad k_0 \text{ given.}
\]

In recursive form:
\[
V(k_t) = \max_{k_{t+1}, h_t} \{ u(w_t h_t (1 - \phi) + r_t k_t - k_{t+1} + (1 - \delta) k_t - \tau_t, 1 - h_t) + \beta V(k_{t+1}) \}
\]

FOCs:
\[
h_t : \quad -u_2(t) + u_1(t) w_t (1 - \phi) = 0
\]
\[
k_{t+1} : \quad -u_1(t) + \beta V'(k_{t+1}) = 0
\]

Envelope condition is:
\[
V(k_t) = u_1(t) [r_t + 1 - \delta]
\]

So household’s solution characterized as
\[
-u_2(t) + u_1(t) w_t (1 - \phi) = 0
\]
\[
-u_1(t) + \beta u_1(t + 1) [r_{t+1} + 1 - \delta] = 0
\]

(c) Write down and solve the problem faced by the representative firm.

Firm solves:
\[
\max_{k^f_t, h^f_t} \{ f(k^f_t, h^f_t) - r_t k^f_t - w_t h^f_t \}
\]

which implies for an interior solution that
\[
f_1(t) = r_t, \quad f_2(t) = w_t
\]

(d) Write down the market clearing conditions and the government budget constraint.

Market clearing conditions are:
\[
k_t = k^f_t
\]
\[
h_t = h^f_t
\]
\[
f(k_t, h_t) + (1 - \delta) k_t = k_{t+1} + c_t + g_t
\]

The government budget constraint is
\[
w_t h_t \phi + \tau_t = g_t
\]

(e) Define a competitive equilibrium for this economy.

**Definition:** A Competitive equilibrium is an allocation, \( \{k_t, c_t, h_t, l_t, k^f_t, h^f_t\} \), prices, \( \{r_t, w_t\} \) and taxes, \( \{\tau_t\} \) such that (i) given prices, the allocation solves the household’s and firm’s problems, (ii) markets clear and (iii) the government budget constraint holds.
(f) Solve for a system of three difference equations in $k_t$, $c_t$ and $h_t$ that characterize equilibrium.

The system boils down to

$$-u_2(c_t, 1 - h_t) + u_1(c_t, 1 - h_t)(1 - \phi)f_2(k_t, h_t) = 0$$

$$-u_1(c_t, 1 - h_t) + \beta u_1(c_{t+1}, 1 - h_{t+1}) [f_1(k_{t+1}, h_{t+1}) + 1 - \delta] = 0$$

$$f(k_t, h_t) + (1 - \delta)k_t - k_{t+1} - c_t - g_t = 0$$

(g) Under what conditions on the tax code does the competitive equilibrium allocation coincide with the Planner’s allocation? Briefly explain.

Solutions coincide when $\phi = 0$. As leisure is not taxed while work is, it distorts the labor leisure choice. With $\phi > 0$ welfare will be lower than in the Planner’s model. Whether output is higher or lower would depend on the relative strengths of the income and substitution effects.