1. In this question, we will use the framework of the real business cycle model to study consumption in the face of anticipated productivity shocks. Welfare of the representative agent is given by

\[ \sum_{t=0}^{\infty} \beta^t E_0 [\ln C_t + \psi_t \ln (1 - \ell_t)], \quad 0 < \beta < 1, \]  

(U)

where \( C_t \) denotes consumption, and \( \ell_t \) denotes labor supply. Assuming the household is endowed with one unit of time, then \( (1 - \ell_t) \) represents leisure. The household’s budget constraint is given by

\[ C_t + K_{t+1} = (1 + r_t) K_t + w_t \ell_t, \]  

(FBC)

with \( K_t \) denoting capital, \( C_t \) consumption, \( r_t \) the real interest rate, and \( w_t \) the wage rate. Note that we have simplified by setting capital depreciation to zero. The firm chooses capital and labor to maximize profits given by

\[ \text{profits} = Y_t - w_t \ell_t - r_t K_t, \quad \text{where } Y_t = A_t K_t^{\alpha} \ell_t^{1-\alpha} \]  

(PRO)

where \( Y_t \) is output, and \( A_t \) is productivity. We assume that at some unknown future date, there will be an adverse productivity shock. Specifically, there is a fixed probability \( \rho \) that \( A_{t+1} < A_t \) and a probability \( 1 - \rho \) that \( A_{t+1} = A_t \). Assume initially that the adverse shock lasts for a single period and then reverts to its constant prior value. This possible future adverse productivity shock is the only source of uncertainty in the economy.

(a) Set up the Bellman equation and derive first order conditions for households. Use recursive notation, dropping time subscripts, and denoting future values with primes. Solve for the agent’s Euler equation in consumption and for the equilibrium relationship between consumption and leisure.

(b) Assume that the firm picks the capital stock and labor to maximize profits and solve for first order conditions on both capital and labor.

(c) List the equations of the model determining the non-stochastic steady-state values of the variables with \( A \) fixed at its initial value, and solve for steady state values of \( C, Y, K, \ell, r, w \).

(d) Linearize the Euler equation and the labor-leisure choice about this steady state for which \( A_t = A \), letting lower-case letters with hats denote percent deviations from this steady state, for all variables except \( \hat{r} \), which denotes the level deviation, not the percent deviation.

(e) Define permanent income and the permanent income theory of consumption. Compare permanent income with income in periods prior to the adverse shock. Use the permanent income theory of consumption to compare consumption to income in periods prior to the adverse shock and in the period when the shock actually occurs. Explain.
(f) Use the agent’s flow budget constraint to explain how capital is changing over time. What does the change in capital imply about the growth of current and expected future consumption?

(g) Does the Euler equation exhibit precautionary savings? Explain. Does the linearized version of the Euler equation exhibit precautionary savings? Explain. In which version would you expect capital to grow faster prior to the shock? Explain.

(h) Before the adverse shock, $E_t\hat{c}_{t+1} - \hat{c}_t = E_t\hat{r}_{t+1} < 0$ (since the capital stock is higher than in the steady-state with $A = 1$), Therefore, expected future consumption is lower than current consumption even though consumption is growing over time prior to the shock. Explain this.

(i) In periods prior to the adverse shock, explain the wealth effect of this expected shock on the decision to supply labor and relate your answer to the first order equation for leisure.

2. One-sided search with continuing wage adjustment.

Time: Discrete, infinite horizon.

Demography: Single worker who lives forever.

Preferences: The worker is risk-neutral. He discounts the future at the rate $r$.

Endowments:

When unemployed, the worker receives the value of leisure, $b$, per period. Also with probability $\alpha$ he gets an offer of employment at a wage $w \sim F(.)$ on $[0, \bar{w}]$ where $\bar{w} > b$.

Whenever employed, the wage (in the same job) is subject to change with probability $\gamma$. New wages are also drawn from $F(.)$. Note: (i) the worker does not get a choice to stay on the old wage but can quit to unemployment, (ii) this is not a one-time wage adjustment – as long as the worker remains employed at the firm he is subject to further wage adjustments.

Also when employed with probability $\lambda$ the job is destroyed and the worker moves to unemployment. Assume that the events of getting a wage adjustment and job destruction are mutually exclusive (i.e. not independent) and that $\lambda + \gamma < 1$.

(a) Let $U$ be the value to unemployment and $V(w)$ be the value to employment at current wage, $w$. Write down the flow asset value equations and briefly explain each one.

(b) Show that the asset value equation for $V(w)$ can be rewritten to make it clear that the reservation wage, $w^*$, above which the worker accepts a job is the same as the one below which he quits. Briefly explain that result.

(c) Solve for $U$ in terms of parameters and $w^*$.

(d) Use this to obtain the reservation wage equation. (Alternatively you can do it using integration by parts and obtaining $V'(w)$ from the asset value equation for $V(w)$.)

(e) What happens to the reservation wage if $\gamma > \alpha$? Briefly interpret this result.

(f) If this person were one out of a unit mass (continuum) of similar workers, obtain an expression for $u$, the steady state share of the population in unemployment, in terms of parameters and $F(w^*)$. 

2
3. Consider the following version of a Lucas tree model. Welfare of the representative agent is given by

\[ E_0 \left( \sum_{t=0}^{\infty} \beta^t u(c_t) \right), \quad 0 < \beta < 1, \quad u(c_t) \geq 0, \quad u'(c_t) \geq 0, \quad u''(c_t) \leq 0 \]

where \( c_t \) denotes consumption. There are two types of trees in this economy. The first is a safe tree which produces \( d \) dividends each period. The second is a risky tree which produces \( 2d \) dividends with probability 0.5 and no dividends with probability 0.5. The dividends from both trees are not storable. The economy starts off with each household owning one such tree. Let \( p^s_t \) be the price at time \( t \) of a title to all future dividends from a safe tree and \( p^r_t \) the price of all future dividends from a risky tree.

Let \( R^{-1}_t = R^{-1}_t(d_t) \) be the time-\( t \) price of a risk-free discount bond that pays one unit of consumption at time \( t + 1 \). Finally, let \( x_t \) denote the consumer’s financial resources, which she allocates between risk-free bonds \( (b_t) \), safe stocks \( (s^s_t) \), risky stocks \( (s^r_t) \) and consumption \( (c_t) \).

(a) Write down the consumer’s problem in recursive form (Bellman equation) and find the first order conditions. Write expressions for the bond Euler equation and for the stock Euler equation.

(b) What is the equilibrium value for consumption in periods when risky dividends are high? In periods when risky dividends are low? Label each with superscripts \( h \) and \( l \).

(c) Use the values for equilibrium consumption in your first order conditions to take the expectations writing out the first order conditions in terms of probabilities of each state and the marginal utility of consumption in those states. Label the future price of each type of stock with an additional superscript, \( h, l \) to denote its value when the price is high versus low. Rearrange your first order conditions to express prices of both stocks and the price of bonds \( (R^{-1}_t) \) on the left-hand side.

(d) Will the interest rate (not the price of bonds) be higher or lower if the current period is one with high dividends? Explain.

(e) Now, recognize that the current price of stocks is state-dependent and is not explicitly dependent on time. Therefore, \( p^{s,h}_t = p^{s,h}_{t+1} \) and so forth. Drop the time subscripts and solve for the state dependent prices of each stock price, where the state is high or low dividends.

(f) Compare your solutions for the prices of the risky and safe asset for the high-income state and determine whether the risky asset or the safe asset has the higher price. Explain.
4. Optimal growth with a payroll tax

**Time:** Discrete, infinite horizon.

**Demography:** Single representative household and representative firm owned by the household. (Both firm and household take prices as given.) Both live forever.

**Preferences:** Household preferences are represented by the instantaneous utility function, \( u(c_t, l_t) \) where \( c_t \) is period \( t \) consumption and \( l_t \) is leisure. The function \( u(\ldots) \) is bounded above, twice differentiable, increasing in both arguments, and strictly concave with \( \lim_{c \to 0} u_1(c, l) = \infty \), and \( \lim_{l \to 0} u_2(c, l) = \infty \) where \( u_i(\ldots), i = 1, 2 \) is the derivative w.r.t. the \( i \)th argument. The household’s discount factor is \( \beta < 1 \).

**Technology:** The firm has access to the production function, \( f(k, h) \) where \( k \) is its capital input and \( h \) is its labor input. The function \( f(\ldots) \) exhibits constant returns to scale, is twice differentiable, concave and increasing in both arguments. Standard Inada conditions apply. Capital depreciates with use at the rate \( \delta \).

**Endowments:** The Household is initially endowed with \( k_0 \) units of capital and each period has one unit of time to distribute between work and leisure.

**Institutions:** There is a government which has to meet expenditures, \( g_t \), each period. Government expenditures are thrown in the ocean. The government also has the power to tax. It will use two taxes, a time varying lump sum tax, \( \tau_t \), and a fixed proportional tax on labor income, \( \phi \). (Both taxes are levied on the household not the firm; \( g_t \) is assumed to be less than the productive capacity of the economy.)

(a) Write down and solve the Planner’s problem for this economy. (The solution should be in the form of a system of three difference equations in \( k_t, c_t \) and \( h_t \).)

(b) Write down and solve the problem faced by the representative household.

(c) Write down and solve the problem faced by the representative firm.

(d) Write down the market clearing conditions and the government budget constraint.

(e) Define a competitive equilibrium for this economy.

(f) Solve for a system of three difference equations in \( k_t, c_t \) and \( h_t \) that characterize equilibrium.

(g) Under what conditions on the tax code does the competitive equilibrium allocation coincide with the Planner’s allocation? Briefly explain.