Section 1. (Suggested Time: 45 Minutes) For 3 of the following 6 statements, state whether the statement is true, false, or uncertain, and give a complete and convincing explanation of your answer.

1. An increase in the real interest rate signals an increase in the likelihood of a recession in the near future.

2. If two countries have the same preferences and technologies, they must experience the same rate of per-capita output growth.

3. Monopolistic competition eliminates monetary neutrality in equilibrium.

4. The more a country saves, the faster it grows.

5. The contemporaneous correlation between consumption and real interest rates is negative.

6. A positive contemporaneous correlation between money and output proves that money is not neutral.
Section 2. (Suggested Time: 2 Hours, 15 minutes) Answer any 3 of the following 4 questions.

7. We are considering a two-person, two-period economy. Art’s preferences are given by:

\[ E_0 \{ \ln(c_0^A) + \ln(c_1^A) \} . \]

Each period, Art inelastically supplies 1 unit of labor, earning \( \pi_t^A w_t \), where \( \pi_t^A \) is Art’s labor productivity, and \( w_t \) is the market wage for effective labor. In period 0, Art’s labor productivity is 1. In period 1, however, Art’s labor productivity is stochastic: with probability \( \frac{1}{2} \), his labor productivity is \( 1 + \gamma \), \( 0 < \gamma < 1 \), and with probability \( \frac{1}{2} \), his labor productivity is \( 1 - \gamma \).

Art uses his period-0 income to purchase 2 assets: contingent claims that pay when Art’s period-1 productivity is high, selling at the price \( p_H \); and contingent claims that pay when Art’s period-1 productivity is low, selling at the price \( p_L \). It follows that in period 0, Art’s flow budget constraint is

\[ w_0 = c_0^A + p_H x_H^A + p_L x_L^A, \]

where \( x \) denotes quantities of contingent claims. Art’s period 1 flow budget constraint is

\[ \pi_1^A w_1 + x_H^A \times 1 \{ \pi_1^A = 1 + \gamma \} + x_L^A \times 1 \{ \pi_1^A = 1 - \gamma \} = c_1^A, \]

where \( 1 \{ \mathcal{A} \} \) is the indicator function that returns 1 if event \( \mathcal{A} \) occurs and 0 otherwise. We assume that \( \pi_1^A \) and \( w_1 \) are independent.

(a) Inserting the budget constraints, Art’s maximization problem can be written as

\[ \max_{\{x_H^A, x_L^A\}} \ln \left( w_0 - p_H x_H^A - p_L x_L^A \right) + \frac{1}{2} \ln \left( (1 + \gamma) w_1 + x_H^A \right) + \frac{1}{2} \ln \left( (1 - \gamma) w_1 + x_L^A \right). \]

The first-order conditions are

\[ \frac{1}{c_0^A} p_H = \frac{1}{2} \left( \frac{1}{(1 + \gamma) w_1 + x_H^A} \right), \]
\[ \frac{1}{c_0^A} p_L = \frac{1}{2} \left( \frac{1}{(1 - \gamma) w_1 + x_L^A} \right). \]

(b) Bart’s preferences and endowments are completely analogous to Art’s—you can replace A’s with B’s—except that when Art’s productivity is high, Bart’s is low, and when Art’s productivity is low, Bart’s is high. The aggregate production function is given by

\[ Y_t = \theta \left[ \pi_t^A \ell_t^A + \pi_t^B \ell_t^B \right]; \quad \theta > 0, \]

where \( \ell \) denotes labor supply. Output is non-storable.

1. We know that \( \pi_0^A + \pi_0^B = 2 \). Because Art’s and Bart’s productivities are offsetting, \( \pi_1^A + \pi_1^B = 2 \) in period 1 as well. Recalling that \( \ell_t^A = \ell_t^B = 1 \), we have

\[ Y_t = 2\theta \]
in both periods. With a price-taking firm, the wage for effective labor, $\pi_t^A \ell_t^A$ or $\pi_t^B \ell_t^B$, equals its marginal product, so that

$$w_t = \theta$$

in both periods.

2. Since output cannot be stored, it must be the case that

$$x_H^A + x_H^B = x_L^A + x_L^B = 0,$$

3. With complete markets, and convex preferences and technologies, we can find the equilibrium allocations using a social planner’s problem, and back out prices. Since aggregate output is the same in all periods, and Art and Bart are identical at time 0, we get an equal-weight allocation, where Art and Bart consume half of aggregate output in every period-1 state:

$$c_L^A = (1 - \gamma) \theta + x_L^A = (1 + \gamma) \theta + x_H^A = c_H^A = \theta,$$

so that

$$x_L^A = \gamma \theta, \quad x_H^A = -\gamma \theta.$$

Market clearing immediately implies that

$$x_H^B = -x_H^A = \gamma \theta, \quad x_L^B = -x_L^A = -\gamma \theta.$$

Note that Art and Bart consume $\theta$ units in period 0 as well. Inserting this into Art’s Euler equations, we get

$$\frac{1}{\theta} p_H = \frac{1}{2} \left( \frac{1}{\theta} \right),$$

$$\frac{1}{\theta} p_L = \frac{1}{2} \left( \frac{1}{\theta} \right),$$

so that $p_H = p_L = p$. Since Bart’s problem is symmetric to Art’s, it’s easy to confirm that his Euler equations are satisfied as well.

4. Note that we could construct a 1-period risk-free bond by acquiring 1 contingent claim under each of the aggregate states. This implies that

$$R^{-1} = p_H + p_L = 2p = 1.$$
8. We are considering an OLG model with CRRA preferences and a Cobb-Douglas production function. For an agent born in period \( t \), utility is given by:

\[
U_t = \frac{c_{1t}^{1-\theta} - 1}{1 - \theta} + \left( \frac{1}{1 + \rho} \right) \frac{c_{2t+1}^{1-\theta} - 1}{1 - \theta}.
\]

Each young agent inelastically supplies one unit of labor, earning \( w_t (1 - \tau_t) \), where \( \tau_t \) is the tax rate on labor income. The agent’s budget constraints in each period of life are given by:

\[
s_t + c_{1t} = w_t (1 - \tau_t),
\]

\[
c_{2t+1} = (1 + r_{t+1}) s_t,
\]

Aggregate output is given by:

\[
Y_t = AK_t^\alpha L_t^{1-\alpha}.
\]

We assume that population, \( L_t \), is fixed over time. The government spends all of its tax revenue each period such that:

\[
G_t = L_t w_t \tau_t.
\]

(a) Plugging in the budget constraints, the problem facing a young agent is

\[
\max_{s_t} \left( w_t (1 - \tau_t) - s_t \right)^{1-\theta} - 1 + \left( \frac{1}{1 + \rho} \right) ([1 + r_{t+1}] s_t)^{1-\theta} - 1,
\]

The first order condition for this problem is

\[
(w_t (1 - \tau_t) - s_t)^{-\theta} = \frac{1}{1 + \rho} ([1 + r_{t+1}] s_t)^{-\theta} [1 + r_{t+1}],
\]

or

\[
c_{1t}^{-\theta} = \frac{1}{1 + \rho} c_{2t+1}^{-\theta} [1 + r_{t+1}],
\]

\[
\frac{c_{2t+1}}{c_{1t}} = \left( \frac{1}{1 + \rho} [1 + r_{t+1}] \right)^{1/\theta}
\]

The last expression shows that \( \theta^{-1} \) gives the intertemporal elasticity of substitution for consumption: as \( \theta^{-1} \) increases, the pattern of consumption becomes more sensitive to the real interest rate.

(b) When \( \theta = 1 \), the Euler equation can be written as

\[
\frac{1}{w_t (1 - \tau_t) - s_t} = \left( \frac{1}{1 + \rho} \right) \frac{1}{s_t},
\]

or

\[
s_t = \frac{1}{1 + \rho} [w_t (1 - \tau_t) - s_t]
\]

\[
= \frac{1}{2 + \rho} w_t (1 - \tau_t).
\]
Imposing the budget constraints immediately shows that

\[ c_{1t} = \frac{1 + \rho}{2 + \rho} w_t (1 - \tau_t), \]
\[ c_{2t+1} = \frac{1 + r_{t+1}}{2 + \rho} w_t (1 - \tau_t). \]

Note that saving does not depend on the real interest rate. This is because with logarithmic preferences \((\theta = 1)\) and no period 2 labor income, the income and substitution effects of an interest rate change exactly offset.

(c) The representative price-taking firm solves

\[ \max_{K_t, L_t} AK_t^\alpha L_t^{1-\alpha} - w_t L_t - (r_t + \delta) K_t, \]

where \(\delta\) is the depreciation rate. Letting \(k_t \equiv K_t/N_t\) and \(\ell_t \equiv L_t/N_t\) denote per-worker quantities, we find that

\[ w_t = (1 - \alpha) Ak_t^\alpha, \]
\[ r_{t+1} + \delta = \alpha A k_{t+1}^{\alpha-1}, \]
\[ 1 + r_{t+1} = \alpha A k_{t+1}^{\alpha-1} - \delta. \]

In equilibrium, it must be case that

\[ k_{t+1} = s_t = \left( \frac{1}{2 + \rho} \right) w_t (1 - \tau_t) = \frac{A}{2 + \rho} k_t^\alpha (1 - \tau_t). \]

In a steady state, \(k_{t+1} = k_t = k_{SS}\), and this expression simplifies to

\[ k_{SS} = \left[ A \frac{1 - \alpha}{2 + \rho} (1 - \tau_{SS}) \right]^{1/\alpha}. \]

(d) Note that

\[ \tau_{SS} = gw_{SS}^{-1} = \frac{g}{(1 - \alpha) A_{SS}} \]

where \(g \equiv G_{SS}/L\) is per-capita government spending. For a fixed value of \(k_{SS}\), \(G_{SS} \uparrow \Rightarrow \tau_{SS} \uparrow\). This in turn implies that \(k_{SS} \downarrow\). (The fact that \(k_{SS} \downarrow \Rightarrow \tau_{SS} \uparrow\) only accentuates the result.) This differs from the Ramsey model, where in the absence of capital-tax-finance, the steady-state capital stock is independent of government spending. The reason why is that in the Ramsey model, infinite-lived agents can over time accumulate enough assets to support any capital stock, so that the steady-state capital stock is determined by the marginal product of capital and the rate of time preference. In the OLG model, all the saving is done by young agents whose only endowment is their labor. Because higher government spending raises labor taxes, it impoverishes these young agents, reducing their saving and the aggregate capital stock.
(e) In the context of OLG models, dynamic inefficiency occurs when the capital stock in the competitive equilibrium exceeds the golden rule capital stock: $k_{SS} > k_{GR}$, where $f'(k_{GR}) \equiv \delta$. Note that

$$f'(k_{SS}) = \alpha A \left( \frac{1 - \alpha}{2 + \rho} (1 - \tau_{SS}) \right)^{1/\alpha-1}$$

$$= \frac{\alpha}{1 - \alpha} \left( \frac{2 + \rho}{1 - \tau_{SS}} \right) > 0.$$ 

If $\delta = 0$, then $f'(k_{SS}) > f'(k_{GR}) \iff k_{SS} < k_{GR}$, and the economy is dynamically efficient. Even if $\delta > 0$, we have shown that $k_{SS}$ decreases in the labor tax rates, with $\lim_{\tau_{SS} \to 1} k_{SS} = 0$, so that for sufficiently high tax rates, the economy will be dynamically efficient.
9. We are considering a representative agent, money-in-the-utility-function (MIU) model. Utility is given by:

\[ U_t = \int_{t=0}^{\infty} e^{-\rho t} (\gamma \ln c_t + (1 - \gamma) \ln m_t) \, dt, \]

where \( c_t \) denotes real consumption and \( m_t = \frac{M_t}{P_t} \). The representative agent inelastically supplies one unit of labor services, earning \( w_t \). To simplify, assume that there is a single agent. Output is given by:

\[ Y_t = AK_t^\alpha. \]

Production is assumed to be perfectly competitive. The agent’s budget constraint is given by:

\[ \dot{K}_t + \left( \frac{1}{P_t} \right) \dot{M}_t = r_t K_t + w_t - c_t + x_t, \tag{FBC} \]

where \( r_t \) is the real interest rate and \( x_t \) is government transfers. A dot denotes a time derivative. Note that capital does not depreciate. The government’s budget constraint is given by:

\[ \left( \frac{1}{P_t} \right) \dot{M}_t = \theta m_t = x_t, \tag{GBC} \]

where \( \theta \equiv \frac{\dot{M}}{M} \) is constant.

(a) Placing money in the utility function can be justified as a short-cut for a shopping-time model, where money reduces the time spent buying and selling; these time reductions increase leisure and hence total utility. On the other hand, the MIU assumption can be criticized for being a reduced form. Policy analyses done with this specification can run afoul of the Lucas critique. It is also difficult to specify a generic utility function under the MIU assumption—what should be the sign of the cross-partial \( \partial^2 u / \partial c \partial m \)?

(b) First, note that

\[ \dot{m}_t = \frac{d}{dt} \left( \frac{M_t}{P_t} \right) \]

\[ = \left( \frac{1}{P_t} \right) \dot{M}_t - \frac{M_t}{P_t} \left( \frac{\dot{P}_t}{P_t} \right) \]

\[ = \frac{\dot{M}_t}{P_t} - m_t \pi_t, \]

where \( \pi_t \) is the inflation rate. Combining this result with equation (FBC), we get

\[ \dot{a}_t = \dot{K}_t + \dot{m}_t \]

\[ = \dot{K}_t + \frac{\dot{M}_t}{P_t} - m_t \pi_t \]

\[ = r_t K_t + w_t - c_t + x_t - m_t \pi_t \]

\[ = r_t (K_t + m_t) + w_t - c_t + x_t - m_t \pi_t - r_t m_t \]

\[ = r_t a_t + w_t - c_t + x_t - i_t m_t, \]

\[ i_t \equiv m_t + r_t. \]
(c) Using the revised budget constraint from above, the Hamiltonian for the consumer’s problem is

\[ H_t = e^{-\rho t} \{ \gamma \ln c_t + (1 - \gamma) \ln m_t + \lambda_t [r_t a_t + w_t - c_t + x_t - i_t m_t] \}. \]

The first order conditions are

\[ \frac{\gamma}{c_t} = \lambda_t, \tag{FOC1} \]
\[ (1 - \gamma) \frac{1}{m_t} = \lambda_t i_t, \tag{FOC2} \]
\[ \frac{d}{dt} (e^{-\rho t} \lambda_t) = -e^{-\rho t} \lambda_t r_t. \tag{FOC3} \]

(d) Equation (FOC3) can be rewritten as

\[ -\rho e^{-\rho t} \lambda_t + e^{-\rho t} \dot{\lambda}_t = -e^{-\rho t} \lambda_t r_t, \]

or

\[ \frac{-\dot{\lambda}_t}{\lambda_t} = r_t - \rho. \]

Combining this with equation (FOC1) yields

\[ -\frac{\dot{\lambda}_t}{\lambda_t} = \frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{EE} \]

Combining equations (FOC1) and (FOC2) yields the money demand equation

\[ (1 - \gamma) \frac{1}{m_t} = \gamma \frac{1}{c_t} i_t, \]

or

\[ m_t = \frac{1 - \gamma}{\gamma} \left( \frac{c_t}{i_t} \right). \]

(e) With perfect competition and constant returns to scale, it must be the case that

\[ r_t = \alpha A K_t^{\alpha - 1}, \]
\[ r_t K_t + w_t = Y_t = A K_t^\alpha, \]

Combining this result with equations (FBC) and (GBC) shows that

\[ \dot{K}_t + \left( \frac{1}{P_t} \right) \dot{M}_t = r_t K_t + w_t - c_t + x_t \]
\[ = A K_t^\alpha - c_t + \left( \frac{1}{P_t} \right) \dot{M}_t, \]

so that

\[ \dot{K}_t = A K_t^\alpha - c_t \tag{CA} \]
Substituting for \( r_t \), equilibrium Euler equation is

\[
\frac{\dot{c}_t}{c_t} = \alpha AK_t^{\alpha-1} - \rho \quad \text{(EE')} 
\]

The system formed by equations (CA) and (EE') is the same one as in the standard Ramsey model. It is therefore saddle-path stable.

(f) The steady-state capital stock is found by setting \( \dot{c}_t/c_t = 0 \) in equation (EE').

\[
K_{ss} = \left( \frac{\alpha A}{\rho} \right)^{1/(1-\alpha)} 
\]

Neither the level of money, \( M_t \), nor it’s growth rate, \( \theta \), affect the behavior of capital or consumption. Money is thus both neutral and superneutral.

(g) Since money has no social cost, the private cost of holding money, \( i_t = r_t + \pi_t \), should equal zero as well. It follows from the basic quantity theory that in a steady state, where output and real money demand are constant, \( \pi_t = \dot{M}_t/M_t = \theta \). Then \( i_t \) will equal zero when \( \theta = -r_t = -\rho \).
10. The preferences of the representative household are

\[ E_0 \left( \sum_{t=0}^{\infty} \beta^t \left[ \ln \left( C_t \right) - \Gamma L_t \right] \right), \]

The production function for this economy is given by

\[ Y_t = L_t^\eta, \quad 0 < \eta < 1, \quad \text{(PRF)} \]

Each period, the government gives the household the lump-sum transfer \( S_t \). The government pays for this transfer by with income taxes, which are levied on labor income and profits. The income tax rate, \( \tau_t \), obeys a balanced budget rule:

\[ \tau_t Y_t = S_t. \quad \text{(BB)} \]

The log of transfers follows an AR(1) process:

\[ \tilde{s}_t \equiv \ln \left( \frac{S_t}{S} \right) = \phi \tilde{s}_{t-1} + \varepsilon_t, \quad 0 \leq \phi < 1, \quad \text{(TS)} \]

where \( S \) is steady-state government spending and \( \{ \varepsilon_t \} \) is an exogenous stationary martingale difference sequence.

(a) We set about finding the competitive allocation.

1. Recall that the representative producer solves

\[ \max_{L_t \geq 0} \Pi_t = Y_t - W_t L_t, \]

so that the first order condition for profit maximization is

\[ \eta L_t^{\eta-1} = W_t, \quad \text{(PM)} \]

Note that the household’s flow budget constraint is

\[ C_t + K_{t+1} = (1 + r) K_t + (1 - \tau_t) \left[ W_t L_t + \Pi_t \right] + S_t, \quad \text{(FBC)} \]

so that the first order conditions for utility maximization are

\[ \frac{1}{C_t} W_t (1 - \tau_t) = \Gamma, \]

\[ \frac{1}{C_t} = \beta (1 + r) \frac{1}{C_{t+1}} = \frac{1}{C_{t+1}}. \quad \text{(EE)} \]

2. We have already found the Euler equation. Using equation (PM) to eliminate wages, the labor allocation condition becomes

\[ \frac{1}{C_t} \eta L_t^{\eta-1} (1 - \tau_t) = \Gamma. \]

Combining equations (BB) and (PRF) imply that

\[ \tau_t = S_t L_t^{-\eta}, \]
so that the labor allocation condition becomes
\[
\frac{1}{C_t} (1 - S_t L_t^{-\eta}) = \Gamma L_t^{1-\eta}. \quad \text{(LL)}
\]

The capital accumulation equation can be written as
\[
K_{t+1} = (1 + r) K_t + L_t^\eta - C_t, \quad \text{(CA)}
\]

This constraint can either be derived directly as a resource constraint, or found by combining the consumer’s, firm’s and government’s budget sets.

(b) Let’s examine the labor allocation condition more closely.

1. Logging both sides of equation (LL) yields
\[
-\ln (C_t) + \ln (1 - \exp (\ln S_t - \eta \ln (L_t))) = \ln \left( \frac{\Gamma}{\eta} \right) + (1 - \eta) \ln (L_t).
\]
Implicitly differentiating this expression yields
\[
-d \ln (C_t) + \frac{-\exp (\ln S_t - \eta \ln (L_t))}{1 - \exp (\ln S_t - \eta \ln (L_t))} \left[ -\eta d \ln (L_t) + d \ln S_t \right] = (1 - \eta) d \ln (L_t),
\]
so that
\[
-\hat{c}_t + \frac{P}{1 - P} \left[ \eta \hat{l}_t - \hat{s}_t \right] = (1 - \eta) \hat{l}_t, \quad P \equiv S/Y_{ss},
\]
Rearranging, we get
\[
\hat{l}_t \approx -\theta \left[ \lambda \hat{s}_t + \hat{c}_t \right], \quad \text{(LL')}
\]
\[
\theta \equiv \frac{1}{1 - \eta / \left[ 1 - P \right]},
\]
\[
\lambda \equiv \frac{1}{P - 1}.
\]
It follows from equation (PRF) that
\[
\exp (\hat{y}_t) = \frac{Y_t}{Y_{ss}} = \frac{L_t^\eta}{L_{ss}^\eta} = \exp \left( \eta \hat{l}_t \right)
\]
Logging both sides and inserting equation (LL’) yields
\[
\hat{y}_t \approx -\alpha \left[ \lambda \hat{s}_t + \hat{c}_t \right], \quad \text{(PRF')}
\]
\[
\alpha \equiv \eta \theta.
\]
2. As \( P \) goes from 0 to 1, \( \theta \) goes from \( 1 / (1 - \eta) > 0 \) to \( +\infty \), then from \( -\infty \) to 0. Over the same interval \( \lambda \) goes from 0 to \( +\infty \). These effects reflect the self-reinforcing nature of the tax system. A higher level of labor/output means that the tax rate needed to finance transfers is smaller. But a lower tax rate encourages workers to supply more labor, leading to more output. If this tax externality is high enough, it can reverse the usual labor market relationships. But the tax externality will be biggest when the government’s relative financing needs, given by \( P \), are highest.
(c) Implicitly differentiate equation (CA):

\[
\exp(\ln(K_{t+1})) = (1 + r) \exp(\ln(K_t)) + \exp(\ln(Y_t)) - \exp(\ln(C_t)),
\]

\[
\exp(\ln(K_{t+1})) d \ln(K_{t+1}) = (1 + r) \exp(\ln(K_t)) d \ln(K_t) + \exp(\ln(Y_t)) d \ln(Y_t) - \exp(\ln(C_t)) d \ln(C_t).
\]

so that

\[
\frac{K_{ss}}{K_{t+1}} \approx (1 + r) K_{ss} \hat{k}_t + Y_{ss} \hat{y}_t - C_{ss} \hat{c}_t,
\]

\[
\hat{k}_{t+1} \approx (1 + r) \hat{k}_t + \frac{Y_{ss}}{K_{ss}} \hat{y}_t - \frac{C_{ss}}{K_{ss}} \hat{c}_t.
\]

Since \(C_{ss}/K_{ss} = \psi\), and \(Y_{ss}/K_{ss} = \psi - r\), this equation can be written as:

\[
\hat{k}_{t+1} \approx (1 + r) \hat{k}_t + (\psi - r) \hat{y}_t - \psi \hat{c}_t.
\]

Then insert (PRF') to get

\[
\hat{k}_{t+1} \approx (1 + r) \hat{k}_t - (\psi - r) \alpha [\lambda \hat{s}_t + \hat{c}_t] - \psi \hat{c}_t,
\]

so that the log-linear approximation for the capital accumulation equation is

\[
\hat{k}_{t+1} = (1 + r) \hat{k}_t - \omega_1 \hat{s}_t - \omega_2 \hat{c}_t, \quad (\text{CA'})
\]

\[
\omega_1 = \alpha \lambda (\psi - r), \quad \omega_2 = \alpha (\psi - r) + \psi.
\]

(d) One can log-linearize the Euler equation and solve the resulting system (which also includes (TS) and (CA')) to express consumption as a function of capital and transfers

\[
\hat{c}_t = \chi \hat{k}_t - \mu \hat{s}_t, \quad (\text{CF})
\]

with

\[
\chi = \frac{r}{\omega_2}; \quad \mu = \frac{r \omega_1}{\omega_2 (1 - \phi + r)}.
\]

For low values of \(P\), \(\alpha\) is positive, so that \(\omega_2\) and \(\chi\) are positive as well. Just above the bifurcation point, \(P = 1 - \eta\), \(\alpha\) is an extremely large negative number, so that \(\omega_2\) and \(\chi\) are negative. As \(P\) approaches 1, \(\alpha\) approaches 0 from below, so that \(\omega_2\) and \(\chi\) are positive again.