Instructions. This exam consists of two parts. The first part contains extended problems; please answer any two of the three problems provided. The second part contains short-answer-type questions; please answer any three of the five questions provided. Be sure to write the numbers of the questions you answer on the covers of your bluebooks.

You have 180 minutes to complete this exam. You should allocate about one hour to each extended problem, and the remaining time to the shorter questions. Good luck!

Part I. Please answer any 2 of the following 3 questions.

1. (From McCallum, 1989.) Consider the following version of a stochastic growth model.

There are a fixed number of price-taking producers that solve

\[
\max_{K_t \geq 0, L_t \geq 0} \Pi_t = Y_t - R_t K_t - W_t L_t, \]

\[
Y_t = Z_t L_t^\alpha K_t^{1-\alpha}, \quad 0 < \alpha < 1, \quad \text{(PRF)}
\]

where: \( \Pi_t \) is profit; \( Z_t \) is exogenous total factor productivity; \( L_t \) is labor; \( K_t \) is capital; \( R_t \) is the rental rate for capital; and \( W_t \) is the real wage. Productivity, \( Z_t \), follows an AR(1) process in logs:

\[
z_t \equiv \ln (Z_t) = \phi z_{t-1} + \varepsilon_t, \quad 0 \leq \phi < 1, \quad \text{(TS)}
\]

where \( \{\varepsilon_t\} \) is an exogenous stationary martingale difference sequence. There is no population or productivity growth, and the population is normalized to 1, so that upper case letters denote intensive as well as aggregate quantities.

The representative household solves the following problem:

\[
\max_{\{C_t, K_t+1, L_t\}} \mathbb{E}_0 \left( \sum_{t=0}^{\infty} \beta^t [\ln(C_t) + \chi \ln(1 - L_t)] \right), \quad 0 < \beta < 1, \quad \chi > 0,
\]

s.t. \( C_t + K_{t+1} = R_t K_t + W_t L_t + \Pi_t, \quad \text{(CA)}\)

\[
\lim_{J \to \infty} \mathbb{E}_t \left( \frac{\beta^J}{C_{t+J}} K_{t+J+1} \right) = 0, \quad \text{(TVC)}
\]

\( L_t \in [0, 1]; \quad K_0 \text{ given}; \quad C_t \geq 0, \)

where \( C_t \) is consumption. The consumer takes prices and profits as given. Note that capital depreciates at a rate of 100 percent.
(a) Find the first order conditions for profit and utility maximization. Imposing equilibrium, eliminate $W_t$ and $R_t$ to find: the labor allocation condition; the Euler equation; and the capital accumulation equation.

(b) With “log-log” preferences, 100 percent depreciation and a Cobb-Douglas production function, we can find an analytical solution to this model. In particular, we will conjecture that the income and substitution effects of productivity shocks exactly cancel, so that labor hours are constant: $L_t = L, \forall t$. We will conjecture further that aggregate expenditures on consumption and investment are constant shares of output, so that

\[
\begin{align*}
C_t &= \Pi_1 Z_t L_t^\alpha K_t^{1-\alpha} = \Pi_1 Z_t L_t^\alpha K_t^{1-\alpha} \equiv \Pi_1 Z_t K_t^{1-\alpha}, \quad \text{(CRULE)} \\
K_{t+1} &= \Pi_2 Z_t L_t^\alpha K_t^{1-\alpha} = \Pi_2 Z_t L_t^\alpha K_t^{1-\alpha} \equiv \Pi_2 Z_t K_t^{1-\alpha}. \quad \text{(KRULE)}
\end{align*}
\]

Imposing these conjectures on your answers to part (a), show that

\[
\begin{align*}
\Pi_1 &= [1 - (1 - \alpha) \beta] L^\alpha, \\
\Pi_2 &= (1 - \alpha) \beta L^\alpha.
\end{align*}
\]

Using these results, show that

\[
L = \frac{\alpha}{\alpha + \chi [1 - (1 - \alpha) \beta]},
\]

thus confirming our conjecture.

(c) Let lower case variables denote logs, so that $x_t = \ln (X_t)$. Recalling equations (TS) and (PRF), use the results from part (b) to show that

\[
\begin{align*}
\kappa_{t+1} &= (\phi + 1 - \alpha) \kappa_t - \phi (1 - \alpha) \kappa_{t-1} + (1 - \phi) \pi_2 + \varepsilon_t, \\
\epsilon_{t+1} &= (\phi + 1 - \alpha) \epsilon_t - \phi (1 - \alpha) \epsilon_{t-1} + \pi_3 + \varepsilon_{t+1}, \\
y_{t+1} &= (\phi + 1 - \alpha) y_t - \phi (1 - \alpha) y_{t-1} + \pi_4 + \varepsilon_{t+1},
\end{align*}
\]

where $\pi_3$ and $\pi_4$ are complicated constants that do not need to be explicitly solved for. (Note the timing of the innovation $\varepsilon$!)

(d) Let’s consider the economy’s response to a technology shock. Let “~” denote logged deviations from steady state values.

1. Suppose that $\bar{y}_0 = \hat{k}_0 = \hat{z}_0 = \varepsilon_0 = 0, \varepsilon_1 = 1, \text{ and } \varepsilon_2 = \varepsilon_3 = \ldots = 0$, and that $\phi = 0.9$ and $\alpha = 0.7$. Using your answer to part (c), trace out the effects of this shock by filling in the following table.

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<th>$t$</th>
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2. Does this economy exhibit any propagation mechanisms? If so, briefly explain how they work.
3. Does the impulse response function for output resemble the impulse response function observed in the data? Briefly explain the mechanics that make it match or not match.
4. Recall that capital depreciation is thought to be around 6-8% per year, rather than the 100% that we have assumed here. Would changing the depreciation rate change the shape of the impulse response function for output? Briefly explain.

2. Consider the following infinite-horizon economy consisting of a representative consumer and a representative firm. The consumer has preferences given by
\[\sum_{t=0}^{\infty} \beta^t [u(c_t) + v(\ell_t)], \quad 0 < \beta < 1,\]
where \(c_t\) is consumption and \(\ell_t\) is leisure. The consumer is endowed with one unit of time in each period which can be allocated between work and leisure. The representative firm has access to a production technology
\[y_t = z_t n_t\]
where \(y_t\) is output, \(n_t\) is labor, and \(z_t\) determines aggregate labor productivity. There is a government that purchases \(g_t\) units of output in period-\(t\) and throws them into the ocean. Purchases are financed through lump-sum taxation and by issuing one-period bonds. The period-\(t\) government budget constraint is
\[g_t + (1 + r_t) b_{t+1}^g = \tau_t + b_{t+1}^p\]
where \(b_{t+1}^p\) is the quantity of bonds issued by the government in period \(t\), \(\tau_t\) is period-\(t\) taxes, and \(r_t\) is the real interest rate on a bond purchased in period \(t - 1\). The representative consumer is also permitted to issue bonds. Letting \(b_{t+1}^p\) denote the quantity of bonds purchased by the consumer in period \(t\), the period-\(t\) budget constraint of the consumer is written
\[c_t + b_{t+1}^p = w_t (1 - \ell_t) - \tau_t + (1 + r_t) b_t^p\]
where \(w_t\) is the real wage. Note that government bonds and private bonds are perfect substitutes. Assume that the discounted quantity of debt is zero in the limit:
\[\lim_{T \to \infty} \frac{b_T^p}{\prod_{i=1}^{T-1} (1 + r_i)} = 0.\]
Also, assume that private and public debt is initially zero: \(b_0^p = b_0^g = 0\). Note that the time path of taxes \(\{\tau_t\}_{t=0}^\infty\) satisfies the intertemporal government budget constraint.

(a) Derive a condition that implicitly determines the Pareto-optimal quantity of leisure in period \(t\) as a function of period-\(t\) productivity and period-\(t\) government purchases.
(b) Suppose that \( g_t = g > 0 \) and \( z_t = z > 0 \) for \( t = 0, 1, 2, \ldots \). Determine the competitive equilibrium interest rate in each period.

(c) Suppose that \( z_t \) increases temporarily with \( g_t = g \) as in part (b). That is, suppose that \( z_t = \bar{z} > z \) for \( t = T \) and \( z_t = z \) for \( t = 0, 1, 2, \ldots, T - 1, T + 1, T + 2, \ldots \). How does the time path of interest rates \( \{r_t\}_{t=0}^{\infty} \) differ from that of part (b)? Be as specific as you can. (Hint: It is useful to draw a diagram that illustrates the qualitative features of the time path.)

(d) Now, suppose that \( z_t \) increases permanently with \( g_t = g \) as in parts (b) and (c). That is, suppose that \( z_t = \bar{z} > z \) for \( t = T, T + 1, T + 2, \ldots \) and \( z_t = z \) for \( t = 0, 1, 2, \ldots, T - 1 \). How does the time path of interest rates \( \{r_t\}_{t=0}^{\infty} \) differ from that of part (b)? Be as specific as you can. (Again, it is useful to make a diagram that illustrates the qualitative features of the time path.)

(e) What is the economic intuition behind the differences in parts (c) and (d)?

3. Consider the following variant of Lucas tree model. There are two types of trees: apple trees, which will be indexed by “a”; and banana trees, which will be indexed by “b”. Each tree gives off \( d^i_t \), \( i \in \{a, b\} \), units of non-storable fruit (dividends), and we will assume that consumers are indifferent between consuming a unit of apple fruit or a unit of banana fruit. The preferences of the representative consumer are

\[
E_0 \left( \sum_{t=0}^{\infty} \beta^t \ln (c_t) \right), \quad 0 < \beta < 1,
\]

where \( c_t \) denotes total consumption of apples and bananas. Let \( p^a_t \) (\( p^b_t \)) denote the price at time \( t \) of a title to all future dividends from an apple (banana) tree, and let \( s^a_t \) (\( s^b_t \)) be the number of apple (banana) trees that the consumer owns at time \( t \). The economy starts off with each household owning one apple and one banana tree.

The dividend processes for the two trees are given by

\[
\begin{align*}
(d^a_t - d) &= \phi (d^a_{t-1} - d) + \varepsilon^a_t, \\
(d^b_t - d) &= \phi (d^b_{t-1} - d) + \varepsilon^b_t,
\end{align*}
\]

with \( 0 \leq \phi < 1 \) and \( d > 0 \). We will further assume that \( \{\varepsilon^a_t\} \) is an i.i.d. white noise process and that

\[
\varepsilon^b_t = \alpha \varepsilon^a_t,
\]

with \(-1 < \alpha < 0\). This means that positive shocks to the apple “crop” are accompanied by negative shocks to banana crop, but that the effect on the apple crop dominates aggregate output. The values of \( d \), \( \phi \) and \( \alpha \), and the process for \( \{\varepsilon^a_t\} \) are such that both dividends are strictly positive in each period.

(a) Let \( S_t \) denote the state vector of the aggregate economy. It turns out that \( S_t \) can be reduced to a single variable. What is this variable? Justify your choice, both formally and intuitively.
(b) Assume that the state variable $S_t$ follows a Markov chain described by the density $f(S_{t+1}, S_t)$. Let $q(S', S)$ be the kernel used to price the one-step-ahead contingent claim that delivers one unit of the consumption good when $S_{t+1} = S'$ and $S_t = S$, and let $y(S')$ denote the “purchasing kernel” that identifies how many state-$S'$ contingent claims the consumer purchases at time $t$. Continuing, let $x_t$ denote the consumer’s financial resources, which she allocates between contingent claims, stocks and consumption. Using this notation, write down the consumer’s problem in recursive form, and find the first order conditions.

(c) Derive the equilibrium pricing function for an apple stock, $p^a_t = p^a(S_t)$. (This function should not include any expected future prices.) Be sure to point out any restrictions that you impose in deriving this function. Replacing $d^a_t$ with $d^b_t$, write down the pricing function for a banana stock, $p^b_t = p^b(S_t)$.

(d) Let $R^{-1}_t = R^{-1}(S_t)$ be the time-$t$ price of a risk-free discount bond that pays one unit of consumption at time $t+1$ under any state. Use the pricing kernel to find $R^{-1}(S_t)$.

(e) Define the return on stocks, $R^i_t$, by

$$R^i_t = \frac{p^i_{t+1} + d^i_{t+1}}{p^i_t}, \quad i \in \{a, b\},$$

and define the equity premium, $e^i_t$, by $e^i_t = E_t(R^i_t) - R_t$. Find $e^i_t$ as function of $R^i_t$, $R_t$ and $S_{t+1}$. Intuitively, would you expect $e^a_t$ or $e^b_t$ to be bigger? (Hint: You might want to recall the Capital Asset Pricing Model.) What happens to the premia as $\alpha$ approaches $-1$?

Part II. Please answer any 3 of the following 5 questions.

4. Consider the following issues of national income accounting.

(a) In the national income and product accounts, is a new automobile bought by a consumer treated as a consumption good or as an investment good? Describe and explain the treatment used.

(b) If the automobile were instead treated in the other way, how would national product be affected? Explain what would be the effect on net national product and gross national product.

(c) A recent change in the national income and product accounts is that the purchase of microcomputer software by a business is now treated as investment, rather than as the purchase of an intermediate good. How will this change affect net national product and gross national product? Contrast the effect in the short run and in the long run.

5. Birthdays and Weddings. It is well known that the amount of resources people are willing to expend on social occasions such as weddings or birthday parties depend on the resources that other people spend. Consider the following two models. In both cases, there are $N$ identical agents. Each individual solves
\[
\max_{e_i \in [0, E]} \ln (e_i + e_0 + \alpha e) + (E - e_i),
\]
where \(e_i\) is individual \(i\)'s expenditures on the social event, \(e\) is the average expenditure of other individuals in the economy, and \(E > 1\) is individual \(i\)'s resources. The goal of this problem is to analyze different values of \(\alpha\) and \(e_0\).

(a) Suppose first that \(e_0 = 1\) and \(\alpha = -1\), so that individuals view these events as competitions. Show that any \(e \in [0, E]\) can be supported as a symmetric Nash equilibrium. Are equilibria with higher levels of \(e\) socially preferable? Briefly discuss, using the concepts developed by Cooper and John (1988).

(b) Now suppose that \(0 < e_0 < 1\) and \(\alpha = 1\), so that individuals view expenditures as substitutable across social events. Show that there is a unique symmetric Nash equilibrium, and that it is socially suboptimal. Briefly discuss, again using the concepts developed by Cooper and John.

6. Consider a \(T\)-period economy with a single agent. The agent has a utility function of the form

\[
E_0 \sum_{t=1}^{T} \beta^{t-1} \left[ \ln c_t + \ln (1 - n_t) \right]
\]

where \(c_t\) is period-\(t\) consumption and \(n_t\) is the fraction of the period worked by the agent in period \(t\). The agent owns a technology that converts time into output according to the formula

\[
y_t = z_t n_t^\alpha, \quad 0 < \alpha < 1,
\]

where \(y_t\) is period-\(t\) output and \(z_t > 0\) is an i.i.d. random variable with positive support. There is a government that imposes an income tax \(\tau_t\) in each period \(t\). The agent’s after-tax income is therefore \((1 - \tau_t) z_t n_t^\alpha\). The government throws the proceeds of the tax into the ocean.

(a) Derive a formula determining the agent’s work effort \(n_t\) as a function of the state \(z_t\) and the tax rate \(\tau_t\). Graph the government’s revenue from the tax versus the tax rate (holding the state constant).

(b) Now suppose that the government chooses a tax rate at the beginning of each period to raise a constant amount of revenue \(G\) in each period. Note that the government takes into account the agent’s response to its choice of tax rate [from part (a)]. Derive a formula determining the government’s choice of \(\tau_t\) as a function of \(z_t\) and \(G\).

(c) An econometrician observes the entire history of this economy (as determined by the realized vector of technology shocks) and graphs the time-series of government tax revenue versus time-series of the tax rate and gets a horizontal line. He writes an article which concludes that government revenue is insensitive to the rate of taxation. Briefly critique his article (i.e. what did he get wrong?).
7. A standard formulation behind the Solow residual measure of disembodied technological change is

\[ g_Y = \alpha g_K + \beta g_L + g_A, \]

where \( g_Y \), \( g_K \), \( g_L \) and \( g_A \) are the growth rates of output (\( Y \)), capital (\( K \)), labor (\( L \)) and total factor productivity (\( A \)), respectively. The goal of this problem is consider the weights \( \alpha \) and \( \beta \).

(a) In Solow’s standard formulation, \( \alpha \) and \( \beta \) equal the national income shares of capital and labor. Derive this result.

(b) Using the standard formulation, compute the Solow residual, based just on the following information. The labor force is 10, and the real wage is 10; find the labor income. Profit (capital income) is half of national income, and labor income is half. During the year, labor rises by 12%, capital rises by 18%, and real output rises by 20%. What is the measured rate of technological change?

(c) Alternatively, suppose that the economy is not perfectly competitive, but there is significant monopoly. With monopoly, the output price is a markup over input cost, and the weights \( \alpha \) and \( \beta \) are given by the cost shares of capital and labor. Show this result, using the following. First, total cost is labor cost plus capital cost,

\[ c = wL + rK, \]

where \( c \) is total cost, \( w \) is the real wage, and \( r \) is the real interest rate. Second, cost minimization by the firm implies that the marginal cost equals the input price divided by the marginal product,

\[ mc = \frac{w}{mp_L} = \frac{r}{mp_K}, \]

where \( mc \) is the marginal cost, \( mp_L \) is the marginal product of labor, and \( mp_K \) is the marginal product of capital. Third, with constant returns to scale, the average cost \( ac \) equals marginal cost \( mc \).

(d) Using the result in (c), recompute the Solow residual, based on the following. Half of the profit is the actual cost of capital, whereas the other half is a monopoly markup over total input cost. What is the measured rate of technological change? Explain intuitively why this value is higher or lower than what you calculated in (b).

8. Suppose that the government can increase the growth rate of per capita consumption by increasing subsidies to education, thereby encouraging individuals to increase the amount of time spent in educational activities, and hence, the rate at which they accumulate human capital. Ignoring all distributional issues, discuss whether the government should increase subsidies to education.