1. The Laffer Curve states that as the tax rate rises from a low level, tax revenue initially rises, reaches a peak, and then falls. This question asks you to derive a steady-state Laffer Curve for an income tax. Utility for the representative agent is given by

\[ U = \sum_{t=0}^{\infty} \beta^t \left[ \ln C_t + \eta \ln G_t + \chi \frac{(1 - \ell_t)^{1-\gamma}}{1-\gamma} \right], \quad 0 < \beta < 1, \quad \eta, \chi > 0 \]  

(U)

where \( C_t \) represents consumption, \( G_t \) is government spending, and \( 1 - \ell_t \) is leisure, with unity as the time endowment and \( \ell_t \) as labor supply. Output is produced with capital \( (K_t) \) and labor according to:

\[ Y_t = K_t^{\alpha} \ell_t^{1-\alpha}. \]  

(Y)

The government’s budget constraint is given by

\[ G_t = \tau_t Y_t. \]  

(G)

\( \tau_t \) is the income tax rate. The agent’s budget constraint is

\[ K_{t+1} = K_t (1 - \delta) + (r_t K_t + w_t \ell_t) (1 - \tau_t) - C_t, \]  

(ABC)

where \( \delta \) is the rate of depreciation of capital. (Recall that \( Y_t = r_t K_t + w_t \ell_t \) since production is constant returns to scale.)

(a) Derive the first order conditions and write them as an Euler equation, labeled (EE) and a labor leisure choice labeled (LL).

\[ \frac{1}{C_t} = \lambda_t \]

\[ -\lambda_t + \beta \lambda_{t+1} [(1 - \tau) r_{t+1} + 1 - \delta] = 0 \]

Substituting for the multipliers yields the Euler equation as

\[ \frac{1}{C_t} = \frac{\beta}{C_{t+1}} [(1 - \tau) r_{t+1} + 1 - \delta] \]  

(EE)

\[ -\chi (1 - \ell_t)^{-\gamma} + \lambda_t (1 - \tau_t) w_t = 0 \]

Substituting for the multipliers yields the labor leisure tradeoff as

\[ \frac{(1 - \tau_t) w_t}{C_t} = \chi (1 - \ell_t)^{-\gamma}. \]  

(LL)

(b) Write the equation for the resource constraint in steady state.

\[ Y = C + G + \delta K \]  

(RC)
(c) Write the profit-maximizing problem for a representative firm using capital and labor to produce output. Take first order conditions with respect to $K_t$ and $\ell_t$.

\[
\begin{align*}
\text{Profits} &= Y_t - (r_t K_t + w_t \ell_t) \\
&= K_t^{\alpha} \ell_t^{1-\alpha} - (r_t K_t + w_t \ell_t)
\end{align*}
\]

\[
\frac{\partial \text{Profits}}{\partial K_t} = \alpha \frac{Y_t}{K_t} - r_t = 0;
\]

\[
\frac{\partial \text{Profits}}{\partial \ell_t} = (1 - \alpha) \frac{Y_t}{\ell_t} - w_t = 0 \quad \text{(FOCfirm)}
\]

(d) List the equations required to solve for the endogenous variables in equilibrium. Includes all first order conditions as well as agent and government budget constraints.

(e) Denote steady-state values by dropping the time subscripts and solve for equilibrium quantities. How does the tax rate affect $\ell$ and $Y$? [Hint: Begin by solving for $\frac{Y}{K}$ and $\frac{C}{K}$. Then express the labor-leisure choice as a non-linear function of $\ell$ and parameters. Since this is non-linear in $\ell$, you do not have to actually solve the equation for $\ell$. Solve for levels of $Y, C,$ and $K$ as a function of $\ell$ and parameters including $\tau$.]

\[
\frac{Y}{K} = \frac{1 - \beta (1 - \delta)}{\alpha (1 - \tau)}
\]

from equations (EE) and (FOCfirm). Then use the resource constraint and the solution above to yield

\[
\frac{C}{K} = \frac{Y - G}{K} - \delta
= \frac{Y}{K} (1 - \tau) - \delta
= \frac{1 - \beta (1 - \delta)}{\alpha} - \delta
\]

Now use the labor leisure choice together with the firm’s first order condition on the wage to yield

\[
\frac{(1 - \tau) (1 - \alpha) Y}{C} = \chi \ell (1 - \ell)^{-\gamma}
\]

Use the equations above to write

\[
\frac{Y}{C} = \frac{Y}{K} \times \frac{K}{C}
= \frac{1 - \beta (1 - \delta)}{\alpha (1 - \tau)} \times \frac{\alpha}{1 - \beta (1 - \delta) - \alpha \delta}
\]

\[
= \frac{1 - \beta (1 - \delta)}{(1 - \tau) [1 - \beta (1 - \delta) - \alpha \delta]}
\]

Substituting into the labor leisure decision yields

\[
\frac{(1 - \alpha) [1 - \beta (1 - \delta)]}{1 - \beta (1 - \delta) - \alpha \delta} = \chi \ell (1 - \ell)^{-\gamma}
\]

Note that steady-state labor and leisure are not affected by the tax rate. Continue to solve for levels using $\ell$ as the steady-state value of labor. Use the production function to solve for $K$.

\[
\begin{align*}
Y &= K^{\alpha} \ell^{1-\alpha} \\
\frac{Y}{K} &= \left( \frac{K}{\ell} \right)^{\alpha - 1} \\
\frac{K}{\ell} &= \left( \frac{Y}{K} \right)^{\frac{1}{1-\tau}}
\end{align*}
\]
Substituting for \( \frac{Y}{K} \)

\[
K = t \left( \frac{\alpha(1-\tau)}{1-\beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}
\]

Using the production function

\[
Y = t \left( \frac{\alpha(1-\tau)}{1-\beta(1-\delta)} \right)^{\frac{\alpha}{1-\alpha}}
\]

(f) Tax revenue is given by \( \tau Y \). Explain how tax revenue varies as \( \tau \) rises from zero to something close to unity. Explain how capital and labor react to the tax rate generating the response of income to the tax rate. Do we have a Laffer Curve? If so, derive the tax-revenue maximizing value of the tax rate.

\[
\frac{\partial}{\partial \tau} \left[ \tau \left(1-\tau\right)^{\frac{\alpha}{1-\alpha}} \right] = (1-\tau)^{\frac{\alpha}{1-\alpha}} - \tau \frac{\alpha}{1-\alpha} (1-\tau)^{\frac{\alpha}{1-\alpha}-1} = 0
\]

\[
(1-\tau)^{\frac{\alpha}{1-\alpha}} \left[ 1 - \frac{\tau}{1-\tau} \frac{\alpha}{1-\alpha} \right] = 0
\]

\[
(1-\tau)^{\frac{\alpha}{1-\alpha}} \left(1 - \alpha \frac{\alpha}{1-\alpha} \right) = \alpha \tau
\]

\[
\frac{\tau}{1-\alpha} = \alpha \tau
\]

\[
\tau = 1 - \alpha
\]

Capital reacts negatively to the tax rate and labor does not change.

2. **Endowment model with "cash" and "credit" goods (inspired by Lucas and Stokey [1987])**

Consider the following economy:

**Time**: discrete, infinite horizon, \( t = 1, 2, \ldots \)

**Demography**: A single representative household (takes prices as given)

**Preferences**: The household likes to consume two goods. Good 1 is a cash good which requires cash in advance of the period in which the purchase is made. Good 2 is a "credit" good which does not require cash for its acquisition. The household gets utility \( u(c_{1,t}, c_{2,t}) \) from consumption of \( c_{i,t} \) units of good \( i = 1, 2 \) at time \( t \). Assume that \( u \) is strictly increasing in both arguments and strictly concave with

\[
\lim_{c_{1,t} \to 0} u_1(c_{1,t}, c_{2,t}) = \lim_{c_{2,t} \to 0} u_2(c_{1,t}, c_{2,t}) = \infty
\]

The household’s discount factor is \( \beta < 1 \).

**Endowments**: In period \( t \), the household received \( y_t \) units of generic perishable consumption good and can choose freely how much of it to take to the market as a cash good, \( y_{1,t} \) and how much to take to the market as a credit good, \( y_{2,t} \) (\( y_{1,t} + y_{2,t} = y_t \)). The usual assumption that you cannot consume your own output applies so you have to bring the goods to market. **Note**: This assumption means that both goods will always trade at the same price, \( p_t \). (If not, people would only bring the higher priced good to market and our assumptions on preferences rule out such a corner solution.)

**Institutions**: There is a government that issues currency so that the stock at time \( t \) is \( H_t \), the time \( t \) growth rate of the money supply is \( \sigma_t \) so that \( H_{t+1} = (1+\sigma_t)H_t \). The money is distributed (or collected if \( \sigma_t < 0 \)) lump sum so that household’s period \( t \) transfer
(tax) is \( \tau_t \). The household can only acquire the cash good using money which has to have been acquired in the previous period. The credit good can be acquired on credit in that the household can use the current period’s proceeds from sales to obtain it. This means they can effectively swap their own goods for credit goods and do not need to go through cash.

(a) Write down the problem faced by the representative household

In recursive form the household will solve:

\[
V(M_t) = \max_{c_{1,t}, c_{2,t}, M_t} \{ u(c_{1,t}, c_{2,t}) + \beta V(M_{t+1}) \} \\
\text{subject to:} \quad y_{1,t} + y_{2,t} + \frac{M_t}{p_t} = c_{1,t} + c_{2,t} + \frac{M^d_t}{p_t} \\
\text{and:} \quad \frac{M_t}{p_t} \geq c_{1,t} \quad \text{(CIA)}
\]

where \( M_t \) are nominal money holdings, \( M^d_t \) is the end-of-period nominal money demand and \( M_{t+1} = M^d_t + \tau_t \).

(b) Obtain the first-order, complementary slackness and envelope conditions.

The appropriate Lagrangian is

\[
\mathcal{L} = u(c_{1,t}, c_{2,t}) + \beta V(M_{t+1}) + \lambda_t \left[ y_t + \frac{M_t}{p_t} - c_{1,t} - c_{2,t} - \frac{M^d_t}{p_t} \right] + \gamma_t \left[ \frac{M_t}{p_t} - c_{1,t} \right]
\]

F.O.C.s:

\[
\begin{align*}
    c_{1,t} & : \quad u_1(c_{1,t}, c_{2,t}) - \lambda_t - \gamma_t = 0 \\
    c_{2,t} & : \quad u_2(c_{1,t}, c_{2,t}) - \lambda_t = 0 \\
    M^d_t & : \quad \beta V'(M_{t+1}) - \frac{\lambda_t}{p_t} = 0
\end{align*}
\]

Complementary slackness:

\[
\gamma_t \left[ \frac{M_t}{p_t} - c_{1,t} \right] = 0, \quad \gamma_t \geq 0
\]

Envelope:

\[
V'(M_t) = \frac{\lambda_t + \gamma_t}{p_t}
\]

(c) Obtain an intertemporal (Euler type) condition that relates the marginal utility of consuming good 2 today and good 1 tomorrow. Explain this expression.

\[
V'(M_{t+1}) = \frac{\lambda_{t+1} + \gamma_{t+1}}{p_{t+1}} = \frac{u_1(t+1)}{p_{t+1}}
\]

Into \( M^d_t \) equation yields

\[
\beta \frac{u_1(t+1)}{p_{t+1}} = \frac{\lambda_t}{p_t} = \frac{u_2(t)}{p_t}
\]
so

\[ u_2(c_{1,t}, c_{2,t}) = \beta \frac{p_t}{p_{t+1}} u_1(c_{1,t+1}, c_{2,t+1}) \]

If I give up a unit of consumption of the credit good today I lose \( u_2(c_{1,t}, c_{2,t}) \) utils and save \( p_t \) units of money that I can use to acquire the cash good tomorrow. To do so I need to buy the cash good at the price \( p_{t+1} \) so that I get \( \frac{p_t}{p_{t+1}} \) units of the cash good tomorrow for every unit of credit good I give up today. The utility I get tomorrow is then \( \frac{p_t}{p_{t+1}} u_1(c_{1,t+1}, c_{2,t+1}) \) which I have to discount by \( \beta \). For the consumption values to be optimal the marginal cost of this switch has to equal the marginal benefit from it.

(d) Write down the market-clearing conditions and the government budget constraint. Define a competitive equilibrium.

Market clearing conditions:

- Goods: \( y_t = c_{1,t} + c_{2,t} \)
- Money: \( M_t = H_t \)
- GBC: \( H_{t+1} = H_t + \tau_t \Rightarrow \tau_t = \sigma_t H_t \)

**Definition:** A competitive equilibrium is an allocation, \( \{c_{1,t}, c_{2,t}, M_t\}_{t=1}^{\infty} \) a sequence of prices, \( \{p_t\}_{t=1}^{\infty} \) and a sequence of transfers \( \{\tau_t\}_{t=1}^{\infty} \) such that given transfers and prices, the allocation solves the household’s problem; markets clear and GBC holds for all \( t \).

(e) Now suppose \( \sigma_t = \sigma \) and \( y_t = y \) for all \( t \) and that we are in a steady state so that \( c_{1,t} = c_1 \) and \( c_{2,t} = c_2 \) for all \( t \). What does this mean for the real return on money, \( p_t/p_{t+1} \) and the optimality of the Friedman rule? Explain your answer.

The real return on money will be given by

\[ \frac{p_t}{p_{t+1}} = \frac{1}{1 + \sigma}. \]

The Friedman rule sets the growth rate of money, \( \sigma \), so that \( 1 + \sigma = \beta \). When that happens the return on money exactly offsets the discount rate and the CIA constraint no longer binds. For the household we get

\[ u_2(c_1, c_2) = u_1(c_1, c_2) \]

which specifies the optimal consumption pattern in absence of any cash-in-advance requirement. The Friedman rule therefore represents optimal monetary policy.

3. Consider a two-period world with **uncertainty**. Utility for the representative agent is given by

\[ U = u(C_1) + \beta u(C_2) \quad \beta < 1 \]

Production in each period \( t = 1, 2 \) is given by

\[ Y_t = A_t K_t^\alpha L_t^{1-\alpha}, \]

where we assume

\[ A_1, K_1 \text{ are given} \quad L_1 = L_2 \]

\[ A_2 = (1 + g) A_1 \text{ where } g \text{ is normally distributed with } E(g) > 0 \]

Agents believe that they can save and dissave using capital and risk-free bonds, denoted by \( B \) (view \( B \) as an asset). Risk-free bonds earn an endogenous interest rate of \( r \). Assume that capital depreciates at rate \( \delta \) and that the agent begins period 1 with no risk-free bonds.
(a) Write equations for the agent’s period 1 and period 2 budget constraints, where the agent knows there is no period 3. Assume that the agent chooses $K_2$ and $B_2$ to maximize expected utility and write Euler equations, one using bonds, and one using capital.

**Period 1:**

$$K_2 + B_2 = (1 - \delta) K_1 + A_1 K_1^{\alpha} - C_1$$

**Period 2:**

$$0 = (1 - \delta) K_2 + A_2 K_2^{\alpha} + (1 + r) B_2 - C_2$$

**Bond Euler:**

$$u'(C_1) = \beta (1 + r) Eu'(C_2)$$

**Capital Euler:**

$$u'(C_1) = \beta E \left[ \left( 1 + \frac{aY_2}{K_2} - \delta \right) u'(C_2) \right]$$

(b) Consider four alternative specifications of utility:

- **Linear:**
  $$u(C) = aC$$

- **Quadratic:**
  $$u(C) = aC - \frac{b}{2} C^2 \quad \text{for} \quad C < \frac{a}{b}$$

- **Log:**
  $$u(C) = \ln C$$

- **CES:**
  $$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \quad \text{for} \quad \sigma > 0$$

Define certainty equivalence. Which utility functions exhibit certainty equivalence? Which bond Euler equations have certainty equivalence?

**Definition:**

$$u[E(C)] = E[u(C)]$$

**Linear:**

$$aE(C) = E[aC] \quad \text{yes}$$

**Quadratic:**

$$aE(C) - \frac{b}{2} [E(C)]^2 \quad \text{compared with} \quad E[aC] - \frac{b}{2} EC^2$$

$$[E(C)]^2 \neq EC^2$$

**Log:**

$$\ln[E(C)] \neq E[\ln C]$$

**CES:**

$$\frac{(EC)^{1-\sigma}}{1-\sigma} \neq E \left[ \frac{C^{1-\sigma}}{1-\sigma} \right] \quad \text{for} \quad \sigma > 0$$

The only utility function which exhibits certainty equivalence is Linear. For the bond Euler equation to have certainty equivalence,

$$u'[E(C_2)] = E[u'(C_2)]$$

For linear utility, marginal utility is constant, so it has certainty equivalence. List marginal utility for the remaining three and evaluate whether marginal utility exhibits certainty equivalence.

**Quadratic:**

$$a - bEC = E[a - bC] \quad \text{yes}$$

**Log:**

$$\frac{1}{EC} \neq E \frac{1}{C}$$

**CES:**

$$(EC)^{-\sigma} \neq E \left[ C^{-\sigma} \right]$$

The Euler equation has certainty equivalence only for the linear and quadratic cases.
(c) Linearize the marginal utility of consumption in the CES case about $C_2 = C_1$. Evaluate whether the linearized bond Euler equation has certainty equivalence. What are the implications of your answer for whether or not the linearized RBC model has certainty equivalence?

Linearize $C_2^{-\sigma}$ about $C_2 = C_1$

Take natural log

$$-\sigma \ln C_2$$

Differentiate with respect to $C_2$ and evaluate at $C_2 = C_1$.

$$-\sigma \frac{1}{C_1} dC_2 \approx -\sigma \frac{C_2 - C_1}{C_1} = -\sigma \hat{C}_2$$

Since

$$-\sigma E \left( \hat{C}_2 \right) = E \left[ -\sigma \hat{C}_2 \right],$$

and this is the only place in the bond Euler equation in which the expectation enters, the linearized bond Euler equation will exhibit certainty equivalence. Since linear equations have certainty equivalence, the linearized RBC model has certainty equivalence.

(d) Define risk aversion and evaluate which utility functions are characterized by risk aversion. What characteristic of the utility function characterizes risk aversion?

Definition: $U \left[ E \left( C \right) \right] > E \left[ U \left( C \right) \right].$

In words, agents prefer the certain outcome at the mean to the uncertain outcome. We saw above that quadratic, log, and CES all fail certainty equivalence. And the inequality goes in the correct direction for all to have risk aversion. For an agent to be risk averse, the derivative of marginal utility with respect to consumption (the second derivative of the utility function) must be negative.

(e) Define precautionary saving and evaluate which utility functions are characterized by precautionary savings. How does precautionary savings differ from risk aversion? Precautionary savings is savings to offset uncertainty about the future. It requires a positive third derivative. Risk aversion only requires a negative second derivative. Consider the bond Euler equation

$$u'(C_1) = \beta (1 + r) E u'(C_2).$$

With precautionary savings $E u'(C_2) > u'(EC_2).$ This requires that $u'(C_1)$ be greater with precautionary savings, implying that $C_1$ is lower and saving greater.

(f) Write an expression for the risk-premium on capital (relative to the risk-free interest rate) using the bond and capital Euler equations with generic utility $(U(C))$. Define the gross return on capital as

$$1 + R = 1 + \frac{\alpha Y_2}{K_2} - \delta,$$

and the risk premium as

$$ER - r.$$  

Use your expression to explain the sign of the risk premium. Do all of the utility functions above have a positive risk premium?

Bond Euler: $u'(C_1) = \beta (1 + r) E u'(C_2)$
Capital Euler: \[ u' (C_1) = \beta E \left[ \left( 1 + \frac{\alpha Y_2}{K_2} - \delta \right) u' (C_2) \right] \]

Combining yields

\[ (1 + r) E u' (C_2) = E \left[ \left( 1 + \frac{\alpha Y_2}{K_2} - \delta \right) u' (C_2) \right] \]

Take the expectation of the rhs

\[ (1 + r) E u' (C_2) = E \left( 1 + \frac{\alpha Y_2}{K_2} - \delta \right) E u' (C_2) + COV \left( 1 + \frac{\alpha Y_2}{K_2} - \delta, u' (C_2) \right) \]

\[ (1 + r) = E (1 + R) + \frac{COV \left( 1 + \frac{\alpha Y_2}{K_2} - \delta, u' (C_2) \right)}{Eu' (C_2)} \]

Solving for the risk premium yields

\[ ER - r = -\frac{COV \left( \frac{\alpha Y_2}{K_2}, u' (C_2) \right)}{Eu' (C_2)}, \]

where we can simplify the covariance since the \( 1 - \delta \) is constant. We need to sign the covariance term. When \( A_2 \) is high, the marginal product of capital will be high and consumption will be high, implying its marginal utility is low. Therefore, the covariance term is likely to be negative implying a positive risk premium. The linear utility function has a constant marginal utility of consumption, yielding a zero covariance term. Therefore, linear utility does not have a risk premium.

4. Diamond-Mortensen-Pissarides with two types of worker

**Time:** Discrete, infinite horizon

**Demography:** A mass of 1 of workers with infinite lives. A proportion \( \psi \) of the workers are high, \( h \), productivity the rest are low, \( l \), productivity workers. A worker’s type, \( i = h, l \) is observable by the firm when they meet and it never changes. There is a large mass of firms who create individual and identical vacancies. The number of vacancies is controlled by free-entry.

**Preferences:** Workers and firms are risk neutral (i.e. \( u(x) = x \)). The common discount rate is \( r \). To simplify the algebra the value of leisure for workers (usually referred to as \( b \)) is set to zero. The cost of holding a vacancy for firms is \( a \) utilis per period.

**Productive Technology:** A firm matched to a worker of productivity \( i \) produces \( p_i > 0 \) \( i = h, l \) units of the consumption good per period (\( p_h > p_l \)). With probability \( \lambda \) each period, jobs (filled or vacant) experience a catastrophic productivity shock and the job is destroyed.

**Matching Technology:** With probability \( m(\theta) \) each period unemployed workers encounter vacancies. Here \( \theta = v/u \), \( v \) is the mass of vacancies, \( u = u_h + u_l \) is the total mass of unemployed workers and \( u_i \) is the mass of type \( i = h, l \) unemployed workers. The function \( m(.) \) is increasing concave and \( m(\theta) < 1 \) for all \( \theta \). Also \( \lim_{\theta \to 0} m'(\theta) = 1 \), \( \lim_{\theta \to \infty} m'(\theta) = 0 \), and \( m(\theta) > \theta m'(\theta) \). The rate at which vacancies encounter unemployed workers is then \( m(\theta)/\theta \) which is decreasing in \( \theta \). (Assume that job destruction and matching are mutually exclusive so \( m(\theta) + \lambda < 1 \).) **Note:** as everyone matches and loses their jobs at the same rate the proportion of unemployed workers who are high productivity will be the same as in the whole population, i.e. \( u_h = \psi u \) and \( u_l = (1 - \psi)u \).
Institutions: The terms of trade are determined by symmetric Nash bargaining. That is, both firms and workers having bargaining power \( \frac{1}{2} \).

(a) Let \( V_u^i \), and \( V_e^i \), be respectively the present discounted expected values of a type \( i \) unemployed worker and a type \( i \) employed worker. Let \( V_f^i \) be the value to a firm of having hired a type \( i \) worker and let \( V_v^i \) be the value to a firm of holding a vacancy (free-entry will drive this to zero.) Write down the flow value or Bellman type equations for workers and firms based on a type \( i = h,l \) worker receiving wage \( w_i \). Hint: the vacancies will meet with workers according their population shares. Assume all meetings lead to matches.

There are actually going to be 7 equations but using \( i = h,l \) they reduce to 4:

For workers,
\[
\begin{align*}
rv_u^i &= m(\theta)(V_e^i - V_u^i) \\
rv_e^i &= w_i - (V_e^i - V_u^i)
\end{align*}
\]

For firms,
\[
\begin{align*}
(r + \lambda)V_v^i &= \frac{m(\theta)}{\theta} \left[ \psi(V_f^h - V_v) + (1 - \psi)(V_f^l - V_v) \right] - a \\
(r + \lambda)V_f^i &= p_i - w_i
\end{align*}
\]

(b) Symmetric Nash bargaining means that the surplus is split evenly between workers and firms so that
\[
(V_f^i - V_v^i) = \frac{1}{2} \left( (V_f^i - V_v^i) + (V_e^i - V_u^i) \right).
\]

By reference to this bargaining equation and the value equations define a free-entry search equilibrium.

**Definition:** A free-entry search equilibrium is a set of value functions, \( V_u^i, V_e^i, V_f^i, V_v^i \), \( i = h,l \) a pair of wages \( w_i \), \( i = h,l \) and market tightness, \( \theta \) such that taking wages and market tightness as given, the value functions solve the value equations above. Wages are obtained from the bargaining equation above and market tightness is consistent with free-entry (i.e. \( V_v^i = 0 \)).

(c) Solve for an expression that characterizes equilibrium in terms of the market tightness \( \theta \).

Free entry plus the bargaining equation imply that
\[
V_f^i = V_e^i - V_u^i. \tag{1}
\]

The workers’ value equations imply
\[
V_e^i - V_u^i = \frac{w_i}{r + \lambda + m(\theta)}
\]

Substitution into (1) for \( V_e^i - V_u^i \) and \( V_f^i \) yields
\[
w_i = \frac{(r + \lambda + m(\theta)) p_i}{2(r + \lambda) + m(\theta)}
\]
so that

\[ V_f^i = \frac{p_i}{2(r + \lambda) + m(\theta)}. \]

Equilibrium market tightness then solves

\[
\frac{m(\theta)}{\theta} \left[ \psi p_h + (1 - \psi)p_l \right] = a. \quad (2)
\]

(d) How does equilibrium market tightness vary with the proportion of the population who are high productivity, \( \psi \)? What does this mean for the value to being an unemployed low productivity person. Explain the result and how this might influence education policy.

Call LHS(2) \( \Omega \). Then total differentiation implies that

\[
\frac{d\theta}{d\psi} = -\frac{\partial \Omega}{\partial \psi} \frac{\partial \Omega}{\partial \theta}
\]

Clearly \( \frac{\partial \Omega}{\partial \psi} \) is positive. As \( \frac{m(\theta)}{\theta} \) is decreasing in \( \theta \) and \( m(\theta) \) which is increasing appears in the denominator, \( \frac{\partial \Omega}{\partial \theta} < 0 \). This means that \( \frac{d\theta}{d\psi} \) is positive. Now

\[
rV_u^l = m(\theta)(V_e^l - V_u^l)
\]

\[
= \frac{m(\theta)w_l}{r + \lambda + m(\theta)}
\]

\[
= \frac{m(\theta)p_l}{2(r + \lambda) + m(\theta)}
\]

so \( V_u^l \) increases in \( \theta \) and therefore also increases in \( \psi \). More high productivity workers is good for the low productivity workers. This is because firms create more vacancies when the workforce is more highly skilled and this benefits the low skilled too. If education could be used to provide more high skilled workers it could benefit everyone who operates in that labor market.