Answer any 3 of the following 4 questions.

1. The Laffer Curve states that as the tax rate rises from a low level, tax revenue initially rises, reaches a peak, and then falls. This question asks you to derive a steady-state Laffer Curve for an income tax. Utility for the representative agent is given by

\[ U = \sum_{t=0}^{\infty} \beta^t \left[ \ln C_t + \eta \ln G_t + \chi \frac{(1 - \ell_t)^{1-\gamma}}{1-\gamma} \right], \quad 0 < \beta < 1, \quad \eta, \chi > 0 \]  

where \( C_t \) represents consumption, \( G_t \) is government spending, and \( 1 - \ell_t \) is leisure, with unity as the time endowment and \( \ell_t \) as labor supply. Output is produced with capital \( (K_t) \) and labor according to:

\[ Y_t = K_t^\alpha \ell_t^{1-\alpha} \]  

(Y)

The government’s budget constraint is given by

\[ G_t = \tau_t Y_t \]  

(G)

\( \tau_t \) is the income tax rate. The agent’s budget constraint is

\[ K_{t+1} = K_t (1 - \delta) + (r_t K_t + w_t \ell_t) (1 - \tau_t) - C_t, \]  

(ABC)

where \( \delta \) is the rate of depreciation of capital. (Recall that \( Y_t = \tau_t K_t + w_t \ell_t \) since production is constant returns to scale.)

(a) Derive the first order conditions and write them as an Euler equation, labeled (EE) and a labor leisure choice labeled (LL).

(b) Write the equation for the resource constraint in steady state.

(c) Write the profit-maximizing problem for a representative firm using capital and labor to produce output. Take first order conditions with respect to \( K_t \) and \( \ell_t \).

(d) List the equations required to solve for the endogenous variables in equilibrium.

(e) Denote steady-state values by dropping the time subscripts and solve for equilibrium quantities. How does the tax rate affect \( \ell \) and \( Y \)? [Hint: Begin by solving for \( Y_K \) and \( C_K \). Then express the labor-leisure choice as a non-linear function of \( \ell \) and parameters. Since this is non-linear in \( \ell \), you do not have to actually solve the equation for \( \ell \). Solve for levels of \( Y, C, \) and \( K \) as a function of \( \ell \) and parameters including \( \tau \).]

(f) Tax revenue is given by \( \tau Y \). Explain how tax revenue varies as \( \tau \) rises from zero to something close to unity. Explain how capital and labor react to the tax rate generating the response of income to the tax rate. Do we have a Laffer Curve? If so, derive the tax-revenue maximizing value of the tax rate.
2. **Endowment model with "cash" and "credit" goods (inspired by Lucas and Stokey [1987])**

Consider the following economy:

**Time:** discrete, infinite horizon, \( t = 1, 2, ... \)

**Demography:** A single representative household (takes prices as given)

**Preferences:** The household likes to consume two goods. Good 1 is a cash good which requires cash in advance of the period in which the purchase is made. Good 2 is a "credit" good which does not require cash for its acquisition. The household gets utility \( u(c_{1,t}, c_{2,t}) \) from consumption of \( c_i,t \) units of good \( i = 1, 2 \) at time \( t \). Assume that \( u \) is strictly increasing in both arguments and strictly concave with

\[
\lim_{c_{1,t} \to 0} u_1(c_{1,t}, c_{2,t}) = \lim_{c_{2,t} \to 0} u_2(c_{1,t}, c_{2,t}) = \infty
\]

The household’s discount factor is \( \beta < 1 \).

**Endowments:** In period \( t \), the household receives \( y_t \) units of generic perishable consumption good and can choose freely how much of it to take to the market as a cash good, \( y_{1,t} \) and how much to take to the market as a credit good, \( y_{2,t} \) (\( y_{1,t} + y_{2,t} = y_t \)). The usual assumption that you cannot consume your own output applies so you have to bring the goods to market. **Note:** This assumption means that both goods will always trade at the same price, \( p_t \). (If not, people would only bring the higher priced good to market and our assumptions on preferences rule out such a corner solution.)

**Institutions:** There is a government that issues currency so that the stock at time \( t \) is \( H_t \), the time \( t \) growth rate of the money supply is \( \sigma_t \) so that \( H_{t+1} = (1 + \sigma_t)H_t \). The money is distributed (or collected if \( \sigma_t < 0 \)) lump sum so that household’s period \( t \) transfer (tax) is \( \tau_t \). The household can only acquire the cash good using money which has to have been acquired in the previous period. The credit good can be acquired on credit in that the household can use the current period’s proceeds from sales to obtain it. This means they can effectively swap their own goods for credit goods and do not need to go through cash.

(a) Write down the problem faced by the representative household. (**Note:** there is no capital in this model so the household’s only state variable will be \( M_t \) their nominal money holdings.)

(b) Obtain the first-order, complementary slackness and envelope conditions.

(c) Obtain an intertemporal (Euler type) condition that relates the marginal utility of consuming good 2 today and good 1 tomorrow. Explain this expression.

(d) Write down the market-clearing conditions and the government budget constraint. Define a competitive equilibrium.

(e) Now suppose \( \sigma_t = \sigma \) and \( y_t = y \) for all \( t \) and that we are in a steady state so that \( c_{1,t} = c_1 \) and \( c_{2,t} = c_2 \) for all \( t \). What does this mean for the real return on money, \( p_t/p_{t+1} \) and the optimality of the Friedman rule? Explain your answer.
3. Consider a two-period world with uncertainty. Utility for the representative agent is given by

\[ U = u(C_1) + \beta u(C_2) \quad \beta < 1 \]

Production in each period \( t = 1, 2 \) is given by

\[ Y_t = A_t K_t^\alpha L_t^{1-\alpha}, \]

where we assume

\[ A_1, K_1 \text{ are given} \quad L_1 = L_2 = 1 \]

\[ A_2 = (1 + g) A_1 \text{ where } g \text{ is normally distributed with } E(g) > 0 \]

Agents believe that they can save and dissave using capital and risk-free bonds, denoted by \( B \) (view \( B \) as an asset). Risk-free bonds earn an endogenous interest rate of \( r \). Assume that capital depreciates at rate \( \delta \) and that the agent begins period 1 with no risk-free bonds.

(a) Write equations for the agent’s period 1 and period 2 budget constraints, where the agent knows there is no period 3. Assume that the agent chooses \( K_2 \) and \( B_2 \) to maximize expected utility and write Euler equations, one using bonds, and one using capital.

(b) Consider four alternative specifications of utility:

- **Linear:** \( u(C) = aC \)
- **Quadratic:** \( u(C) = aC - \frac{b}{2} C^2 \) for \( C < \frac{a}{b} \)
- **Log:** \( u(C) = \ln C \)
- **CES:** \( u(C) = \frac{C^{1-\sigma}}{1-\sigma} \) for \( \sigma > 0 \)

Define certainty equivalence. Which utility functions exhibit certainty equivalence? Which bond Euler equations have certainty equivalence?

(c) Linearize the marginal utility of consumption in the CES case about \( C_2 = C_1 \). Evaluate whether the linearized bond Euler equation has certainty equivalence. What are the implications of your answer for whether or not the linearized RBC model has certainty equivalence?

(d) Define risk aversion and evaluate which utility functions are characterized by risk aversion. What characteristic of the utility function characterizes risk aversion?

(e) Define precautionary saving and evaluate which utility functions are characterized by precautionary savings. How does precautionary savings differ from risk aversion?

(f) Write an expression for the risk-premium on capital (relative to the risk-free interest rate) using the bond and capital Euler equations with generic utility \( (U(C)) \). Define the gross return on capital as

\[ 1 + R = 1 + \frac{\alpha Y_2}{K_2} - \delta, \]

and the risk premium as

\[ ER - r. \]

Use your expression to explain the sign of the risk premium. Do all of the utility functions above have a positive risk premium?
4. Diamond-Mortensen-Pissarides with two types of worker

**Time:** Discrete, infinite horizon

**Demography:** A mass of 1 of workers with infinite lives. A proportion \( \psi \) of the workers are high, \( h \), productivity the rest are low, \( l \), productivity workers. A worker’s type, \( i = h, l \) is observable by the firm when they meet and it never changes. There is a large mass of firms who create individual and identical vacancies. The number of vacancies is controlled by free-entry.

**Preferences:** Workers and firms are risk neutral (i.e. \( u(x) = x \)). The common discount rate is \( r \). To simplify the algebra the value of leisure for workers (usually referred to as \( b \)) is set to zero. The cost of holding a vacancy for firms is \( a \) utils per period.

**Productive Technology:** A firm matched to a worker of productivity \( i \) produces \( p_i > 0 \), \( i = h, l \) units of the consumption good per period ( \( p_h > p_l \) ). With probability \( \lambda \) each period, jobs (filled or vacant) experience a catastrophic productivity shock and the job is destroyed.

**Matching Technology:** With probability \( m(\theta) \) each period unemployed workers encounter vacancies. Here \( \theta = v/u \), \( v \) is the mass of vacancies, \( u = u_h + u_l \) is the total mass of unemployed workers and \( u_i \) is the mass of type \( i = h, l \) unemployed workers. The function \( m(.) \) is increasing concave and \( m(\theta) < 1 \) for all \( \theta \). Also \( \lim_{\theta \to 0} m'(\theta) = 1 \), \( \lim_{\theta \to \infty} m'(\theta) = 0 \), and \( m(\theta) > \theta m'(\theta) \). The rate at which vacancies encounter unemployed workers is then \( m(\theta)/\theta \) which is decreasing in \( \theta \). (Assume that job destruction and matching are mutually exclusive so \( m(\theta) + \lambda < 1 \).) **Note:** as everyone matches and loses their jobs at the same rate the proportion of unemployed workers who are high productivity will be the same as in the whole population, i.e. \( u_h = \psi u \) and \( u_l = (1 - \psi)u \).

**Institutions:** The terms of trade are determined by symmetric Nash bargaining. That is, both firms and workers having bargaining power \( \frac{1}{2} \).

(a) Let \( V^u_i \), and \( V^e_i \), be respectively the present discounted expected values of a type \( i \) unemployed worker and a type \( i \) employed worker. Let \( V^f_j \) be the value to a firm of having hired a type \( i \) worker and let \( V^v \) be the value to a firm of holding a vacancy (free-entry will drive this to zero.) Write down the flow value or Bellman type equations for workers and firms based on a type \( i = h, l \) worker receiving wage \( w_i \). **Hint:** the vacancies will meet with workers according their population shares. Assume all meetings lead to matches.

(b) Symmetric Nash bargaining means that the surplus is split evenly between workers and firms so that

\[
(V^f_j - V^e) = \frac{1}{2} [(V^f_j - V^e) + (V^e_i - V^f_i)].
\]

By reference to this bargaining equation and the value equations define a free-entry search equilibrium.

(c) Solve for an expression that characterizes equilibrium in terms of the market tightness \( \theta \).

(d) How does equilibrium market tightness vary with the proportion of the population who are high productivity, \( \psi \)? What does this mean for the value to being an unemployed low productivity person. Explain the result and how this might influence education policy.