Ph. D. Comprehensive Examination: Macroeconomics
August, 2017

Answer any 3 of the following 4 questions.

1. Consider an economy composed of heterogeneous agents who live for two periods. Agents derive utility from consumption $c$ according to utility function $u(c)$, and discount utility of the second period at a discount rate of $\beta$. Agents enter the economy with zero asset. In the first period, they receive an endowment $w_1 = w$; and in the second period, they have a probability of 0.5 to receive endowment $w_2 = (1 + \delta)w$, and a probability of 0.5 to receive endowment $w_2 = (1 - \delta)w$, where $0 < \delta < 1$. Agents can save but cannot borrow. Let $a$ denote the amount of assets carried to the second period, and assets earn a gross return of $1 + r$. Let further denote the first period consumption as $c_1$ and the second period consumption as $c_2$.

(a) Write down the agent problem. Your solution should contain the choice variables and the budget constraint for each period.

$$\max_{c_1, c_2, a} u(c_1) + \beta E_w u(c_2)$$

subject to

$$c_1 + a = w_1$$
$$c_2 = w_2 + (1 + r)a$$
$$a \geq 0$$

(b) Find the Euler equation

$$u'(c_1) = \beta(1 + r)E_w u'(c_2)$$

where $u'(\cdot)$ denote the marginal utility of consumption.

(c) Under the assumption $(1 + r)\beta = 1$ and $u(c) = \eta_1 c + \eta_2 c^2$, where $\eta_1 > 0$, $\eta_2 < 0$, and $c < \frac{-\eta_2}{2\eta_1}$, what is the optimal level of asset carried to the second period.

The optimal asset is 0 as the solution to the below equation:

$$\eta_1 + 2\eta_2(w - a) = 1/2(\eta_1 + 2\eta_2(w(1 + \delta) + (1 + r)a)) + 1/2(\eta_1 + 2\eta_2(w(1 - \delta) + (1 + r)a))$$

(d) Under the assumption $(1 + r)\beta = 1$ and $u(c) = c^{1-\sigma}/(1-\sigma)$, where $\sigma > 0$, is the optimal level of asset carried to the second period equal to 0 or greater than 0. Provide a proof and an intuitive explanation for your answer.

The optimal asset is positive.

**Proof.** Suppose the optimal asset is zero, then $c_1 = w_1$ and $c_2 = w_2$. $u^{1-\sigma}$ is a convex function. According to Jensen’s inequality we have $E_w u'(c_2) > u'(E_w(c_2)) = u(w) = u(c_1)$, which means the MB of saving is greater than the MC of savings, and saving zero amount is not optimal. ■

This example illustrates that when agents are prudent $u'' > 0$, they want to accumulate precautionary savings to insure against income shocks. (They do not accumulate precautionary savings when they have quadratic utility $u'' = 0$)
(e) Under the assumption $(1 + r)\beta = 1$ and $u(c) = e^{1-\sigma}/(1 - \sigma)$, where $\sigma > 0$, suppose now agents face greater uncertainty about future income, i.e. have a probability of 0.5 to receive endowment $w_2 = (1 + \delta')w$, and a probability of 0.5 to receive endowment $w_2 = (1 - \delta')w$, where $0 < \delta < \delta' < 1$. Would agents save more, save less, or save the same amount compared to the case in the previous question. Provide a proof and an intuitive explanation.

Agents will save more, since they are prudent and want to take actions to insure against greater uncertainty of future income.

**Proof.** Denote that optimal saving of the previous case as $a^*$. We need to show that the MB of saving $a^*$ in the new economy is greater than the marginal cost of saving $a^*$, since this implies agents want to save more than $a^*$, an increase in savings:

\[
\frac{1}{2}u'(1 - \delta^*) + \frac{1}{2}u'((1 + \delta^*) > u^*)
\]

\[
= \frac{1}{2}u'' + \frac{1}{2}u^*
\]

\[
\iff u'((1 - \delta^*) - u^*) + u'((1 + \delta^*) - u^*) > 0
\]

Thus,

\[
u'((1 - \delta^*) - u^*) + u'((1 + \delta^*) - u^*)
\]

\[
> (u'' - u^*)(\delta - \delta')w > 0
\]

where the last inequality holds because $u''' > 0$. ■

2. **Diamond OG model with proportional taxation**

We are going to look into optimal proportional taxes in the Diamond OG model.

**Time:** discrete, infinite horizon, $t = 1, 2, \ldots$

**Demography:** A mass $N$ of newborns enter every period (no population growth). Everyone lives for 2 periods except for the first generation of old people who live for one.

**Preferences:** for the generations born in and after period 1;

\[
U(c_{1,t}, c_{2,t+1}) = \ln(c_{1,t}) + \beta \ln(c_{2,t+1})
\]

where $c_{i,t}$ is consumption in period $t$ and stage $i$ of life. For the initial old generation $\bar{U}(c_{2,1}) = \ln(c_{2,1})$.

**Productive technology:** The production function available to firms is $F(K_t, N)$ where $K_t$ is the time $t$ capital stock and $N$ is the number of workers employed (i.e. the number of young people). $F(.,.)$ is twice differentiable, concave, has constant returns to scale and satisfies the Inada conditions. You may find it convenient to use the implied per young person production function, $f(k_t) \equiv F(k_t, 1)$ where $k_t \equiv K_t / N$ is the time $t$ capital stock per worker.
Endowments: Everyone has one unit of labor services when young. The initial old share an endowment, $K_1$ of capital so they have $k_1$ units each.

Institutions: There are competitive markets every period for labor and capital. (You can think of a single collectively owned firm which takes wages and interest rates as given.) There is a government that is required to meet an exogenous sequence of expenditures, $\{g_t\}_{t=1}^\infty$ (which are thrown away). It will do so by taxing first period income (wages) at the proportional rate $\tau^w_t$ and second period income, rent on capital, at the proportional rate $\tau^k_t$.

(a) Write down and solve the problem faced by the individuals born in period $t$.

Individual’s problem is

$$\max_{c_{1,t},c_{2,t+1},s_{t+1}} \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$$

s.t. $w_t(1 - \tau^w_t) = c_{1,t} + s_{t+1}$

$$R_{t+1}s_{t+1}(1 - \tau^k_{t+1}) = c_{2,t+1}$$

After substituting out consumption the FOC w.r.t $s_{t+1}$ is

$$\frac{-1}{w_t(1 - \tau^w_t) - s_{t+1}} + \frac{\beta R_{t+1}(1 - \tau^k_{t+1})}{R_{t+1}s_{t+1}(1 - \tau^k_{t+1})} = 0.$$ 

So

$$s_{t+1} = \frac{\beta}{1 + \beta} w_t(1 - \tau^w_t)$$

(b) Write down and solve the representative firm’s problem in period $t$. (N.B. firms are not taxed.)

Firms solve

$$\max_{k_t} f(k_t) - w_t - R_t k_t.$$ 

The first order condition plus c.r.s of $F$ imply

$$R_t = f'(k_t)$$

$$w_t = f(k_t) - f'(k_t)k_t$$

(c) Write down the market clearing condition for capital and goods. Write down the government’s budget constraint. Define a competitive equilibrium for given sequence of tax rates $\{\tau^w_t, \tau^k_t\}$.

Market Clearing:

For capital: $s_t = k_t$

For goods: $f(k_t) = c_{1,t} + c_{2,t} + g_t + k_{t+1}$

Government budget constraint (GBC):

$$w_t \tau^w_t + R_{t+1}s_{t+1}\tau^k_{t+1} = g_t$$

Definition 1 A competitive equilibrium is a sequence of prices $\{w_t, R_t\}$ and an allocation $\{k_t, c_{1,t}, c_{2,t+1}\}$ such that given taxes $\{\tau^k_t, \tau^w_t\}$ and prices, the allocation solves the households’ optimization problems, markets clear and GBC is satisfied.
(d) Solve for an equation that characterizes the dynamics of the competitive equilibrium capital stock (as a function of the tax rates).

\[ k_{t+1} = \left( \frac{\beta}{1+\beta} \right) (f(k_t) - f'(k_t)k_t) (1 - \tau_t^w) \]

(e) Now suppose the tax rates are fixed at \( \tau^w \) and \( \tau^k \) and government spending \( g_t = g \) for all \( t \). Write down the equation that characterizes a steady-state equilibrium capital stock, \( k^* \) in terms of \( \tau^w \) and \( \tau^k \). Write down the steady-state government budget constraint.

Steady state capital stock is given by

\[ (1 + \beta)k^* - \beta(1 - \tau^w) \left( f(k^*) - f'(k^*)k^* \right) = 0. \tag{1} \]

The steady state GBC is

\[ g_t - \tau^w \left( f(k^*) - f'(k^*)k^* \right) - \tau^k f'(k^*)k^* = 0. \tag{2} \]

(f) Solve for steady state consumption as a function of \( k^* \), \( \tau^w \) and \( \tau^k \).

\[ c_1 = [1 - \tau^w][f(k^*) - f'(k^*)k^*] - k^* \]
\[ c_2 = [1 - \tau^k]f'(k^*)k^* \]

(g) We want to consider optimal tax policy. Assume the government weights each generation equally (i.e. does not discount). Ignore the first generation of old so that you can assume that optimal policy maximizes steady state utility. Using the solutions to part f to eliminate consumption, write out (but do not solve) the government’s problem for a given constant expenditure, \( g \).

The government problem is

\[
\max_{\tau^w, \tau^k} \ln \left( [1 - \tau^w][f(k^*) - f'(k^*)k^*] - k^* \right) + \beta \ln \left( [1 - \tau^k]f'(k^*)k^* \right)
\]

subject to (1) and (2).

3. Consider an economy where there are two agents, indexed by \( i = 1, 2 \), living for one period. Let \( s \in \{A, B\} \) denote the state of the world for this period. Ex-ante, each state has an equal probability (0.5) to happen. Ex-post, when \( s = A \), agent 1 receives endowment \( w^1_A \), and agent 2 receives endowment \( w^2_A \), where we use superscript to indicate the agent number; when \( s = B \), agent 1 receives endowment \( w^1_B \), and agent 2 receives endowment \( w^2_B \). Suppose the market is complete, and agents make time-0 (before knowing \( s \)) trading of arrow securities. Denote the price of arrow security that pays one unit of goods for state \( s \) as \( p_s \) and holdings of this security as \( q^i_s \). Similarly, let \( c^i_s \) denote the consumption for agent \( i \) in state \( s \). The period utility function is \( \ln(c) \), the same for both agents.

(a) Define a competitive equilibrium, in particular, explicitly write market clearing condition(s).

**Definition 2** Given the distribution of \( s \), and the endowment \( w^i_s \), a competitive equilibrium consists of a set of pricing functions \( \{p_A, p_B\} \) and a set of plans for consumption and security demand.
i. Given prices, the plans solve the consumer problem for all agents.

ii. The market for each type of security is clear:

\[ q^1_A + q^2_A = 0 \]
\[ q^1_B + q^2_B = 0 \]

(b) Write the consumer problem for agent 1. Note that you need to specify the choice variables and the constraints that this agent faces.

\[
\max_{c^1_A, c^2_A, q^1_A, q^1_B, q^2_A, q^2_B} \frac{1}{2} \ln(c^1_A) + \frac{1}{2} \ln(c^1_B)
\]

subject to

\[ c^1_A + p^A q^1_A + p^B q^1_B = w^1_A + q^1_A \]
\[ c^1_B + p^A q^1_A + p^B q^1_B = w^1_B + q^1_B \]

(c) Find the first order conditions for agent 1.

\[
\mathcal{L} = \frac{1}{2} \ln(w^1_A + q^1_A - p^A q^1_A - p^B q^1_B) + \frac{1}{2} \ln(w^1_B + q^1_B - p^A q^1_A - p^B q^1_B)
\]
\[
\frac{\partial \mathcal{L}}{\partial q^1_A} = 0 \iff \frac{1}{c^1_A} (1 - p^A) = \frac{1}{c^1_B} (1 - p^B)
\]
\[
\frac{\partial \mathcal{L}}{\partial q^1_B} = 0 \iff \frac{1}{c^1_A} (p^B) = \frac{1}{c^1_B} (1 - p^B)
\]

(d) Note that by replacing \( i = 1 \) with \( i = 2 \) in the previous parts, we will have the first order conditions for agent \( i = 2 \). Given these first order condition, will the share of total endowment consumed by agent \( i \) differ across two states, i.e. whether \( \frac{c^1_i}{w^1_i + w^2_i} \) equals \( \frac{c^2_i}{w^1_i + w^2_i} \)?

This share is constant, since \( \frac{c^1_i}{c^2_i} = \frac{c^2_i}{c^1_i} = \frac{1 - p^A}{p^A} = \frac{p_B}{1 - p_B} \).

(e) According to the first welfare theorem, as the preference is locally no-satiated, the competitive equilibrium is Pareto optimal. This means, we can find a set of Pareto weight, denoted by \( \lambda^i \), that makes the solution to the social planner problem to be the same as the solution to the competitive equilibrium. To apply this theorem, first define the social planner problem.

\[
\max_{c^1_A, c^2_A, c^1_B, c^2_B} \frac{1}{2} (\lambda^1 \ln(c^1_A) + \lambda^2 \ln(c^2_A)) + \frac{1}{2} (\lambda^1 \ln(c^1_B) + \lambda^2 \ln(c^2_B))
\]

subject to

\[ c^1_A + c^2_A = w^1_A + w^2_A \]
\[ c^1_B + c^2_B = w^1_B + w^2_B \]
(f) Solve the social planner problem and express consumption as a function of endowment and Pareto weight.

The first order conditions are:

\[
\frac{\lambda^1}{c_A} = \frac{\lambda^2}{c_B} = 2\theta_A
\]

\[
\frac{\lambda^1}{c_A} = \frac{\lambda^2}{c_B} = 2\theta_B
\]

where \( \theta_s \) is the shadow price of the resource constraint for state \( s \).

Combining the above conditions with the resource constraints, we have:

\[
c_1^s = \frac{\lambda^1}{\lambda^1 + \lambda^2} (w_1^s + w_2^s)
\]

\[
c_2^s = \frac{\lambda^2}{\lambda^1 + \lambda^2} (w_1^s + w_2^s)
\]

(g) Use the information derived from previous steps to solve the prices as a function of endowments. What is the relationship between price \( p_s \) and the total endowment of state \( s \). Provide an intuitive explanation for your finding.

\[
\frac{c_A}{p_A} = \frac{1-p_A}{p_A} = \frac{p_B}{1-p_B} = \frac{w_1^A + w_2^A}{w_1^B + w_2^B} \Rightarrow p_A = \frac{w_1^A + w_2^A}{w_1^A + w_3^A + w_1^B + w_2^B}, \text{ and } p_B = \frac{w_1^A + w_2^A}{w_3^A + w_1^B + w_2^B}.
\]

\( p_s \) is decreasing in the total endowment of state \( s \), since the marginal utility consumption, the benefit of purchasing the arrow security for that state, is decreasing in the total endowment of state \( s \).

4. One-sided search with learning-by-doing

**Time:** Discrete, infinite horizon.

**Demography:** A single worker who faces the possibility of death with probability \( \delta \) each period.

**Preferences:** The worker is risk neutral (i.e. \( u(x) = x \)) and discounts the future at the rate \( r \). When dead he gets 0 utility forever.

**Endowments:**

The worker can be high or low skilled. He is born unemployed and low-skilled. He becomes high-skilled only by being employed as a low-skilled worker.

- When unemployed (regardless of skill level) he receives the "benefit", \( b \), each period.
- When first unemployed he has low skill level and with probability \( \alpha \) he gets a low-skilled job that pays \( w_L \) each period.
- Employed low-skilled workers become high-skilled with probability \( \gamma \) each period.
- High skilled employed workers receive \( w_H \) each period.
- Once the worker becomes high-skilled he is high skilled for life.
- Employed workers (regardless of skill level) loose their job with probability \( \lambda \).
- Unemployed high-skilled workers get high-skilled jobs (paying \( w_H \)) with probability \( \alpha \).
Note 1: Events that can happen to an individual are mutually exclusive (i.e. not independent). Assume that $\delta + \gamma + \lambda < 1$ and $\delta + \alpha < 1$.

Note 2: The worker can be in one of 4 possible states: unemployed low-skilled, employed low-skilled, unemployed high-skilled and employed high-skilled.

a. Write down the flow asset value (flow Bellman) equations for the worker. (Notation: use $V^x_y$ for the value to being in skill level, $x = L, H$ and employment status $y = u, e$)

\[
\begin{align*}
rv^L_u &= b + \alpha (v^L_e - v^L_u) + \delta (0 - v^L_u) \\
rv^L_e &= w_L + \gamma (v^H_e - v^L_e) + \lambda (v^L_u - v^L_e) + \delta (0 - v^L_e) \\
rv^H_e &= w_H + \lambda (v^H_u - v^H_e) + \delta (0 - v^H_e) \\
rv^H_u &= b + \alpha (v^H_e - v^H_u) + \delta (0 - v^H_u)
\end{align*}
\]

b. Obtain an expression for $v^H_e - v^L_e$ in terms of the model parameters.

First obtaining expressions for differences between value functions:

\[
\begin{align*}
(r + \alpha + \delta + \lambda) (v^L_e - v^L_u) &= w_L - b + \gamma (v^H_e - v^L_e) \\
(r + \gamma + \delta) (v^H_e - v^L_e) &= w_H - w_L + \lambda (v^L_u - v^L_e) - \lambda (v^H_e - v^H_u) \\
(r + \alpha + \delta + \lambda) (v^H_e - v^H_u) &= w_H - b
\end{align*}
\]

Eliminating $(v^H_e - v^H_u)$ in second equation leads to

\[
(r + \gamma + \delta) (v^H_e - v^L_e) = w_H - w_L - \frac{\lambda (w_H - b)}{r + \alpha + \delta + \lambda} + \lambda (v^L_e - v^L_u)
\]

Eliminating $(v^L_e - v^L_u)$ leads to

\[
v^H_e - v^L_e = \frac{(r + \alpha + \delta) (w_H - w_L)}{(r + \gamma + \delta) (r + \alpha + \delta + \lambda) - \gamma \lambda}
\]

c. Use the result from part b. to obtain a restriction on the value of $w_L$ such that the low-skilled worker will not reject a job offer.

For low skilled workers to accept job offers we need that $v^L_e - v^L_u \geq 0$. That is we need

\[
w_L - b + \gamma (v^H_e - v^L_e) \geq 0
\]

or

\[
w_L - b + \frac{\gamma (r + \alpha + \delta) (w_H - w_L)}{(r + \gamma + \delta) (r + \alpha + \delta + \lambda) - \gamma \lambda} \geq 0
\]

d. Now suppose there is mass one of such individuals and that anyone who dies is replaced by a newborn. (Newborns are unskilled and unemployed.) Draw a diagram to show the population flows between the 4 states.
e. Write down a system of equations that can be solved for the steady state populations (you don’t need to solve them).

Any 4 of the following 5 equations will work:

\[
\begin{align*}
    u_L + u_H + e_H + e_L &= 1 \\
    \alpha u_L &= (\gamma + \delta + \lambda)e_L \\
    \alpha u_H + \gamma e_L &= (\delta + \lambda)e_H \\
    \lambda e_H &= \delta u_H + \alpha u_H \\
    \delta(u_H + e_H) + (\delta + \lambda)e_L &= \alpha u_L
\end{align*}
\]