7. Consider an economy in which agents can live for a maximum of two periods. Let $s$ denotes the survival rate from the first to the second period. Agents derive utility from consumption $c$ according to utility function $\ln(c)$ if alive, and derive warm-glow utility from leaving bequests $b$ according to a utility function $\lambda \ln(b + \kappa)$ if decease, where $\lambda$ measures the strength of the bequest motive, and $\kappa$ measures to what extent bequest is a luxury good. For simplicity, we set interest rate to zero and the discount factor to 1. In the first period, agents receive an endowment $w_1$, and choose consumption $c_1$, savings $a_1$, and life insurance amount $q_1$, which could be purchased as an unit price of $0 < p < 1$. Bequests $b_1$ left at the end of the first period if decease are the sum of life insurance face value $q_1$ and savings $a_1$. In the second period, if alive, agents receive endowment $w_2$, and choose consumption $c_2$ and savings $a_2$. Note that no insurance contracts are available at the second period, since the death is certain. Bequests $b_2$ left at the end of the second period are entirely composed of savings $a_2$.

Formally, the agent life-cycle problem is written as follows:

$$\max_{c_1, c_2, a_1, a_2, q_1, b_1, b_2} \ln(c_1) + (1 - s)\lambda \ln(b_1 + \kappa) + s \ln(c_2) + s \lambda \ln(b_2 + \kappa)$$

subject to

$$\begin{align*}
c_1 + pq_1 + a_1 &= w_1 \\
b_1 &= q_1 + a_1 \\
c_2 + a_2 &= w_2 + a_1 \\
b_2 &= a_2
\end{align*}$$

(a) Write the first order conditions. Note that we want to start with a general case, and do not restrict the solution to be internal, i.e., the conditions you write should contain inequities. Intuitively explain the meaning of the left hand side and right hand side of each condition.

There are three first order conditions:

1. Trade-off between consumption and purchasing life insurance in the first period:

$$\frac{p}{c_1} \geq (1 - s)\lambda \frac{1}{b_1 + \kappa}$$

(1)
2. Trade-off between consumption and saving in the first period

\[
\frac{1}{c_1} \geq (1 - s) \lambda \frac{1}{b_1 + \kappa} + s \frac{1}{c_2}
\]  

(2)

marginal cost of saving  
marginal benefit of saving

3. Trade-off between consumption and saving in the second period

\[
\frac{1}{c_2} \geq \lambda \frac{1}{b_2 + \kappa}
\]  

(3)

marginal cost of saving  
marginal benefit of saving

(b) For the below analysis, we assume that the solution is internal, and solve the problem recursively. As a first step, write the problem for an agent who is alive at the second period, and solve \(c_2\) and \(a_2\) as a function of the state variables in the second period. Discuss the partial effect (fixing state variables) of sequentially increasing \(\lambda\) and \(\kappa\) on the allocation between consumption and saving in the second period.

\[
\max_{c_2, a_2, b_2} \ln(c_2) + \lambda \ln(b_2 + \kappa)
\]

subject to

\[
c_2 + a_2 = w_2 + a_1 \\
b_2 = a_2
\]

Combining condition (3) with the two constraints, it is easy to show that

\[
c_2 = \frac{w_2 + a_1 + \kappa}{1 + \lambda}
\]

(4)

\[
b_2 = a_2 = \frac{\lambda(w_2 + a_1) - \kappa}{1 + \lambda}
\]

Given \(a_1\), the amount of bequests \(b_2\) increases in \(\lambda\) and decreases in \(\kappa\), reflecting the change in marginal utility of leaving bequests.

(c) Verify that \(b_1\) can be expressed in the form of \(b_1 = Aw_1 + Bw_2 + C\) with \(A > 0\), \(B > 0\). Intuitively explain why the amount of bequests chosen in the first period is increasing in endowments for both periods.

From conditions (1) and (2), we derive:

\[
c_1 = \frac{p}{(1 - s)\lambda}(b_1 + \kappa)
\]

\[
c_2 = \frac{sp}{(1 - s)\lambda(1 - p)}(b_1 + \kappa)
\]
Inserting these and equation 4 into the first period budget constraint, we have:

\[ b_1 = \frac{(1 - s)\lambda}{p(1 + \lambda + s)}w_1 + \frac{(1 - s)\lambda(1 - p)}{p(1 + \lambda + s)}w_2 + \frac{\lambda(1 - s - p) - p(1 + s)}{p(1 + \lambda + s)}k. \]

Define \( A = \frac{(1 - s)\lambda}{p(1 + \lambda + s)} \), \( B = \frac{(1 - s)\lambda(1 - p)}{p(1 + \lambda + s)} \), and \( C = \frac{\lambda(1 - s - p) - p(1 + s)}{p(1 + \lambda + s)}k \). We have \( b_1 = Aw_1 + Bw_2 + C \), with \( A > 0 \) and \( B > 0 \). The amount of bequests increases with respect to both \( w_1 \) and \( w_2 \), because bequests are luxury goods, and so people want to leave a greater bequest as available resource increases.

(d) Find \( q_1 \) and specify a condition under which agents decrease life insurance holdings \((q_1)\) in response to an increase in first period endowment \(w_1\). Provide an intuitive explanation.

\[ q_1 = b_1 - a_1 = b_1 - \left( \frac{(1 + \lambda)s}{(1 - s)\lambda(1 - p)}(b_1 + k) - w_2 - k \right) \]

\[ = \frac{\lambda(1 - s - p) - ps}{(1 - s)\lambda(1 - p)}b_1 + w_2 + \frac{\lambda(1 - s - p) - ps}{(1 - s)\lambda(1 - p)}k \]

\[ = \frac{\lambda(1 - s - p) - ps}{p(1 + \lambda + s)}w_1 + \frac{\lambda(1 - s) + p}{p(1 + \lambda + s)}w_2 + \frac{\lambda(1 - s)(\lambda(1 - s - p) - ps)}{p(1 + \lambda + s)(1 - s)\lambda(1 - p)}k \]

\( q_1 \) is a decreasing function of \( w_1 \) if and only if \( \frac{\lambda(1 - s - p)}{\lambda + p} < 0 \), or equivalently, \( s > -\frac{\lambda(1 - p)}{\lambda + p} > 1 - p \). This condition implies that when survival rate is high (relatively to the available price of life insurance), an increase in current endowments decreases holdings of life insurance. This is because agents prefer to save in the form of risk-free assets, which also increases their future consumption if alive. In contrast, when survival rate is low, an increase in current endowments raises life insurance demand, since the relative price of life insurance is cheaper compared with consumption. If we extend this model to a heterogeneous agent setting, in which agents differ by their mortality risks, this channel implies that the degree of adverse selection in the life insurance market worsens when agents have more initial resources (\( w_1 \) in this model).

(e) How does a reduction in second period endowment \((w_2)\) affects life insurance demand \((q_1)\), and intuitively explain your findings?

It transpires that \( q_1 \) increases in \( w_2 \), and a reduction in second period endowment reduces life insurance demand. This is because a drop in \( w_2 \) reduces the opportunity cost of saving and hence agents are more willing to save to leave a bequest. Combining this channel with the channel discussed in the previous part indicates that if we classify the life insurance market into segments of different age groups, it is likely that adverse selection plays a larger role in the market for older age groups than for younger age groups, since older individuals have more assets and fewer future non-capital income.

8. Diamond OLG model proportional taxes
We will investigate the possibility that the government want to find the most efficient way to meet its spending obligations using proportional taxes.

**Time**: discrete, infinite horizon, \( t = 1, 2, \ldots \)

**Demography**: A mass \( N_t \equiv N_0(1 + n)^t \) of identical newborns enter in period \( t \). Everyone lives for 2 periods except for the first generation of old people who live for one. A mass (normalized to 1) of competitive (price taking) firms.

**Preferences**: for the generations born in and after period 1;

\[
U(c_{1,t}, c_{2,t+1}) = \ln(c_{1,t}) + \beta \ln(c_{2,t+1})
\]

where \( c_{i,t} \) is consumption in period \( t \) and stage \( i \) of life. For the initial old generation \( \tilde{U}(c_{2,1}) = \ln(c_{2,1}) \).

**Productive technology**: The production function available to firms is \( F(K, N) = zK^\alpha N^{1-\alpha} \), \( 0 < \alpha < 1 \), where \( K \) is the capital stock and \( N \) is the number of workers employed. You may find it convenient to use the implied per young person production function, \( f(k) = zk^\alpha \) where \( k \) is the capital stock per worker.

**Endowments**: Everyone has one unit of labor services when young. The initial old share an endowment, \( K_1 \) of capital so they have \( (1 + n)k_1 \) units each.

**Government**: The government has to meet an exogenous expenditure schedule \( \{g_t\} \). It does so by levying proportional taxes at rates, \( \{\tau_{1,t}\} \) and \( \{\tau_{2,t}\} \) on labor and capital earnings respectively in each period.

**Institutions**: There are competitive markets every period for labor and capital. (You can think of a single collectively owned firm which takes wages and interest rates as given.)

(a) Write down and solve the problem faced by the individuals born in period \( t \).

\[
\max_{c_{1,t}, c_{2,t+1}, s_{t+1}} \ln(c_{1,t}) + \beta \ln(c_{2,t+1})
\]

s.t.

\[
c_{1,t} = w_t(1 - \tau_{1,t}) - s_{t+1}
\]

\[
c_{2,t+1} = R_{t+1}s_{t+1}(1 - \tau_{2,t+1})
\]

After substitution the FOC wrt \( s_{t+1} \) is

\[
\frac{-1}{w_t(1 - \tau_{1,t}) - s_{t+1}} + \frac{\beta R_{t+1}(1 - \tau_{2,t+1})}{R_{t+1}s_{t+1}(1 - \tau_{2,t+1})} = 0
\]

which leads to

\[
s_{t+1} = \frac{\beta w_t(1 - \tau_{1,t})}{1 + \beta}
\]

(b) Write down and solve the representative firm’s problem in period \( t \). 

\[
w_t = (1 - \alpha)zk_t^\alpha, \quad R_t = \alpha zk_t^{\alpha - 1}
\]
(c) Write down the market clearing condition for capital, and define a competitive equilibrium (given taxes).

\[(1 + n)k_{t+1} = \frac{(1 - \alpha)\beta(1 - \tau_{1,t})zk_t^\alpha}{1 + \beta}\]

**Definition:** A competitive equilibrium is an allocation, \(\{c_{1,t}, c_{2,t+1}, k_t\}\) and a sequence of prices, \(\{w_t\}\) and \(\{R_t\}\) such that given prices the allocation solves the households’ and firm’s problems and markets clear.

(d) Obtain an expression for the non-trivial steady-state level of the capital stock and, given that the tax rates are less than 1, show that the steady-state is stable. What impact do each of the tax rates have on the steady-state capital stock?

\[k^* = \left[\frac{(1 - \alpha)\beta(1 - \tau_{1,t})zk_t^\alpha}{1 + \beta}\right]^{\frac{1}{1 - \alpha}}\]

\[\frac{dk_{t+1}}{dk_t} = \frac{\alpha(1 - \alpha)\beta(1 - \tau_{1,t})zk_t^\alpha}{1 + \beta}\]

\[\left.\frac{dk_{t+1}}{dk_t}\right|_{k=k^*} = \alpha\]

Which is between zero and 1 so the steady-state is stable with monotone dynamics. An increase in \(\tau_{1,t}\) will lower the steady-state capital stock while \(\tau_{2,t+1}\) has no impact on it.

(e) Now assume the government does not discount the future so that we can look at steady-states only (and drop time subscripts). The government treats all households alike and wants to maximize welfare by choosing the path of tax rates to meet a constant expenditure, \(g\), for all time. *Without solving it*, write down the problem faced by the government in terms of the tax rates and the parameters of the model.

Problem is

\[\max_{\tau_1, \tau_2} \ln(c_1^*) + \beta \ln(c_2^*)\]

s. t. \(w^*\tau_1 + R^*k^*\tau_2 = g\)

where the * refers to steady-state values. In terms of the parameters this becomes:

\[\max_{\tau_1, \tau_2} \ln((1 - \alpha)zk^{*\alpha} - k^*) + \beta \ln(\alpha zk^{*\alpha}(1 - \tau_2))\]

s. t. \(z[(1 - \alpha)\tau_1 + \alpha\tau_2]k^* = g\)

where \(k^*\) is given above.

9. Consider the following version of a stochastic growth model. There are a fixed number of price-taking producers that solve

\[\max_{\Pi_t} \Pi_t = Y_t - W_t L_t,\]
\[ Y_t = L_t^{1-\zeta}, \quad 0 < \zeta < 1, \quad \text{(PRF)} \]

where: \( \Pi_t \) is profit; \( L_t \) is labor; \( W_t \) is the real wage; and \( Y_t \) is output.

The preferences of the representative household over consumption, \( C_t \), and labor are given by
\[
E_0 \left( \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\nu} \left( C_t^\alpha (1 - L_t)^{1-\alpha} \right) \right),
\]
\[ 0 < \beta < 1, \quad \nu > 1, \quad 0 < \alpha < 1. \]

Households receive labor income and profits from firms, and pay consumption taxes to the government. Households can store their assets \( K_t \), and earn zero returns on the stored assets. As usual, assume that assets held at the beginning of period \( t+1 \), \( K_{t+1} \), are chosen in period \( t \). Note that assets are used only as a storage device, and not as a factor of production. Households face the usual initial, non-negativity and No-Ponzi-Game conditions.

The government collects taxes from consumption and maintains a balanced budget:
\[ (1 + \tau_t)C_t = G_t. \quad \text{(BB)} \]

Taxes are driven by government spending, which follows an AR(1) process around the log of its steady state value:
\[ \hat{g}_t \equiv \ln \left( \frac{G_t}{G_{ss}} \right) = \phi \hat{g}_{t-1} + \varepsilon_t, \quad 0 \leq \phi < 1, \quad \text{(TS)} \]

where \( \{\varepsilon_t\} \) is an exogenous i.i.d. process, and \( G_{ss} > 0 \) is steady state government spending. Individual consumers and producers are sufficiently small to take \( \tau_t \) as well as \( G_t \) as given.

(a) Define a competitive equilibrium
Given the set of exogenous stochastic processes \( \{G_t\}_{t=0}^{\infty} \) and initial endowments \( K_0 \), a competitive equilibrium consists of the stochastic processes \( \{C_t, L_t, K_{t+1}, W_t, \tau_t\}_{t=0}^{\infty} \) such that:

1. Given the process of \( \{W_t, \tau_t\} \), \( \{C_t, L_t, K_{t+1}\} \) solves the consumer’s problem;
2. \( \{L_t\} \) solves the firm’s problem
3. Labor market is clear
4. Government maintains a balanced budget each period.

(b) Write down the firm’s problem and find the first order conditions that maximize profits
The firm’s problem is
\[ \max_{L_t} L_t^{1-\zeta} - W_t L_t \]

The FOC for an interior solution is
\[ W_t = (1 - \zeta) L_t^{-\zeta} \quad \text{(WE)} \]
(c) With the specified preference, is consumption \((C_t)\) a substitute or a complement to leisure \((1 - L_t)\)?

The period utility function is:

\[
u(C_t, L_t) = \frac{1}{1 - \nu} (C_t^{\alpha}(1 - L_t))^{1-\nu}
\]

The cross-derivative is:

\[
\frac{\partial^2 u}{\partial C_t(1 - L_t)} = \alpha(1 - \alpha)(1 - \nu)(C_t^{\alpha(1-\nu)-1}(1 - L_t))^{(1-\alpha)(1-\nu)-1} < 0
\]

Thus, consumption and leisure are substitutes.

(d) Write down the household problem in a recursive form (Bellman’s equation and constraints), and find the first order conditions that maximize household utility. For the below analysis, we assume that the problem always has an interior solution.

The household problem is:

\[
V(\tau_t, W_t, K_t) = \max_{C_t \geq 0, L_t \geq 0} \frac{1}{1 - \nu} (C_t^{\alpha}L_t^{1-\alpha})^{1-\nu} + \beta E_t V(\tau_{t+1}, W_{t+1}, K_{t+1})
\]

(Note here \(G_t\) is not a relevant state variable)

subject to:

\[
(1 + \tau_t)C_t + K_{t+1} = W_t L_t + \Pi_t + K_t
\]

\[
\lim_{j \to \infty} E_t(K_{t+j}) = 0
\]

The FOC for interior solution are (where \(\lambda_t\) is the shadow price for period \(t\) budget constraint in terms of period \(t\) consumption goods):

\[
\alpha C_t^{\alpha(1-\nu)-1}(1 - L_t)^{(1-\alpha)(1-\nu)} = (1 + \tau_t)\lambda_t
\]

\[
(1 - \alpha)C_t^{\alpha(1-\nu)}(1 - L_t)^{(1-\alpha)(1-\nu)-1} = \lambda_t W_t
\]

\[
\lambda_t = \beta E_t\left(\frac{\partial V[t+1]}{\partial K_{t+1}}\right)
\]

From the envelop theorem, we have:

\[
\frac{\partial V[t]}{\partial K_t} = \lambda_t
\]

and,

\[
\frac{\alpha C_t^{\alpha(1-\nu)-1}(1 - L_t)^{(1-\alpha)(1-\nu)}}{1 + \tau_t} = \beta E_t\left(\frac{\alpha C_{t+1}^{\alpha(1-\nu)-1}(1 - L_{t+1})^{(1-\alpha)(1-\nu)-1}}{1 + \tau_{t+1}}\right) \quad \text{(EE)}
\]
(e) Let lower-case letters with carats “^” denote deviations of logged variables around their steady state values, and subscript \( ss \) denotes steady state values. Log linearize the wage equation (the FOC of the firm), Euler equation, and the equations that characterizes the labor-leisure trade-off, and the government budget constraint.

Log both sides of equation (WE), we have:

\[
\ln W_t = \ln (1 - \zeta) - \zeta \ln L_t
\]

\[
d \ln W_t = -\zeta d \ln L_t
\]

\[\hat{w}_t = -\zeta \hat{l}_t\]

From the government budget contraint, we have:

\[
\ln (1 + \tau_t) + \ln (C_t) = \ln (G_t) \implies 1 + \tau_t + \hat{c}_t = \hat{g}_t
\]

From the (LL), we have:

\[
\ln (1 + \tau_t) + \ln (1 - \alpha) - \ln (1 - L_t) = \ln \alpha - \ln C_t + \ln W_t
\]

\[1 + \tau_t - (1 - \hat{l}_t) = -\hat{c}_t + \hat{w}_t\]

Finally, from (EE), we derive:

\[
\ln \alpha + (\alpha(1 - \nu) - 1) \ln C_t + (1 - \alpha)(1 - \nu) \ln (1 - L_t) - \ln (1 + \tau_t)
= \beta E_t((\alpha(1 - \nu) - 1) \hat{c}_t + (1 - \alpha)(1 - \nu)(1 - \hat{l}_t) - 1 + \tau_t =
\beta E_t((\alpha(1 - \nu) - 1) \hat{c}_{t+1} + (1 - \alpha)(1 - \nu)(1 - \hat{l}_{t+1}) - 1 + \tau_{t+1})
\]

(f) Suppose the economy is hit by a temporal expansionary fiscal policy shock \((\varepsilon_t > 0)\). Proof that this change would cause agents to reduce labor supply. Provide an intuitive explanation for this response.

We want to proof by contradiction. Suppose labor supply increases, we have \(\hat{l}_t > 0\) and \(1 - \hat{l}_t < 0\). From the three contemporaneous relationships, it is easy to show that

\[\hat{g}_t = -\zeta \hat{l}_t + (1 - \hat{l}_t) < 0\]

which contradicts with the assumption.

We can think this as an allocation problem with two types of goods: consumption and leisure. When consumption and leisure are substitutes, an increase in the relative price of consumption leads agents to consume more leisure, or work less.
10. **One-sided search with a two-tier benefit system**

Unemployment insurance systems around the world pay a high level of benefits for an initial period. Once the initial benefits have expired they pay a lower benefit, called subsistence allowance, indefinitely. That’s what we are looking at here.

**Time:** Discrete, infinite horizon.

**Demography:** A single infinite lived worker.

**Preferences:** The worker is risk neutral (i.e. \( u(x) = x \)) and discounts the future at the rate \( r \).

**Endowments:** The endowments of the worker depend on his state within the labor market. There are 3 possible states determined by employment and the size of benefits received.

- When *unemployed with high benefits* he receives flow income \( b_u \) per period, with probability \( \alpha \) he receives a job offer of wage \( w \sim F \) on \([0, \bar{w}]\). Or, with probability \( \gamma \) his high benefits expire and he shifts to being unemployed with low benefits. (**Note:** getting a job and getting a reduction in benefits are mutually exclusive events. Assume \( \alpha + \gamma < 1 \).)

- When *unemployed with low benefits* he receives flow income \( b_x < b_u \) per period (\( b_x \) is the subsistence allowance, the \( x \) stands for “expiry”). He continues to receive job offers with probability \( \alpha \) per period from the same distribution.

- A worker who gets hired immediately requalifies for high benefits.

- While employed a worker hired at wage \( w \) receives flow income \( w \) per period and with probability \( \lambda \) he loses his job to become unemployed (with high benefits).

a. Write down the flow asset value (flow Bellman) equations for the worker.

\[
\begin{align*}
 rV_e(w) &= w + \lambda \left[ V_u - V_e(w) \right] \\
 rV_u &= b_u + \alpha \mathbb{E}_w \left[ \max \left\{ V_e(w) - V_u, 0 \right\} \right] + \gamma \left[ V_x - V_u \right] \\
 rV_x &= b_x + \alpha \mathbb{E}_w \left[ \max \left\{ V_e(w) - V_x, 0 \right\} \right]
\end{align*}
\]

b. Show that for the state of unemployment with high benefits the usual (fundamental search) equation applies (i.e. \( rV_u = w_u^* \) where \( w_u^* \) is the reservation wage for the high benefit unemployed and \( V_u \) is the discounted present expected value of unemployment with high benefits.)

Define, \( w_u^* \) such that \( V_e(w_u^*) = V_u \) and \( w_x^* \) such that \( V_e(w_x^*) = V_x \). These are the reservation wages for high and low benefit recipients respectively. So,

\[
\begin{align*}
 rV_u &= b_u + \alpha \int_{w_u^*}^{\bar{w}} \left[ V_e(w) - V_u \right] dF(w) + \gamma \left[ V_x - V_u \right] \\
 rV_x &= b_x + \alpha \int_{w_x^*}^{\bar{w}} \left[ V_e(w) - V_x \right] dF(w)
\end{align*}
\]
From \( V_e \) equation,
\[
V_e(w_u^*) = \frac{w_u^* + \lambda V_u}{r + \lambda} = \frac{w_u^* + \lambda V_u(w_u^*)}{r + \lambda} \implies rV_u = w_u^*
\]

c. Obtain the related expression for \( V_x \), the discounted present expected value of unemployment with low benefits, and show that the fundamental search equation does not apply in this case. Briefly explain your result.
\[
V_x = V_e(w_x^*) = \frac{w_x^* + \lambda V_u}{r + \lambda} = \frac{w_x^* + \lambda w_u^*/r}{r + \lambda}
\]
So
\[
rV_x = \frac{rw_x^* + \lambda w_u^*}{r + \lambda}
\]
This is weighted average of the reservation wages.\(^1\) The flow value to receiving low benefits exceeds the reservation wage because of the fact that when they get rehired they become re-entitled to higher benefits again. This makes the people receiving low benefits very eager to get a job. Their reservation wage is very low (can be negative).

d. Draw a flow diagram showing the population movements between states when there is a continuum (mass 1) of similar workers.

\(^1\)I did not ask for it, but for completeness here is the system of reservation wage equations:
\[
(r + \lambda)(w_u^* - b_u) = \alpha \int_{w_u^*}^{\bar{w}} (w - w_u^*) \, dF(w) + \gamma (w_x^* - w_u^*)
\]
\[
 rw_x^* + \lambda w_u^* = (r + \lambda)b_x + \alpha \int_{w_x^*}^{\bar{w}} (w - w_x^*) \, dF(w)
\]
From these it is possible to show that as \( b_u > b_x, w_u^* > w_x^* \).
e. Write down a system of equations that can be solved for the steady state populations (you don’t need to solve them).

Let $u$ be the measure of unemployed on high benefits, $x$ the measure of unemployed on low benefits and $e$ be the measure of employed. Any 3 of the following 4 equations can be used to pin down these measures:

\[
\begin{align*}
    u + e + x &= 1 \\
    \alpha [1 - F(w_u^*)] u + \alpha [1 - F(w_x^*)] x &= \lambda e \\
    \{\alpha [1 - F(w_u^*)] + \gamma\} u &= \lambda e \\
    \alpha [1 - F(w_x^*)] x &= \gamma u 
\end{align*}
\]