Section 1. (Suggested Time: 45 Minutes) For 3 of the following 6 statements, state whether the statement is true, false, or uncertain, and give a complete and convincing explanation of your answer. Note: Such explanations typically appeal to specific macroeconomic models.

1. A tax on consumption is equivalent to a tax on labor income.

2. An increase in government spending without a commensurate increase in taxes will increase national income.

3. In the past few months, the Federal Reserve has dramatically increased the money supply, but output, employment and prices have all increased very slowly. Such an outcome is evidence in favor of the quantity theory.

4. An increase in government spending without a commensurate increase in taxes will increase welfare.

5. A permanent increase in productivity should cause permanent increases in both consumption and employment.

6. The lowering of a country’s sovereign debt rating is equivalent to an increase in everyone’s income tax rate and having the revenues thrown into the ocean.
Section 2. (Suggested Time: 2 Hours, 15 minutes) Answer any 3 of the following 4 questions.

7. The preferences of the representative household are

\[ E_0 \left( \sum_{t=0}^{\infty} \beta^t \ln (C_t + G_t) - \chi L_t \right), \]

while the production function is

\[ Y_t = L_t^{1-\alpha}, \quad 0 < \alpha < 1. \quad \text{(PRF)} \]

Government expenditures, \( G_t \), are driven by fluctuation in taxes:

\[ G_t = (1 - S_t) W_t L_t, \quad \text{(GBC)} \]

with \( S_t = 1 - \tau_t^e \), the ‘after-tax’ rate on labor income, following an AR(1) process around the log of its steady state value:

\[ \hat{s}_t \equiv \ln \left( \frac{S_t}{S} \right) = \phi \hat{s}_{t-1} + \varepsilon_t, \quad 0 \leq \phi < 1, \quad \text{(TS)} \]

where \( \{\varepsilon_t\} \) is a zero-mean i.i.d. process.

(a) We set about finding the competitive allocation. Recall that the representative producer solves

\[ \max_{L_t \geq 0} \Pi_t = L_t^{1-\alpha} - W_t L_t, \]

so that the first order condition for profit maximization is

\[ (1 - \alpha) L_t^{-\alpha} = (1 - \alpha) \frac{Y_t}{L_t} = W_t, \quad \text{(PM)} \]

and profits are

\[ \Pi_t = \alpha Y_t. \]

The household’s flow budget constraint is

\[ C_t + B_t = (1 + r_{t-1}) B_{t-1} + S_t W_t L_t + \Pi_t. \quad \text{(FBC)} \]

Recalling that the household takes government spending as given, the first order conditions for utility maximization are

\[ \frac{1}{C_t + G_t} S_t W_t = \chi, \quad \text{(LL)} \]

\[ \frac{1}{C_t + G_t} = \beta (1 + r_t) E_t \left( \frac{1}{C_{t+1} + G_{t+1}} \right). \quad \text{(EE)} \]
(b) Combining the consumer’s, firm’s and government’s budget constraints yields the capital accumulation equation:

\[
B_t = (1 + r_{t-1}) B_{t-1} + S_t W_t L_t + \Pi_t - C_t
\]

\[
= (1 + r_{t-1}) B_{t-1} + W_t L_t - (1 - S_t) W_t L_t + \Pi_t - C_t
\]

\[
= (1 + r_{t-1}) B_{t-1} + Y_t - G_t - C_t.
\]

With no capital and identical consumers, the equilibrium quantity of bonds is zero, and the resource constraint is

\[
Y_t = G_t + C_t. \quad \text{(RC)}
\]

Combining equations (PM), (LL) and (RC) yields the labor-leisure condition:

\[
S_t (1 - \alpha) \frac{Y_t}{L_t} \left( \frac{1}{C_t + G_t} \right) = \chi
\]

\[
\Rightarrow L_t = \left( \frac{1 - \alpha}{\chi} \right) S_t. \quad \text{(LL')}
\]

Inserting this result into equation (PRF) yields

\[
Y_t = \left( \frac{1 - \alpha}{\chi} \right)^{1-\alpha} S_t^{1-\alpha} \equiv \Lambda S_t^{1-\alpha}. \quad \text{(PRF')}
\]

Using this result, and the resource constraint, we can rewrite the Euler equation as

\[
1 + r_t = \beta^{-1} \left( \frac{1}{\Lambda S_{t+1}^{1-\alpha}} \right) E_t \left( \frac{1}{\Lambda S_t^{1-\alpha}} \right)^{-1}. \quad \text{(EE')}
\]

(c) It follows from equation (PRF') that higher labor income taxes (lower values of \(S_t\)) decrease output. By reducing the after-tax wage, the substitution effect of higher taxes is less work. In equilibrium, there is essentially no income effect, because labor taxes are returned to the consumer in the form of government goods. Because the government goods are perfectly substitutable for private consumption, they are equivalent to lump-sum tax refunds. The only income effect is that in equilibrium the consumer works less and thus has less total consumption. This allows the substitution effect to dominate.

(d) Let lower-case letters with carats “\(^\ast\)” denote deviations of logged variables around their steady state values. It follows from equation (LL') that

\[
L_t = \left( \frac{1 - \alpha}{\chi} \right) S_t
\]

\[
\Rightarrow \exp(\hat{\ell}_t) = \frac{L_t}{L_{ss}} = \left( \frac{1 - \alpha}{\chi} \right) S_t / \left( \frac{1 - \alpha}{\chi} \right) S = \exp(\hat{s}_t).
\]

Logging both sides yields

\[
\hat{\ell}_t = \hat{s}_t. \quad \text{(LL'')}\]
Proceeding similarly, it follows from equation (PRF') that
\begin{align*}
\exp(\tilde{y}_t) &= \exp((1 - \alpha)\tilde{s}_t), \\
\Rightarrow \tilde{y}_t &= \theta \tilde{s}_t, \quad \theta \equiv 1 - \alpha > 0.
\end{align*}

(PRF'')

The log of average labor productivity is
\begin{align*}
\hat{apl}_t &= \tilde{y}_t - \ell_t \\
&= -\alpha \tilde{s}_t.
\end{align*}

(e) Note that average labor productivity can be written as
\begin{align*}
\hat{apl}_t &= -\frac{\alpha}{1 - \alpha} \tilde{y}_t.
\end{align*}

With \(\alpha/(1 - \alpha) > 0\), it immediately follows that increases in \(\tilde{y}_t\) are accompanied by decreases in \(\hat{apl}_t\), so that labor productivity is countercyclical. Fluctuations in output are driven solely by fluctuations in the tax rate. With decreasing returns to labor, however, a increase in labor that cause output to rise will cause average productivity, \(Y/L\), to fall. This feature of the model is inconsistent with the data, where labor productivity is pro-cyclical. One way to improve the models performance would be to introduce labor hoarding. When labor hoarding occurs, firms use their workers more intensively during economic expansions, causing the standard productivity measures, with do not adjust for intensity, to increase.

8. Proportional income tax in simple dynamic model with a government and specific utility functions

Consider the following economy:

**Time:** Discrete; infinite horizon

**Demography:** Continuum of mass 1 of (representative) consumer/worker households, and a large number of profit maximizing firms, owned jointly by the households.

**Preferences:** the instantaneous household utility function is \(c^{1/2}\) where \(c\) is household consumption. The discount factor is \(\beta \in (0, 1)\).

**Technology:** There is a constant returns to scale technology for which labor is the only input so that a firm that hires \(h\) units of labor produces \(zh\) units of output.

**Endowments:** Each household has 1 unit of time per period to allocate how ever they like between work and leisure.

**Institutions:** There is a government that has to meet an exogenous stream of expenditures, \(\{g_t\}\). Government spending is thrown into the ocean. The government can levy taxes and issue bonds in order to meet its expenditure requirement. Taxes are restricted to being proportional to labor income so that in period \(t\), the tax revenue from a household which provides labor services \(h_t\) is then \(\tau_t w_t h_t\) where \(\tau_t\) is the period \(t\) tax rate and \(w_t\) is the wage rate. Every period there are markets for labor, government bonds and consumption goods.
(a) Write down and solve the problems faced by the representative household and the representative firm.

Households solve: (Students might spot immediately that households do not value leisure and so $h_t = 1$ for all $t$ in which case as long as they mention it, they can drop $h_t$ from the household’s problem.)

$$\max_{\{c_t, h_t, s_t\}} \sum_{t=0}^{\infty} \beta^t c_t^{1/2}$$

s.t. $c_t = (1 - \tau_t)w_t h_t + (1 + r_t) s_{t-1} - s_t$

where $s_t$ is the saving (in bonds) by a household in period $t$ and $r_t$ is the interest on bonds

Clearly, $h_t = 1$ for all $t$

and there is an intertemporal optimality condition

$$c_t^{-1/2} = \beta (1 + r_{t+1}) c_{t+1}^{-1/2}$$

Firms solve

$$\max_{h^f_t} \left\{ z h^f_t - w_t h^f_t \right\}$$

So for any firm that hires a strictly positive (but finite) amount of labor,

$$w_t = z$$

for all $t$

(b) Write down the government’s (period by period) budget constraint.

$$g_t + (1 + r_t)b_{t-1} = b_t + \tau_t w_t h_t$$

(c) Define and a characterize a competitive equilibrium.

A competitive equilibrium is a sequence of prices, $\{w_t, r_t\}$ a sequence of tax rates $\{\tau_t\}$ and an allocation $\{c_t, h_t, h^f_t, b_t, s_t\}$ such that, given prices and tax rates, the allocation solves the households’ problem and the firms’ problem; the government budget constraint holds and the markets for labor, consumption goods and bonds all clear.

The market clearing conditions are:

$$h^f_t = h_t = 1$$

$$b_t = s_t$$

$$z = g_t + c_t$$

Using these we get

$$c_t = (1 - \tau_t) z + (1 + r_t) b_{t-1} - b_t$$
and the government budget constraint becomes
\[ g_t + (1 + r_t)b_{t-1} = b_t + \tau_t z \]

Characterization of equilibrium:
\[ 1 + r_{t+1} = \beta \left( \frac{z - g_{t+1}}{z - g_t} \right)^{1/2} \]

(d) Does Ricardian equivalence hold? Explain

Yes, the timing \( \tau_t \) does not affect the equilibrium allocation which is summarized by \( c_t = z - g_t \) for all \( t \).

(e) How would your answer to part (d) change if the utility function was replaced by \( 2c_t^{1/2}(1 - h_t)^{1/2} \)? Explain your answer.

If households care about leisure then the proportional tax will distort the labor-leisure time allocation. Goods market clearing will be \( zh_t = g_t + c_t \) and \( h_t \) will depend on the timing of taxes. The equilibrium wage will still be \( z \) for all \( t \) but the first-order condition for the labor leisure choice will imply
\[ (zh_t - g_t)^{-1/2}(1 - h_t)^{1/2}z(1 - \tau_t) = (zh_t - g_t)^{1/2}(1 - h_t)^{-1/2} \]

So
\[ (1 - h_t)z(1 - \tau_t) = zh_t - g_t \]

and \( c_t = zh_t - g_t \) for all \( t \) but \( h_t \) is a function of \( \tau_t \).

9. (Adapted from a question by Rody Manuelli.) We are considering a Lucas tree model where dividends follow a two-state growth process. The preferences of the representative consumer are
\[ E_0 \left( \sum_{t=0}^{\infty} \beta^t \frac{1}{1 - \sigma} \left[ c_t^{1-\sigma} - 1 \right] \right), \quad 0 < \beta < 1, \quad \sigma > 0. \]

Output is produced by an infinite-lived tree: each period, the tree produces \( d_t \) units of non-storable output. The growth rate of dividends follows a stationary process, with \( G_t \equiv d_t/d_{t-1} \) following a two-state Markov chain. In particular, \( G_t \) can take on two values: 1, and \( \gamma > 1 \). The conditional probabilities are
\[
\begin{align*}
\Pr(G_{t+1} = \gamma | G_t = \gamma) &= \pi, \\
\Pr(G_{t+1} = 1 | G_t = \gamma) &= 1 - \pi, \\
\Pr(G_{t+1} = \gamma | G_t = 1) &= 0, \\
\Pr(G_{t+1} = 1 | G_t = 1) &= 1.
\end{align*}
\]

so that zero growth is permanent.
(a) Writing the consumer’s problem as a Lagrangean, we get

\[ V(x_t, d_t, G_t) = \min_{\{\lambda_t \geq 0\}} \max_{\{c_t \geq 0, s_{t+1}, z(1), z(\gamma)\}} \left\{ \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \right\} \]

\[ + \lambda_t \left( x_t - c_t - p_t s_{t+1} - \sum_{G_{t+1} \in (1, \gamma)} q(G_{t+1}, G_t) z(G_{t+1}) \right) \]

\[ + \beta \sum_{G_{t+1} \in (1, \gamma)} \left\{ [p_{t+1} + d_{t+1}] s_{t+1} + z(G_{t+1}) d_{t+1}, G_{t+1} \right\} f(G_{t+1}, G_t) \}

The FOC for an interior solution are:

\[ c_t^{-\sigma} = \lambda_t, \]
\[ \lambda_t p_t = \beta E_t \left( \frac{\partial V[t+1]}{\partial x_{t+1}} [p_{t+1} + d_{t+1}] \right), \]
\[ \lambda_t q(G_{t+1}, G_t) = \beta \frac{\partial V[t+1]}{\partial x_{t+1}} f(G_{t+1}, G_t), \quad G_{t+1} \in \{1, \gamma\}. \]

Since (following Benveniste-Scheinkman),

\[ \frac{\partial V[t]}{\partial x_t} = \lambda_t = c_t^{-\sigma}, \]

the Euler equations are

\[ q(G_{t+1}, G_t) = \beta \frac{c(G_{t+1}, d_{t+1})^{-\sigma}}{c(G_t, d_t)^{-\sigma}} f(G_{t+1}, G_t), \quad G_{t+1} \in \{1, \gamma\}, \quad (EE1) \]

\[ p_t c_t^{-\sigma} = \beta E_t \left( c_{t+1}^{-\sigma} [p_{t+1} + d_{t+1}] \right). \quad (EE2) \]

(b) Given the random variables \( d_0 \) and \( G_0 \), the conditional density \( f(G_{t+1}, G_t) \), and the initial endowments \( s_0 = 1 \) and \( z_1 = z_\gamma = 0 \), a recursive rational expectations equilibrium consists of pricing functions \( p(d, G) \) and \( q(G', G) \), a value function \( V(x, d, G) \), and decision functions \( c(x, d, G) \), \( s(x, d, G) \), and \( z(x, d, G', G) \) such that:

1. Given the pricing functions \( p(d, G) \) and \( q(G', G) \), the value and policy functions \( V(x, d, G) \), \( c(x, d, G) \), \( s(x, d, G) \), and \( z(x, d, G', G) \) solve the consumer’s problem.

2. Markets clear: for \( x = p(d, G) + d \), \( c(x, d, G) = d \), \( s(x, d, G) = 1 \), and \( z(x, d, 1, G) = z(x, d, \gamma, G) = 0 \).

(c) Noting that \( d_{t+1} = G_{t+1} d_t \), it follows from equation \((EE1)\) that equilibrium contingent claims prices obey

\[ q(G_{t+1}, G_t) = \beta \frac{(G_{t+1} d_t)^{-\sigma}}{d_t^{-\sigma}} f(G_{t+1}, G_t), \]

\[ = \beta G_{t+1}^{-\sigma} f(G_{t+1}, G_t) \quad G_{t+1} \in \{1, \gamma\}, \quad (EE1') \]

It follows from our specification of the stochastic process for \( G_t \) that:
1. When the current state is 1, we have
\[ q(\gamma, 1) = \beta \gamma^{-\sigma} \cdot 0 = 0, \]
\[ q(1, 1) = \beta 1^{-\sigma} \cdot 1 = \beta. \]

2. When the current state is \( \gamma \), we have
\[ q(\gamma, \gamma) = \beta \gamma^{-\sigma} \cdot \pi = \beta \pi \gamma^{-\sigma}, \]
\[ q(1, \gamma) = \beta 1^{-\sigma} \cdot (1 - \pi) = \beta (1 - \pi). \]

(d) It follows from arbitrage arguments that if an asset pays \( w(G') \) units of consumption goods when \( G_{t+1} = G' \), its price is
\[ p_t^w = \sum_{G_{t+1} \in \{1, \gamma\}} w(G_{t+1}) q(G_{t+1}, G_t). \]

When the asset is a risk-free discount bond, \( w(G') = 1 \). Imposing equation (EE1'), it follows that the price of a risk-free bond, \( R^{-1}(G_t) \), is given by
\[ R^{-1}(G_t) = \beta \sum_{G_{t+1} \in \{1, \gamma\}} G_{t+1}^{-\sigma} f(G_{t+1}, G_t) = \beta E_t(G_{t+1}^{-\sigma}). \]

Explicit solutions for the two values of \( G_t \) are
\[ R^{-1}(1) = \beta E_t(\gamma^{-\sigma}|G_t = 1) = \beta, \]
\[ R^{-1}(\gamma) = \beta E_t(\gamma^{-\sigma}|G_t = \gamma) = \beta [\pi \gamma^{-\sigma} + (1 - \pi)]. \]

(e) Let \( q(G', G_t) / f(G', G_t) \) denote the “probability-adjusted” contingent claim price. It follows from equation (EE1') that \( q(G', G_t) / f(G', G_t) = \beta (G')^{-\sigma}. \) With \( \gamma > 1 \) and \( \sigma > 0 \), it follows that
\[ \frac{q(\gamma, \gamma)}{f(\gamma, \gamma)} = \beta \gamma^{-\sigma} < R^{-1}(\gamma) = \beta [\pi \gamma^{-\sigma} + (1 - \pi)] < \frac{q(1, \gamma)}{f(1, \gamma)} = \beta. \]

This rank ordering reflects the diminishing marginal utility of consumption. A unit of output delivered when \( G_{t+1} = \gamma \) is a unit delivered when total consumption is highest and the marginal utility of consumption is lowest. Adjusting for conditional probabilities, this unit of consumption is worth less than a unit of consumption delivered when \( G_{t+1} = 1 \), when the marginal utility of consumption is high. The risk-free bond, being a weighted average, has an intermediate value.

10. **Diamond Coconut Economy with uniform idiosyncratic preferences**

**Time:** Discrete, infinite horizon

**Geography:** A trading island and a production island.

**Demography:** A mass of 1 of ex ante identical individuals with infinite lives.
**Preferences:** The common discount rate is $r$, consumption of own produce yields 0 utils, consumption of someone whose good you like yields $u$ utils. The share of individuals whose goods I like is $\phi$. Whether an individual likes my good or not is independent of whether I like hers. (So she likes my good with probability $\phi$ too.)

**Productive Technology:** On the production island individuals come across a tree with a coconut with probability $\alpha$ each period. The cost of obtaining the coconut is $c$ which is uniformly distributed over $(0, \tilde{c}]$ where $\tilde{c} > u$ (so some trees will get rejected).

**Matching Technology:** On the trading island people with coconuts meet each with probability $f$, a constant.

**Navigation:** Travel between islands is instantaneous.

**Endowments:** Everyone has a boat and starts off with one of their own own coconuts

(a) Write down the asset value (Bellman) equations for this economy. Define the terms you introduce.

The regular asset value equations are

$$V_T = \frac{1}{1+r} \left\{ \gamma \phi^2 (u + V_P) + (1 - \gamma \phi^2) V_T \right\}$$

$$V_P = \frac{1}{1+r} \left\{ \alpha E \left[ \max \{ V_T - c, V_P \} \right] + (1 - \alpha) V_P \right\}$$

where $V_T$ is the value to being on the trading island and $V_P$ is the value to being on the production island

The flow value equations are

$$r V_T = \gamma \phi^2 (u + V_P - V_T)$$

$$r V_P = \alpha E \left[ \max \{ V_T - c - V_P, 0 \} \right]$$

(b) Define a search equilibrium

A search equilibrium is a pattern of trade such that given everyone else conforms to that patterns no individual will wish to deviate from it.

(c) Solve for an implicit equation that specifies the reservation tree “height”, $c^*$ in terms of the parameters of the model.

Let $c^* = V_T - V_P$. Then rewrite $V_P$ as

$$r V_P = \alpha \int_0^{c^*} \left[ V_T - c - V_P \right] dc$$

now subtract $r V_P$ from $r V_T$ to get

$$r \left( V_T - V_P \right) = \gamma \phi^2 u - \gamma \phi^2 \left( V_T - V_P \right) - \alpha \int_0^{c^*} \left[ V_T - c - V_P \right] dc$$

or by definition of $c^*$,

$$(r + \gamma \phi^2) c^* = \gamma \phi^2 u - \alpha \int_0^{c^*} \left[ c^* - c \right] dc$$

so

$$(r + \gamma \phi^2) c^* - \gamma \phi^2 u + \frac{\alpha c^*^2}{2} = 0$$
(d) How does $c^*$ change with respect to $\phi$ (i.e. obtain the sign of the comparative static)?

Define

$$\Psi(c, \phi) \equiv (r + \gamma \phi^2)c - \gamma \phi^2 u + \frac{\alpha c^2}{2}$$

so that $\Psi(c^*, \phi) = 0$. Then

$$\frac{dc^*}{d\phi} = \frac{-\frac{\partial \Psi}{\partial \phi}}{\frac{\partial \Psi}{\partial c}} \bigg|_{c=c^*}$$

Then

$$\frac{\partial \Psi}{\partial \phi} \bigg|_{c=c^*} = 2\phi \gamma (c^* - u) < 0$$

(sign comes from equation (1)). And

$$\frac{\partial \Psi}{\partial c} \bigg|_{c=c^*} = (r + \gamma \phi^2) + \alpha c^* > 0.$$  

So

$$\frac{dc^*}{d\phi} > 0$$

Increasing $\phi$ rapidly improves my chances of getting to trade my good and return to the production island. Investment (in inventory) has a higher return and I am therefore more ready to invest.

(e) Draw a diagram showing the population flows between the islands. Write down the steady-state equations and solve for the population on the trading island as a function of $c^*$.

The diagram is the same as in the class notes for constant $\gamma$ except that $\gamma$ should be replaced with $\gamma \phi^2$ and $F(c^*)$ replaced with $c^*$. Steady-state equations are:

$$\gamma \phi^2 n_T = \alpha c^* n_P$$

$$n_T + n_P = 1$$

where $n_T$ is proportion of the population on the trading island and $n_P$ is the proportion of the population on the production island. Thus

$$n_T = \frac{\alpha c^*}{\gamma \phi^2 + \alpha c^*}$$