1. The NAND operator is defined as

\[ x \text{ NAND } y \equiv \neg(x \land y) \]

Prove, without using truth-tables, that

\[ p \rightarrow (q \text{ NAND } r) \equiv (p \text{ NAND } q) \lor (p \text{ NAND } r) \]

No credit will be given if you use truth-tables. (Remark: Case analysis is no different from using a truth-table.)

\[ p \rightarrow (q \text{ NAND } r) \equiv p \rightarrow (\neg q \lor \neg r) \equiv \neg p \lor \neg q \lor \neg r \equiv (\neg p \lor \neg q) \lor (\neg p \lor \neg r) \equiv (p \text{ NAND } q) \lor (p \text{ NAND } r) \]

2. Is the following quantified formula true over the set of integers \( \mathbb{Z} \)?

\[ \forall x \exists y \ [(x < y) \rightarrow (x^2 < y^2)] \]

Justify your answer clearly.

**True:** Pick \( y = |x| + 1 \) whatever \( x \) is.

3. Let \( A = \{x, y, z\} \). Specify a binary relation \( R \) on \( A \) such that \( R \) is not reflexive, but \( R \circ R \) is reflexive. You should explain for your example why \( R \) is not reflexive but \( R \circ R \) is.

\[ R = \{(x, y), (y, x), (z, z)\} \]

4. Let \( A = \{a, b, c, d\} \). How many binary relations over \( A \) are both symmetric and antisymmetric?

\[ 2^4 \]

5. Let \( \mathbb{N}^+ = \{1, 2, \ldots\} \) denote the set of positive integers. Consider the function \( f \) from \( \mathbb{N}^+ \) to \( \mathbb{N}^+ \) defined as follows:

\[ f(x) = 1 + \text{ the number of 9's in the decimal representation of } x. \]

For example, \( f(1) = 1, f(293) = 2, f(1929) = 3. \)

(a) Is \( f \) one-to-one? Justify your answer.

(b) Is \( f \) onto? Justify your answer.
6. Suppose \( A = \{a, b, c, d\} \) and \( B = \{1, 2, 3\} \). How many functions from \( A \) to \( B \)
(a) map \( a \) to 1, and
(b) are not onto?

In other words, how many functions satisfy both conditions (a) and (b)?

There are 27 functions from \( A \) to \( B \) that also map \( a \) to 1. There are 6 onto functions from \( \{b, c, d\} \) to \( \{1, 2, 3\} \) and 6 onto functions from \( \{b, c, d\} \) to \( \{2, 3\} \). Thus the answer is \( 27 - (6 + 6) = 15 \).

Alternatively, there are 8 functions from \( \{b, c, d\} \) to \( \{1, 2\} \) and 8 functions from \( \{b, c, d\} \) to \( \{1, 3\} \). The function that maps every element in \( A \) to 1 is common to both counts; thus the answer is \( 8 + 8 - 1 = 15 \).

9. Alice hiked for 10 hours and covered a total distance of 35 miles. It is known that she covered 4 miles during the first hour and only 2 miles during the last hour. Prove that she must have hiked at least 8 miles within a certain period of two consecutive hours.

Let \( x_1, x_2, \ldots, x_{10} \) be the distances that Alice covered each hour. Suppose \( x_i + x_{i+1} \leq 7 \) for all \( i < 10 \). Then

\[
(x_1 + x_2) + (x_2 + x_3) + \ldots + (x_9 + x_{10}) \leq 63
\]

Adding \( x_1 \) and \( x_{10} \) to this, we get

\[
2x_1 + \ldots + 2x_{10} \leq 69
\]

and thus

\[
x_1 + \ldots + x_{10} < 35
\]