1. Suppose $p, q$ and $r$ are propositions such that $p \rightarrow q$, $(\neg p) \rightarrow r$ and $r \rightarrow (p \lor q)$ are all true. Show that $q$ is true.

Suppose $q$ is false. Then $p$ also has to be false. This will force $r$ to be true. Now the third formula $r \rightarrow (p \lor q)$ will be false.

2. Is the following quantified formula true over the set of integers $\mathbb{Z}$?

$$\forall x \exists y \left[ (x = y^2) \rightarrow (y \geq 0) \right]$$

Justify your answer clearly.

**True**: Whatever $x$ is, set $y = 0$.

Alternatively, whatever $x$ is, set $y = x + 1$. Now $(x = (x + 1)^2)$ is never true over $\mathbb{Z}$.

3. Suppose $A$, $B$ and $C$ are sets such that $B \subsetneq C$ and $A \cap C = \emptyset$. Prove that $(A \cup B) \subsetneq (A \cup C)$.

$(X \subsetneq Y$ means $X$ is a proper subset of $Y$.)

No credit will be given if you use Venn diagrams to prove this result.

Suppose $x \in (C \setminus B)$. Then $x \in (A \cup C)$ and $x \not\in (A \cup B)$ since $x \not\in A$ (because $A \cap C = \emptyset$).

4. How many different boolean functions $F(x, y, z)$ are there so that $F(x, y, z) = F(z, y, x)$ for all values of the boolean variables $x$, $y$, and $z$? Briefly justify your answer. (No formal proof is needed.)

$2^6$

5. Use induction on $n$ to prove that for all integers $n \geq 8$, postage of $n$ cents can be realized using only 3-cent and 5-cent stamps.

Trivial for $n = 8, 9, 10$ (the base cases). For $n \geq 11$, if $n - 3 = 3p + 5q$ then $n = 3(p + 1) + 5q$.

6. Let $A = \{x, y, z\}$. Specify a binary relation $R$ on $A$ such that $R$ is not reflexive, but $R \circ R$ is reflexive. You should explain for your example why $R$ is not reflexive but $R \circ R$ is.

$$R = \{(x, y), (y, x), (x, z), (z, x)\}.$$ 

7. Let $\mathbb{R}$ denote the set of real numbers. Consider the function $f$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (3x + 2y, x + y)$. Find $f^{-1}$. (You may assume without proof that $f$ is a bijection.)

If $3x + 2y = p$ and $x + y = q$, then $x = p - 2q$ and $y = 3q - p$. Thus

$$f^{-1}(p, q) = (p - 2q, 3q - p).$$
8. How many non-negative integer solutions are there for the equation \(2x_1 + 2x_2 + 2x_3 + x_4 = 15\) if \(x_1 \geq 1\)? (You may leave the answer as an expression consisting of binomial coefficients.)

Note that \(x_4\) has to be an odd number; so let it be equal to \(2 * y_4 + 1\). Similarly, let \(x_1 = y_1 + 1\). Thus we get the equation \(y_1 + x_2 + x_3 + y_4 = 6\). This has \(\binom{9}{3}\) solutions.

9. Recall that a bit string is a string composed of characters 0 and 1. How many bit strings of length 20 have either less than ten 1’s or contain 11 as a substring? (10101010101010101000 is such a string. So is 00000000110000000000.)

The number of bit strings of length 20 that have ten or more 1’s and does not contain 11 as a substring is 2. Thus the answer is \(2^{20} - 2\).

10. Consider a square whose side has length 2. Suppose \(S\) is an arbitrarily chosen set of 5 points from this square. Prove that \(S\) contains two points whose distance is at most \(\sqrt{2}\).

The square can be divided into 4 identical mini-squares whose sides have length 1. Among the 5 points, at least two should be in the same mini-square. The length of the diagonal in each mini-square is \(\sqrt{2}\).