CSI 604 – Spring 2016

Proof of NP-Completeness: An Example

In this handout, we provide a proof that the Minimum Dominating Set problem for graphs is NP-complete. This is accomplished by a reduction from the Minimum Set Cover problem. The formal definitions of these problems are as follows.

Minimum Set Cover (MSC)

Instance: A base set \( Q = \{q_1, q_2, \ldots, q_n\} \), a collection \( C = \{c_1, c_2, \ldots, c_m\} \) where \( c_j \subseteq Q \ (1 \leq j \leq m) \) and an integer \( K \leq m \).

Question: Is there a subcollection \( C' \subseteq C \), with \( |C'| \leq K \), such that \( \bigcup_{c \in C'} c = Q \)?

Minimum Dominating Set (MDS)

Instance: An undirected graph \( G(V, E) \) and an integer \( R \leq |V| \).

Question: Is there a dominating set \( D \subseteq V \) for \( G \) such that \( |D| \leq R \)?

Theorem: MDS is NP-complete.

Proof: MDS is in NP since given a subset \( D \) of vertices we can verify in polynomial time that \( |D| \leq R \) and that for each node \( v \in V - D \), there is a node \( w \in D \) such that \( \{v, w\} \in E \).

To prove NP-hardness, we use a reduction from the MSC problem. Consider an instance \( I \) of MSC given by the set \( Q \), the collection \( C \) and the integer \( K \). An instance \( I' \) of the MDS problem consisting of graph \( G(V, E) \) and the integer \( R \) is constructed as follows.

1. Let \( V = V_1 \cup V_2 \) where \( V_1 = \{v_1, v_2, \ldots, v_n\} \) (i.e., there is one vertex in \( V_1 \) corresponding to each element in \( Q \)) and \( V_2 = \{w_1, w_2, \ldots, w_m\} \) (i.e., there is one vertex in \( V_2 \) corresponding to each set in \( C \)).

2. Let \( E = E_1 \cup E_2 \) where

\[
E_1 = \{\{v_i, w_j\} | q_i \in c_j\} \quad \text{and} \quad E_2 = \{\{w_i, w_j\} | i \neq j\}.
\]

Thus, the edges in \( E_1 \) represent the membership of elements in subsets and the edges in \( E_2 \) connect all the nodes in \( V_2 \) as a clique.

3. The construction is completed by setting the value of \( R \) (the dominating set size) to \( K \) (the size of set cover).

The construction can be carried out in polynomial time since \( V \) has \( m + n \) nodes and \( E \) has \( O(m(m + n)) \) edges (because \( E_1 \) has at most \( mn \) edges and \( E_2 \) has \( m(m - 1)/2 \) edges).
We now show that the resulting MDS instance $I'$ has a solution if and only if the MSC instance $I$ has a solution.

**Part 1:** Suppose the MSC instance $I$ has a set cover of size at most $K$.

**To prove:** $G$ has a dominating set $D$ of size at most $K$.

**Proof:** Let $C' = \{c_{i_1}, c_{i_2}, \ldots, c_{i_r}\}$, where $r \leq K$, denote the given set cover. Consider the set $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_r}\}$ of nodes from $G$. We claim that $D$ is a dominating set for $G$. To see this, first note that any node in $D$ is adjacent to all the nodes in $V_2 - D$ (since the nodes in $V_2$ form a clique). Further, since $C'$ is a set cover, every node in $V_1$ (i.e., an element node) is adjacent to at least one node in $D$ (i.e., a set node). Thus, $D$ is indeed a dominating set. Further, $|D| = r \leq K$ and so $D$ is a dominating set of size at most $K$ for $G$. This completes the proof of Part 1.

**Part 2:** Suppose the MDS instance has a dominating set of size at most $K$.

**To prove:** There is a set cover $C'$ of size at most $K$ for the MSC instance.

**Proof:** Let $D'$ be dominating set of size $r \leq K$ for $G$. Partition $D'$ into $D_1$ and $D_2$ where $D_i \subseteq V_i$, $i = 1, 2$. If $D_1$ is nonempty, we obtain a set of nodes $D$ by repeatedly modifying $D'$ as follows until $D_1$ becomes empty: Let $v$ be a node (element node) in $D_1$. Find a node $w$ (a set node) in $V_2$ such that $\{v, w\} \in E$. Such a node must exist since each element in the MSC instance occurs in at least one set. Delete $v$ from $D_1$ and add $w$ to $D_2$ if $w$ is not already in $D_2$. After each step of this modification, $D_1 \cup D_2$ continues to be a dominating set since all the (set) nodes dominated by the element node $v$ are also dominated by the set node $w$. Obviously, this modification does not increase the size of $D$. At the end of this modification, $D = D_2 \subseteq V_2$.

Let $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_r}\}$. Recall that $r \leq K$. Consider the subcollection $C' = \{c_{i_1}, c_{i_2}, \ldots, c_{i_r}\}$. We will prove that $C'$ is a set cover. To see this, consider any element $q_x \in Q$. We will show that there is a set in $C'$ that contains $q_x$. Recall that $D'$ is the original dominating set for $G$. We have two cases to consider.

**Case 1:** Set $D'$ contains $v_x$, the node corresponding to $q_x$.

In this case, the modified dominating set $D$ was obtained by replacing $v_x$ by a node $w_y \in V_2$ such that $\{w_y, v_x\}$ is an edge in $G$. By our construction and the choice of $C'$, the set $c_y$ corresponding to $w_y$ is in $C'$ and $c_y$ contains $q_x$.

**Case 2:** Set $D'$ does not contain $v_x$, the node corresponding to $q_x$.

In this case, since $D'$ is a dominating set, there must be a node $w_y \in D_1$ such that $\{w_y, v_x\}$ is an edge in $G$. By our construction and the choice of $C'$, the set $c_y$ corresponding to $w_y$ is in $C'$ and $c_y$ contains $q_x$.

Thus, $C'$ is a valid set cover. Further, $|C'| = r \leq K$ and so $C'$ is a solution to the MSC instance. This completes the proof of Part 2 as well as that of the theorem.