CSI 445/660 – Part 9
(Introduction to Game Theory)

Ref: Chapters 6 and 8 of [EK] text.
Game Theory Pioneers

- John von Neumann (1903–1957)
- Ph.D. (Mathematics), Budapest, 1925
- Contributed to many fields including Mathematics, Economics, Physics and Computer Science.
- Taught at the Institute for Advanced Study in Princeton.
- A key participant in the Manhattan Project.

Note: The book “Theory of Games and Economic Behavior” by von Neumann and Morgenstern (which marks the beginning of Game Theory) was first published in 1944.
Game Theory Pioneers

- Oskar Morgenstern (1902–1977)
  - Ph.D. (Political Science), University of Vienna, 1925.
  - Taught at Princeton University and the Institute for Advanced Study at Princeton.
  - Many contributions to Economics and Mathematics.

- John Nash (1928–2015)
  - Many deep contributions to Mathematics.
  - Taught at MIT.
Game Theory: Useful in analyzing situations where outcomes depend on a person’s decisions as well as the choices made by others interacting with the person.

Some Applications:

- Pricing a product (when other companies have a similar product).
- Auctions.
- Choosing routes in transportation networks.
- International relations.

An example of a 2-person game:

- Two students ("players") A and B.
- They have an exam and a joint presentation the next day.
- Each can only prepare for one and not both.
Score for the exam:

- If the student studies, then score = 92.
- If the student doesn’t study, then score = 80.

Score for the presentation:

- If both A and B prepare, then score = 100.
- If only one student prepares, then score = 92.
- If neither A nor B prepares, then score = 84.

A and B cannot contact each other; however, they must make a decision.

Analysis:

1. Both A and B prepare for the presentation.

- Each gets 100 for the presentation.
- Each gets 80 for the exam.
- Average score for each = 90.
Analysis: (continued)

2 Both A and B study for the exam.
   - Each gets 92 for the exam.
   - Each gets 84 for the presentation.
   - Average score for each = 88.

3 A studies for the exam and B prepares for the presentation.
   - A gets 92 for the exam and 92 for the presentation.
     So, average score for A = 92.
   - B gets 80 for the exam and 92 for the presentation.
     So, average score for B = 86.

4 A prepares for the presentation and B studies for the exam.
   - A gets 80 for the exam and 92 for the presentation.
     So, average score for A = 86.
   - B gets 92 for the exam and 92 for the presentation.
     So, average score for B = 92.
Summary of the Analysis – Payoff matrix:

Table shows the actions for A and B.

The payoff value \((x, y)\) means that A’s (average) score is \(x\) and B’s (average) score is \(y\).

**Note:** A’s payoff depends on B’s actions as well.

Basic ingredients of a game:

- A set of players (**Focus:** 2-person games).
- A set of options (**strategies**) for each player.
- A **payoff matrix** that specifies the payoff values for the players for each combination of strategies.

**Note:** The game is completely captured by the payoff matrix.
**Standard Assumptions**

- **One-shot** games: Each player chooses an action (strategy) **without** knowing what the other player will choose.
- Everything players care about is specified in the payoff matrix.
- Each player knows all the possible strategies and the full payoff matrix. (If not, we have games of *incomplete information*.)
- Players behave **rationally**.
  - Each player wants to maximize his/her payoff.
  - Each player succeeds in selecting an optimal strategy.

**Illustration – Reasoning in the Exam-Presentation Game:**

- Consider the reasoning from A’s point of view. (B’s point of view is similar because of symmetry.)
Reasoning in the Exam-Presentation Game

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(90,90)</td>
<td>(86,92)</td>
</tr>
<tr>
<td>E</td>
<td>(92,86)</td>
<td>(88,88)</td>
</tr>
</tbody>
</table>

**Case 1:** Suppose B chooses E.

- If A chooses P, payoff = 86.
- If A chooses E, payoff = 88.
- Due to rationality, A must choose E in this case.

**Case 2:** Suppose B chooses P.

- If A chooses P, payoff = 90.
- If A chooses E, payoff = 92.
- Due to rationality, A must choose E in this case also.

**Conclusion:** No matter what B does, A must choose E to get maximum payoff.
Here, A has a strategy (namely, E) that is strictly better than all of A’s other strategies, no matter what B chooses.

This is an example of a dominant strategy.

By symmetry, B also has the same dominant strategy.

**Consequence:** Both players choose E and each gets a payoff of 88. (Rationality dictates this outcome.)
Rational play (i.e., both players choose E) leads to a payoff of 88 for each.

If they both choose P, note that each of them can get a better payoff (namely, 90).

Based on the rationality assumption, that choice cannot happen. (If A agrees to choose P, B will choose E to get a better payoff of 92.)

**Prisoner’s Dilemma:**

- Idea developed by Merrill Flood and Melvin Drescher in 1950; formalized by Albert Tucker.
- Two prisoners P1 and P2, interrogated in two separate rooms.
- Actions for each: Confess (C) or Not Confess (NC).
Payoff Matrix for Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(-1,−1)</td>
</tr>
<tr>
<td>NC</td>
<td>(-4,−4)</td>
</tr>
</tbody>
</table>

- Payoff value “−4” means a 4 year jail term.
- Maximizing payoff implies less jail time.

Analysis by Prisoner P1:

Case 1: Suppose P2 chooses C.

- If P1 chooses C, then payoff = −4.
- If P1 chooses NC, then payoff = −10.
- So, the rational choice is C.
Analysis by Prisoner P1 (continued):

### Case 2: Suppose P2 chooses NC.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>NC</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(-4,-4)</td>
<td>(0,-10)</td>
</tr>
<tr>
<td>NC</td>
<td>(-10,0)</td>
<td>(-1,-1)</td>
</tr>
</tbody>
</table>

- If **P1** chooses C, then payoff = 0.
- If **P1** chooses NC, then payoff = -1.
- So, the rational choice is again C.

**Consequences:**

- So, the dominant strategy for both is C.
- Each gets a payoff of -4.
- Even though there is a better alternative (namely, the action NC for both), it can’t be achieved through rational play.
Prisoner’s Dilemma (continued)

- Canonical example of situations where cooperation is difficult to establish because of individual self-interest.
- Has been used as a framework to study many real-world situations (generally referred to as arms races).

**Example:** Use of performance enhancing drugs in professional sports.

- **Strategies:** Use drugs (U) and Don’t use drugs (DU).
- Dominant strategy for both players is U with (2, 2) as the payoff.
- The alternative with better payoff (namely, (3, 3)) won’t be reached.
For situations like Prisoner’s Dilemma to arise, payoffs must be chosen in a certain way.

Even small changes to the payoff matrix can change the situation significantly.

**Example:** A modified payoff table for the Exam-Presentation game.

<table>
<thead>
<tr>
<th></th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(98,98)</td>
</tr>
<tr>
<td>E</td>
<td>(96,94)</td>
</tr>
</tbody>
</table>

Now, the dominant strategy for both players is P.
The corresponding payoff is (98, 98).
Some Formal Definitions

Best Response:

- Represents the best choice for a player, given the other player’s choice.

<table>
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</tr>
</thead>
<tbody>
<tr>
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<td>(94,96)</td>
</tr>
<tr>
<td>E</td>
<td>(96,94)</td>
<td>(92,92)</td>
</tr>
</tbody>
</table>

- If B chooses E, A’s best response is P.

Notation:

- $P_1(x, y)$: Represents payoff to Player 1 when Player 1 uses strategy $x$ and Player 2 uses strategy $y$.
- $P_2(x, y)$: Similar but represents payoff to Player 2.
Some Formal Definitions (continued)

**Definition:** A strategy $s$ for Player P1 is a **best response** to strategy $t$ for Player 2 if $P_1(s, t) \geq P_1(s', t)$ for all other strategies $s'$ of P1.

**Note:** Best response strategy for P2 is defined similarly.

**Additional Definitions:**

- In general, there may be more than one best response.
- If there is a **unique** best response, it is a **strict best response**.
- A strategy $s$ for P1 is a **strict best response** for strategy $t$ by P2 if $P_1(s, t) > P_1(s', t)$ for all other strategies $s'$ of P1.
Some Formal Definitions (continued)

Additional Definitions (continued):

- A **dominant strategy** for P1 is a strategy that is a **best response** to every strategy of P2.

- A **strictly dominant strategy** for P1 is a strategy that is a **strict best response** to every strategy of P2.

Example:

<table>
<thead>
<tr>
<th></th>
<th>P</th>
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</tr>
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<tr>
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<tr>
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<td>(92,92)</td>
</tr>
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Here, P is a **strictly dominant strategy** for both players.

Note: When a player has a strictly dominant strategy, the player should be expected to use it (due to rationality).
Strict Dominant Strategies

- **So far:** Games in which both players had strict dominant strategies.
- **Now:** Games in which *only one* player has a strictly dominant strategy.

**The setting:** (Manufacturing/Marketing)

- There are two versions, namely low cost (L) and upscale (U), of a product X. (Strategies: L and U.)
- There are two firms F1 and F2 (the players).
- **Market segment:** 60% of the population will buy L and 40% will buy U.
- F1 and F2 capture 80% and 20% of the market respectively.
- If only one firm manufactures L (or U), it will capture 100% of the corresponding market.
Computing Payoff Matrix:

- Both F1 and F2 manufacture L.
  - Market segment is 60%.
  - F1 captures 80% of the market (i.e., 48% overall) and F2 captures 12%.
  - So, the payoff for this case is (48, 12).
- Other combinations can be computed similarly.

Resulting Payoff Matrix:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>(48,12)</td>
<td>(60,40)</td>
</tr>
<tr>
<td>U</td>
<td>(40,60)</td>
<td>(32,8)</td>
</tr>
</tbody>
</table>
Analysis by F1:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
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<tr>
<td>U</td>
<td>(40,60)</td>
<td>(32,8)</td>
</tr>
</tbody>
</table>

- **Case 1:** F2 chooses L. Here, F1’s strict best response is L.
- **Case 2:** F2 chooses U. Again, F1’s strict best response is L.

**Conclusion:** L is the strictly dominant strategy for F1.

Analysis by F2:

- **Case 1:** F1 chooses L. F2’s strict best response is U.
- **Case 2:** F1 chooses U. F2’s strict best response is L.

**Conclusion:** F2 does not have a strictly dominant strategy.
### What is the outcome of the game?

<table>
<thead>
<tr>
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<th>L</th>
<th>U</th>
</tr>
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<tbody>
<tr>
<td>L</td>
<td>(48,12)</td>
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</tr>
<tr>
<td>U</td>
<td>(40,60)</td>
<td>(32,8)</td>
</tr>
</tbody>
</table>

**Reasoning used by F2:**

- Due to rationality, F1 will choose L, its strictly dominant strategy.
- So, F2’s best response is U and the resulting payoff is (60, 40).

**Note:** F2’s reasoning relies on **common knowledge**:

- Both players know the complete payoff matrix.
- Both players know that each player knows all the rules and will act rationally.
The Concept of Equilibrium

Motivation:

- Suppose we have a game where neither player has a strictly dominant strategy.
- John Nash proposed the concept of equilibrium to predict the outcomes of such games.

Example: Consider the following game.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(4,4)</td>
<td>(0,2)</td>
<td>(0,2)</td>
</tr>
<tr>
<td>B</td>
<td>(0,0)</td>
<td>(1,1)</td>
<td>(0,2)</td>
</tr>
<tr>
<td>C</td>
<td>(0,0)</td>
<td>(0,2)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

- In this game, no player has a strictly dominant strategy.
- Reason: If F2 chooses A, F1’s best response is A; however, if F2 chooses B, F1’s best response is B.
**Definition:** A pair of strategies \((x, y)\) is a **pure Nash equilibrium** (pure NE) if \(x\) is a best response to \(y\) and vice versa.

**Example:**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
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<td>(0, 2)</td>
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</tr>
<tr>
<td>B</td>
<td>(0, 0)</td>
<td>(1, 1)</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>C</td>
<td>(0, 0)</td>
<td>(0, 2)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

- Consider the strategy pair \((A, A)\).
- The payoff is \((4, 4)\).

- If \(F1\) plays \(A\), \(F2\)'s best response is \(A\) and vice versa.
- So, \((A, A)\) is a **pure NE** for this game.
- Once the players choose \((A, A)\), there is **no incentive** for either player to switch to another strategy unilaterally.
Consider the strategy pair (B, B).

The payoff is (1, 1).

If F1 plays B, F2’s best response is C (with payoff = 2).

So, F2 has an incentive to switch and (B, B) is not a pure NE.

Notes:

- Similarly, (B, C) is not a pure NE. (F1 has an incentive to switch to C.)

- In fact, the only pure NE for the game is (A, A).
Remarks on the Equilibrium Concept

- At an equilibrium, there is no force pushing it to a different outcome. (It is bad for a player to switch unilaterally to a different strategy.)

- If a pair of strategies \((x, y)\) is not a pure NE, players cannot believe that this pair would actually be used (since one of the players has an incentive to switch).

- The equilibrium concept is not based on rationality alone.

- It is based on beliefs. (If each player believes that the other player will use a strategy which is part of an NE, then the other player has an incentive to use his/her part of the NE.)
Coordination Games

Example:

- Players A and B are preparing slides for a presentation.
- They can use Power Point (PP) or Keynote (KN).

Payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>PP</th>
<th>KN</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP</td>
<td>(1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>KN</td>
<td>(0,0)</td>
<td>(2,2)</td>
</tr>
</tbody>
</table>

- This is a “coordination game” since the goal is to choose a common strategy by both players.
- For this game, both (PP, PP) and (KN, KN) are pure NEs.
- An unbalanced coordination game – payoffs for the two pure NEs are different.
Contexts for coordination games – Some examples:

- Manufacturing companies work together to decide the unit of measurement (English or Metric) for their machinery.
- Units of an army must decide on a strategy to attack the enemy.
- People trying to meet each other in a shopping mall must decide where to meet.

Which Nash Equilibrium?

- A coordination game may have several pure NEs.
- Which will the players choose?
- Thomas Schelling introduced the idea of a focal point to study this.
- **Basic idea:** There may be natural reasons (possibly external to the payoff matrix) that allow people to choose an appropriate NE.
Example 1: Power Point vs Keynote game.

<table>
<thead>
<tr>
<th></th>
<th>PP</th>
<th>KN</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP</td>
<td>(1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>KN</td>
<td>(0,0)</td>
<td>(2,2)</td>
</tr>
</tbody>
</table>

- The payoff is higher for the (KN, KN) equilibrium.
- So, if the focal point is “higher payoff”, players will prefer (KN, KN).

Example 2: Cars on a (dark) undivided road.

Strategies: L or R.
Example 2 (continued):

- **Note:** “Inf” denotes $\infty$.
- Value $-\infty$ denotes “disaster”.
- Value $\infty$ denotes “ok” (nobody gets hurt).

- Both (L, L) and (R, R) are pure NEs.
- The choice is based on **social convention**.
  - In USA, each driver uses R.
  - In UK, each driver uses L.
Example 3 (Battle of the Sexes):

- Two people want to watch a movie together.
- **Strategies:** Action movie (A) or Romantic comedy (R).
- They want to coordinate on their choice.

\[
\begin{array}{cc|cc}
 & R & A \\
R & (1, 2) & (0, 0) \\
A & (0, 0) & (2, 1) \\
\end{array}
\]

- (R, R) and (A, A) are both pure NE.
- (R, R) is better for P2 while (A, A) is better for P1.

**Consequence:** Additional information (e.g. a convention that exists between the players) is needed to predict which equilibrium will be chosen.
Hawk-Dove Game:

- Dividing a piece of food (weight: 6 lbs) among two animals (players).
- Strategies: Hawk (aggressive behavior) or Dove (passive behavior).

|       | P2
|-------|---
|       | H | D |
| **P1** |    |   |
| D      | (3, 3) | (1, 5) |
| H      | (5, 1) | (0, 0) |

- If both choose H, they “destroy” each other and nobody gets anything.
- (H, D) and (D, H) are both pure NE; these correspond to “anti-coordination”.
- We can’t predict which of these equilibria will be chosen without additional information about the players.
A context for the Hawk-Dove game:

- Two neighboring countries (the players).
- Hawk and Dove represent strategies with respect to foreign policy.
- If both countries are aggressive, they may go to war (which may be disastrous to both).
- If both are passive, then each country has an incentive to switch.

**Equilibrium:** One country is aggressive and the other is passive.
When games have one or more pure NE, we have some information about the outcome (i.e., the players are likely to choose the strategies corresponding to one of the equilibria).

There are games where is there is no pure NE. (Example: Matching Pennies game – to be discussed next.)

The notion of equilibrium for such games is based on randomized strategies (mixed strategies).
A Game Without any Pure Nash Equilibrium

Matching Pennies:

- Two players (P1 and P2), each holding a penny.
- **Strategies:** Head (H) or Tail (T).
- If coins match, P1 loses the penny to P2.
- Otherwise, P2 loses the penny to P1.

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>(-1, +1)</td>
<td>(+1, -1)</td>
</tr>
<tr>
<td>T</td>
<td>(+1, -1)</td>
<td>(-1, +1)</td>
</tr>
</tbody>
</table>

- An example of a **zero sum** game.
- In every outcome, what one player wins is exactly what the other player loses.
A Game Without any Pure Nash Equilibrium

Matching Pennies (continued):

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 H</td>
<td>(−1, +1)</td>
<td>(+1, −1)</td>
</tr>
<tr>
<td>P1 T</td>
<td>(+1, −1)</td>
<td>(−1, +1)</td>
</tr>
</tbody>
</table>

- There is no dominant strategy for either player.
- There is no pure NE in this game.

Reason:

- For each pair of strategies, there is a player with a payoff of −1.
- That player has an incentive to switch.

What should the players do?

- If P1 knows what P2 is going to do, then P1 can always get a payoff of +1.
- So, P2 should make it difficult for P1 to guess what P2 will do; that is, employ randomization.
Basic Ideas:

- Each player chooses a probability for playing H.
- So, each strategy is a real number in \([0, 1]\).
- If probability of H is \(p\), then probability of T = \(1 - p\).
- Players are “mixing” the options H and T (mixed strategies).
- When \(p = 0\) or \(p = 1\), we get the corresponding pure strategy.
- Expected payoffs must be considered.
- Rationality: Players want to maximize their expected payoffs.
Notation:
- P1 and P2 play H with probabilities $p$ and $q$ respectively.
- Each mixed strategy is a probability value (i.e., the probability of playing H).

Definition: If P1’s mixed strategy is $p$, then the best response of P2 is a probability value $q$ that maximizes P2’s expected payoff.

Definition: A mixed Nash equilibrium (mixed NE) is a pair $(p, q)$ of probability values for P1 and P2 such that $p$ is the best response for $q$ and vice versa.

Note: In a mixed equilibrium, no player has an incentive to change his/her mixed strategy (i.e., probability value) unilaterally.
Lemma 1: No pure strategy can be part of a mixed NE for the Matching Pennies game.

Proof sketch:

- We already know that there is no pure NE for the game; that is, both P1 and P2 cannot use pure strategies in an equilibrium.

- Suppose P1 uses pure strategy H while P2 uses mixed strategy $q$, where $0 < q < 1$.

- Now, P2 has the incentive to change the strategy to $q = 1$ (i.e., play H all the time) to ensure a win every time.

- Other cases are handled similarly.

Consequence: In any mixed NE for the Matching Pennies game, the probability values can’t be either 0 or 1.
Computing expected payoff (P2’s Analysis):

<table>
<thead>
<tr>
<th></th>
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<th>T</th>
</tr>
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<tbody>
<tr>
<td><strong>P1</strong></td>
<td><strong>P2</strong></td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>(−1, +1)</td>
<td>(+1, −1)</td>
</tr>
<tr>
<td>T</td>
<td>(+1, −1)</td>
<td>(−1, +1)</td>
</tr>
</tbody>
</table>

- P2 plays H with probability $q$ (and T with probability $1 - q$).

**Case 1:** Suppose P1 plays the pure strategy H.

- P1 loses 1 cent each time P2 plays H, that is, with probability $q$.
- P1 gains 1 cent each time P2 plays T, that is, with probability $1 - q$.
- So, expected payoff for P1 $= -q + (1 - q) = 1 - 2q$. 
Mixed Strategies & Expected Payoff (continued)

Computing expected payoff (continued):

<table>
<thead>
<tr>
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<tbody>
<tr>
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</tr>
<tr>
<td>T</td>
<td>(+1, -1)</td>
<td>(-1, +1)</td>
</tr>
</tbody>
</table>

- P2 plays H with probability $q$ (and T with probability $1 - q$).

Case 2: Suppose P1 plays the pure strategy T.

- P1 gains 1 cent each time P2 plays H, that is, with probability $q$.
- P1 loses 1 cent each time P2 plays T, that is, with probability $1 - q$.
- So, expected payoff for P1 = $q - (1 - q) = 2q - 1$.

Summary:

- P1’s expected payoff when using pure strategy H = $1 - 2q$.
- P1’s expected payoff when using pure strategy T = $2q - 1$. 
Lemma 2 (Generalization): Suppose P1 and P2 use strategies $p$ and $q$ respectively. Then

- The expected payoff for P1 = $(2p - 1)(1 - 2q)$.
- The expected payoff for P2 = $(1 - 2p)(1 - 2q)$.

Lemma 3: If $1 - 2q \neq 2q - 1$, then a pure strategy maximizes P1’s expected payoff.

Proof sketch: Suppose $1 - 2q \neq 2q - 1$. Then either $1 - 2q > 2q - 1$ or $1 - 2q < 2q - 1$.

Case 1: $1 - 2q > 2q - 1$.

- Here, $1 - 2q > 0$.
- In this case, the expected payoff for P1 = $(2p - 1)(1 - 2q)$.
- This function increases as $p$ increases; it is maximized when $p = 1$.
- Thus, using pure strategy H maximizes P1’s expected payoff.
Proof sketch for Lemma 3 (continued)

Case 2: $1 - 2q < 2q - 1$.

- Pure strategy $T$ maximizes $P1$’s expected payoff. (The argument is similar to that of Case 1.)

Lemma 4: If $1 - 2q \neq 2q - 1$, then there is no mixed NE for the game.

Reason:

- When $1 - 2q \neq 2q - 1$, Lemma 3 shows that $P1$’s best response is a pure strategy.

- However, Lemma 1 points out that no pure strategy can be part of a mixed NE for the game.
Consequences of Lemma 4:

- P2 must choose $q$ so that $1 - 2q = 2q - 1$, that is, $q = 1/2$ to get a mixed NE.

- Similarly, P1 must choose $p = 1/2$ for a mixed NE.

- Thus, the only mixed NE for the game is $(1/2, 1/2)$.

Additional Remarks:

- If P2 chooses $q < 1/2$ (i.e., plays T more often than H), then P1 will use the pure strategy H to gain advantage.

- If P2 chooses $q > 1/2$ (i.e., plays H more often than T), then P1 will use the pure strategy T to gain advantage.
Additional Remarks (continued)

- When P2 chooses $q = 1/2$, both the pure strategies (H and T) give the same expected payoff to P1.

- The choice $q = 1/2$ by ensures that neither of the pure strategies offers any advantage to P1 (i.e., makes P1 indifferent between choosing H or T).

**Theorem: [Nash 1950]**

Every game with a finite number of players has at least one mixed equilibrium.
Another example for Mixed NE Computation: Consider the following game.

**P2’s Analysis:** Suppose P2 plays A with probability $q$ (and B with probability $1 - q$).

**Case 1:** P1 chooses pure strategy A.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
<th>Payoff to P1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A,A)</td>
<td>$q$</td>
<td>90</td>
</tr>
<tr>
<td>(A,B)</td>
<td>$1 - q$</td>
<td>20</td>
</tr>
</tbody>
</table>

P1’s expected payoff in Case 1 $= 90 \times q + 20 \times (1 - q) = 70q + 20$. 

**Exercise:** Does this game have one or more pure NE?
Example for Mixed NE Computation (continued):

![Game Matrix]

- **Case 2:** P1 chooses pure strategy B.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
<th>Payoff to P1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B,A)</td>
<td>q</td>
<td>30</td>
</tr>
<tr>
<td>(B,B)</td>
<td>1–q</td>
<td>60</td>
</tr>
</tbody>
</table>

P1’s expected payoff in Case 2 = \(30 \times q + 60 \times (1-q) = -30q + 60\).

To make P1 indifferent with respect to pure strategy, we must have

\[70q + 20 = -30q + 60 \quad \text{or} \quad q = 0.4.\]
Example for Mixed NE Computation (continued):

- A similar calculation shows that P1 must choose $p = 0.3$.
- So (0.3, 0.4) is a mixed NE for this game.

Power Point vs Keynote coordination game:

- This game has two pure Nash equilibria, namely (PP, PP) and (KN, KN).
- It also has a mixed NE.
**P2’s Analysis:** Suppose P2 plays PP with probability $q$ (and KN with probability $1 - q$).

**Case 1:** P1 chooses the pure strategy PP.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
<th>Payoff to P1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(PP,PP)</td>
<td>$q$</td>
<td>1</td>
</tr>
<tr>
<td>(PP,KN)</td>
<td>$1 - q$</td>
<td>0</td>
</tr>
</tbody>
</table>

P1’s expected payoff in Case 1 $= q$.

**Case 2:** P1 chooses the pure strategy KN. P1’s expected payoff in this case $= 2(1 - q)$.

To obtain a mixed NE, we have $q = 2(1 - q)$ or $q = 2/3$.

By symmetry, $p = 2/3$. So, $(2/3, 2/3)$ is a mixed NE for this game.
Complexity of Finding Nash Equilibria

- For the form of games we have considered (called normal form), determining whether a game has a pure NE is efficiently solvable.

- In general, with many players and more complex specifications of strategies, determining whether a game has a pure NE is \textbf{NP}-complete.

- Finding a mixed NE for a game is complete for another complexity class called \textbf{PPAD}.

- The class \textbf{PPAD} contains problems for which we know at least one solution exists but finding a solution is difficult ("needle in a haystack").

- It is believed that the class \textbf{PPAD} is different from the class \textbf{NP}. 
Presentation-Exam Game (discussed earlier):

- **E** is a dominant strategy for both **A** and **B**.
- (E, E) is also a **pure NE**.
- The payoff for (E, E) is (88, 88).
- (P, P) is **not** a pure NE; **A** has an incentive to switch to E.

### Additional Notes:

- Outcome (P, P) **can’t** be reached under rational behavior (i.e., when players optimize **individually**).
- Other mechanisms are needed to allow such outcomes.

**Exercise:** Show that there is **no** mixed NE for the above game when the probability values are required to be strictly between 0 and 1.
Vilfredo Pareto (1848–1923)
Ph.D. (Civil Engineering), University of Turin, Italy.
Pareto Principle (or “80-20 rule”) is named after him.
Made many important contributions to Microeconomics.

Towards a definition of Pareto Optimality:

- The four payoff vectors in the Presentation-Exam game are: 
  \[(90, 90), \ (86, 92), \ (92, 86), \ (88, 88)\]

- The vector \((90, 90)\) is **strictly better** than \((88, 88)\) (since it allows both players to do better).
Suppose we add one more vector \((88, 90)\) to the set to get:

\[(90, 90), (86, 92), (92, 86), (88, 88), (88, 90)\]

The vector \((88, 90)\) is at least as good as \((88, 88)\) since

- no player is worse off choosing \((88, 90)\) over \((88, 88)\) and
- at least one player’s payoff is better off in \((88, 90)\) compared to that in \((88, 88)\).

**Terminology:** Payoff vector \((88, 90)\) dominates the payoff vector \((88, 88)\). (Alternatively, \((88, 88)\) is dominated by \((88, 90)\).)
Definition: A payoff vector \((x_1, y_1)\) dominates another payoff vector \((x_2, y_2)\) if all the following conditions hold:

1. \(x_1 \geq x_2\),
2. \(y_1 \geq y_2\) and
3. at least one of these inequalities is strict (i.e., ‘>’ instead of ‘\(\geq\)’).

Examples:

- The vector \((88, 90)\) dominates \((88, 88)\).
- The vector \((86, 92)\) does not dominate \((88, 88)\).
- A vector \((x, y)\) does not dominate itself.

Representation:

\[
\begin{align*}
(x_1, y_1) & \downarrow \\
(x_2, y_2) & \\
\end{align*}
\]

\((x_1, y_1)\) dominates \((x_2, y_2)\).
Consider the following set $X$ of vectors

$$X = \{(90, 90), (86, 92), (92, 86), (88, 88), (88, 90)\}.$$ 

The domination relationship among these vectors is as follows:

- Vectors which don’t have an incoming edge are “non-dominated”.
- They represent **Pareto optimal** payoffs.

**Definition:** A pair of strategies is **Pareto optimal** if the payoff vector for the pair is not dominated by the payoff vector for any other pair of strategies.
Example:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(90,90)</td>
<td>(86,92)</td>
</tr>
<tr>
<td>E</td>
<td>(92,86)</td>
<td>(88,88)</td>
</tr>
</tbody>
</table>

Here, the Pareto optimal strategy pairs are (P, P), (P, E) and (E, P).

The only pure Nash equilibrium (E, E) is not Pareto optimal. (Interestingly, that is the only strategy pair that is not Pareto optimal!)

How can players reach a Pareto optimal outcome?

- They must sign a binding contract before the game.
- If there is no such contract, some player may have an incentive to switch to another strategy (since a Pareto optimal strategy need not be a pure NE).
Some Pareto optimal strategies provide outcomes that are good for both players (“good for society”).

**Example:** In the Presentation-Exam game, the strategy pair \((P, P)\) (with payoff = \((90, 90)\)) is better for both players than the strategy pair \((E, E)\) (with payoff = \((88, 88)\)).

There are other ways to define **social optimality**.

**Definition:** A pair of strategies \((\alpha, \beta)\) is a **social optimum** (or a **social welfare maximizer**) if it maximizes the sum of the payoffs to the two players.

**Example:** In the Presentation-Exam game, the strategy pair \((P, P)\) (with payoff = \((90, 90)\)) is the **unique** social optimum with a total value of 180.
Lemma: (1) Every social optimum is also Pareto optimal. (2) A Pareto optimal solution need not be a social optimum.

Proof:

Part 1: Suppose a payoff vector \((x, y)\) is a social optimum but not Pareto optimal.

- Then, there must be another payoff vector \((x', y')\) which dominates \((x, y)\).
- Thus, \(x' \geq x, \ y' \geq y\), and at least one inequality is strict.
- Therefore, \(x' + y' > x + y\), and this contradicts the assumption that \((x, y)\) is a social optimum.

Part 2: In the Presentation-Exam game, \((86, 92)\) is Pareto optimal. However, it is not a social optimum (which is \((90, 90)\)).
Note: We consider pure Nash equilibria.

- A pure Nash Equilibrium need not be Pareto optimal.

  Example: In the Presentation-Exam game, (88, 88) is a pure NE but not Pareto optimal (it is dominated by (90, 90)).

- A pure Nash Equilibrium need not be a social optimum.

  Example: In the Presentation-Exam game, (88, 88) is a pure NE but not the social optimum (which is (90, 90)).

Note: We will consider two contexts where we can quantify how the total value of a pure NE compares with the social optimum.

- Traffic in transportation networks.
- Cost-sharing in computer networks.
Example – Traffic in transportation networks:

- Cars want to go from A to B.
- The value on each edge is the travel time.
- On the edges (A, C) and (D, B), travel time is a **linear** function of the number of cars $x$. (These edges are **sensitive to congestion**.)
- Number of cars = 4000.

- If all cars use the route A-C-B, travel time for each car $= (4000/100) + 45 = 85$.
- If all cars use the route A-D-B, travel time for each car is again 85.
- Suppose cars divide evenly between the two routes. Then travel time for each car $= (2000/100) + 45 = 65$. 
The underlying game:

- 4000 players (Drivers)
- **Strategies:** \{A-C-B, A-D-B\}
- Payoff for each player: Travel time

Notes:

- We will **minimize** payoffs.
- There is no dominant strategy for any player; the travel time for a route depends on the number of players using that route.
- There are **many** pure Nash equilibria for this game.
Applying Game Theory ... (continued)

**Theorem:**

1. Every combination of strategies that divides the 4000 cars evenly between the two routes is a pure NE.

2. In every pure NE, each route has the same number of cars.

**Proof sketch for Part 1:** Consider any combination of strategies that has 2000 cars along each route. (Travel time for each player = 65.)

**Question:** Does any single player have an incentive to switch to the other route?

- Suppose one player switches from A-C-B to A-D-B.
- After the switch, there will be 2001 cars along A-D-B.
- New travel time along A-D-B = 45 + (2001/100) > 65; that is, the payoff is worse.
- So, no player has an incentive to switch (unilaterally).
Proof sketch for Part 2: Suppose there a pure NE with \( t \) cars on A-C-B and \( 4000 - t \) cars on A-D-B.

To prove: \( t = 4000 - t \) (which implies that \( t = 2000 \)).

Case 1: \( t > 4000 - t \).

- Here, it is easy to verify that \( 4000 - t \leq t - 2 \).
- Current travel time for player along A-C-B = \( 45 + (t/100) \).
- Switch one player from A-C-B to A-D-B.
- New travel time for the player is

\[
45 + [(4000 - t) + 1]/100 \leq 45 + [(t - 2) + 1]/100 < 45 + (t/100)
\]

- Thus, the player has an incentive to switch and we don’t have a pure NE.

Case 2: \( t < 4000 - t \) : The proof is similar.
Braess’s Paradox

In any pure NE, each of the two routes is used by 2000 players.

Travel time for each player = 65.

After adding the edge (C, D):

- **Strategies:** \{A-C-B, A-C-D-B, A-D-B\}.
- **Surprise:** There is a unique pure NE where every player uses the route A-C-D-B.

Travel time for each player = 80.

Verifying that A-C-D-B a pure NE:

- Suppose a player wants to switch to A-D-B.
- New travel time = 45 + (4000/100) = 85.
- So, no player has an incentive to switch.
Why A-C-D-B is a unique pure NE – A brief explanation:

- Consider the flow pattern with 2000 players using A-C-B and 2000 using A-D-B.
- Travel time for each player = 65.

- Suppose a player X switches from A-C-B to A-C-D-B.
- Travel time for X = \( \frac{2000}{100} + \frac{2001}{100} = 40.01 \).
- So, X has an incentive to switch.
- So, the above flow pattern is not a pure NE.

Note: A similar argument applies to other flow patterns.

Remark: Removing the red edge (C, D) creates a better pure NE.
Braess’s Paradox:

- Travel time in a pure NE increases even though resources were added to the system.
- Named after Dietrich Braess (1938–), a Mathematician from Germany.

Empirical observations supporting Braess’s Paradox:

- In Seoul (South Korea), the destruction of a 6-lane highway (as part of a project called “Cheonggyecheon Restoration”) actually reduced the commute time for many drivers.
- In Stuttgart (Germany), closing a major road actually decreased traffic congestion.
- In 1990, the closing of 42nd Street in New York City significantly reduced traffic congestion.
Braess’s Paradox (continued)

Additional Remarks:

- Braess’s paradox shows a situation where introducing a new choice (strategy) makes the payoff worse for everyone.
- Other such situations also exist.

Example: Prisoner’s dilemma.

If each player is given only one strategy, namely NC, things would be better for both.

Adding a second strategy (C) introduces difficulties.

For each player, C is the strictly dominant strategy. So, the outcome is (C, C), which is worse for both compared to (NC, NC).
No. of players = 4000.

**Pure NE:** All 4000 players use the route A-C-D-B.

**Social optimum:** 2000 players use A-C-B and 2000 use A-D-B.

For the pure NE, travel time for each player = 80.

So, total time (cost) for this pure NE = $4000 \times 80 = 320,000$.

For the social optimum, travel time for each player = 65.

So, cost of social optimum = $4000 \times 65 = 260,000$.

This example shows that cost of pure NE can be larger than that of social optimum.
A General Model for the Problem

Ref: [Roughgarden & Tardos, 2002]

- Road network represented by a directed graph with predefined origin and destination.
- For each edge $e$, a linear travel time function given by
  
  $$T_e(x) = a_e x + b_e$$

  where $a_e$ and $b_e$ are constants and $x$ is the number of cars on the edge $e$.

- A traffic pattern specifies a path for each car. (Paths are assumed to be simple.)

- Social cost of a traffic pattern $Z$ is the sum of the travel times for all the drivers.

Research Question 1: Under this model, is there always a (pure) Nash equilibrium?

Answer: Yes.
Outline of algorithm for producing an equilibrium:

1. Start with any traffic pattern $Z$.
2. while ($Z$ is not an equilibrium) do
   - Move one driver (chosen arbitrarily) to a better path.
   - Let $Z$ denote the new traffic pattern.

Notes:

- The above algorithm always terminates.
- The traffic pattern produced when the algorithm terminates is a Nash equilibrium.

Proof idea: (Potential Function Argument)

- Define a suitable function (called a potential function).
- Argue that every time a driver is moved to a better path, the value of the function decreases.
- Also argue that the value of the function cannot decrease below a lower limit (at which point no switches can occur).
Research Question 2: How does the total travel time at an equilibrium compare with the social optimum?

Theorem: [Roughgarden & Tardos, 2002] There is always an equilibrium travel pattern $Z$ such that the travel time of $Z$ is at most twice the social optimum.

Further improvement: [Anshlevich et al., 2004] There is always an equilibrium travel pattern $W$ such that the travel time of $W$ is at most $4/3$ times the social optimum.

Notes:

- For some non-linear travel cost functions, the cost of an equilibrium can be much larger than that of social optimum.
- For networks with more complicated travel cost functions, an equilibrium may not exist.
A directed graph with a cost $c(e) \geq 0$ for each edge, a designated source node $s$ and $k$ distinct terminal nodes $t_1, t_2, \ldots, t_k$ (one for each player).

Each player $P_i$ wants to set up a directed path from $s$ to terminal $t_i$ ($1 \leq i \leq k$).

Paths chosen by different players may share edges.

If an edge $e$ is shared by $q$ players, then the cost for each player is $c(e)/q$. (So, there is an incentive to share edges.)

Each player wants to minimize the cost of their path.

Social cost of any solution is the sum of the costs of all the players.

Ref: [Anshlevich et al. 2004]
Example:

- Two players $P_1$ and $P_2$.
- Choices for $P_1$: $s \rightarrow t_1$ or $s \rightarrow v \rightarrow t_1$.
- Choices for $P_2$: $s \rightarrow t_2$ or $s \rightarrow v \rightarrow t_2$.
- **Initial choice:** $P_1$ uses the edge $s \rightarrow t_1$ and $P_2$ uses the edge $s \rightarrow t_1$.

Moves:

- $P_1$ notices that switching to $s \rightarrow v \rightarrow t_1$ does **not** decrease the cost.
- $P_2$ notices that switching to $s \rightarrow v \rightarrow t_2$ **does** decrease the cost (from 8 to 6), and does the switch.
- Now, $P_1$ notices that switching to $s \rightarrow v \rightarrow t_1$ **does** decrease the cost (4 to 3.5) because of the **shared** cost for $s \rightarrow v$. 
Equilibrium:

- $P_1$ uses $s \rightarrow v \rightarrow t_1$ and
- $P_2$ uses $s \rightarrow v \rightarrow t_2$.

Now, neither player has an incentive to switch.

Example with multiple equilibria:

- **Equilibrium 1:** $P_1$ uses $s \rightarrow x \rightarrow v \rightarrow t_1$ and $P_2$ uses $s \rightarrow x \rightarrow v \rightarrow t_2$. (Cost for each player $= 1.1/2 = 0.55$.)
- **Equilibrium 2:** $P_1$ uses $s \rightarrow y \rightarrow v \rightarrow t_1$ and $P_2$ uses $s \rightarrow y \rightarrow v \rightarrow t_2$. (Cost for each player $= 2/2 = 1.0$.)
Example – Social optimum need not be an equilibrium:

- **Social optimum**: $P_1$ uses $s \rightarrow v \rightarrow t_1$ and $P_2$ uses $s \rightarrow v \rightarrow t_2$.

- Total cost = 7. (Cost for each player = 3.5.)

- This is **not** an equilibrium.

Moves:

- $P_1$ has an incentive to switch to $s \rightarrow t_1$ (since the cost decreases from 3.5 to 3).

- Once $P_1$ switches, $P_2$ has an incentive to switch to $s \rightarrow t_2$ (since the cost decreases from 6 to 5).

- The situation where
  - $P_1$ uses $s \rightarrow t_1$ and
  - $P_2$ uses $s \rightarrow t_2$ is an equilibrium.

- Social cost at this equilibrium = 8.
Ref: [Anshlevich et al. 2004], [Kleinberg & Tardos, 2006]

Research Question 1: Does every multicast problem have a Nash equilibrium?

Answer: Yes. (Proof uses the potential function technique.)

Research Question 2: How does the cost of a best equilibrium compare with the social optimum?

Answer: For any $k \geq 2$ players, the cost of a best equilibrium is at most $H_k$ times the social optimum, where

$$H_k = 1 + (1/2) + (1/3) + \ldots + (1/k)$$

is the $k^{th}$ Harmonic Number.

Note: $\ln k < H_k < \ln k + 1$. 