Ref: Chapter 5 of [Easley & Kleinberg].
Positive and Negative Relationships

- So far: Edges in a network represent **friendship** information (positive relationships).

- We also need to consider **conflicts** (negative relationships).

- The combination leads to the notion of **structural balance**.

- Provides another illustration of how **local** structure (i.e., a property involving a few nodes at a time) may have a **global** effect.

**Model:**

- The underlying graph is a **clique**; that is, each person has a positive or negative relationship with every other person. (General graphs will be considered later.)

- Each edge has a **label**: ‘+’ (indicating a positive relationship) or ‘-’ (indicating a negative relationship).

- A common model for studying international conflicts.
Model (continued):

- Ideas developed (in the sociological context) by Fritz Heider.
  
  - Fritz Heider (1896–1988)
  - Austrian Sociologist.
  - Taught at the University of Kansas for many years.

- The mathematical development is due to Dorwin Cartwright and Frank Harary.
Dorwin Cartwright (1915–2008)
Areas: Psychology and Mathematics.
One of the founders of Group Dynamics.
University of Michigan, Ann Arbor, MI.

Frank Harary (1921–2005)
Mathematician who specialized in Graph Theory and its Applications.
University of Michigan, Ann Arbor, MI and later New Mexico State University, Las Cruces, NM.
Structural Balance

Possible Edge Labelings for Three People:

- Labelings (1) and (2) have an odd number of ‘+’ labels.
- Labelings (3) and (4) have an even number of ‘+’ labels.
- Labeling (1): Three mutual friends; causes no problem.
- Labeling (2): Two friends and they both dislike the third; causes no problem.
- So, Labelings (1) and (2) have structural balance.
Labeling (3): A has two friends who don’t like each other. This may be a source of “stress” for A. (It may cause A to lose the friendship with B or C.)

Note: Recall (from the slides for Part 2) the study by Bearman & Moody [2004] involving the health records of teenage girls.

Labeling (4): Here, two of the people may “team up” against the third person (i.e., there may be forces to change the label of one of the edges to ‘+’).

So, Labelings (3) and (4) have structural imbalance.
Balance condition for Three People:

- A labeled triangle is **balanced** if and only if the number of ‘+’ labels is **odd**.

Extension – Structural Balance for all Cliques:

- A labeled clique is **balanced** if and only if each of its triangles is balanced (i.e., in each triangle, the number of ‘+’ labels is **odd**).

Example:

- 4-clique.
- Not balanced.
- Triangle BCD has two edges labeled ‘+’ (and so does triangle ABC).
Testing the Structural Balance – An Easy Algorithm:

**Input:** A clique $G$ with $n$ nodes where each edge has a ‘+’ or ‘−’ label.

**Output:** “Yes” if $G$ is balanced and “No” otherwise.

**Outline of the Algorithm:**

1. for each triple of nodes $x$, $y$ and $z$ do
   
   if (triangle $\{x, y, z\}$ is not balanced)
   
   Output “No” and stop.

2. Output “Yes”.

**Running time:** $O(n^3)$ (since there are $\binom{n}{3} = O(n^3)$ triangles in a clique with $n$ nodes).
Characterizing Structural Balance

**Note:** The following trivial cases are ignored in the discussion.

- All edges of $G$ are labeled ‘+’: $G$ is balanced.
- All edges of $G$ are labeled ‘-’: $G$ is unbalanced.

**Idea of Battling Factions:**

- Suppose we can partition the nodes of $G$ into two sets $X$ and $Y$ such that the following conditions hold:
  - All edges inside $X$ or inside $Y$ are labeled ‘+’ and
  - all edges that join a node in $X$ to a node in $Y$ are labeled ‘-’.

![Diagram showing mutual friends with green edges and opposing factions with red edges.](image-url)
Characterizing Structural Balance (continued)

X and Y are called **battling factions**.

In this structure, every triangle is balanced (to be explained in class).

**Key idea:** In any balanced clique, such a structure exists.

**Terminology:**

- **Internal edge:** An edge that joins two nodes in X or two nodes in Y.
- **External edge:** An edge that joins a node in X to a node in Y.
Theorem: [Cartwright & Harary]
If a labeled complete graph $G$ is balanced, then

- either all the edge labels in $G$ are ‘$+$’ or
- the nodes of $G$ can be partitioned into two sets $X$ and $Y$ such that
  1. each internal edge is labeled ‘$+$’ and
  2. each external edge is labeled ‘$-$’.

Example:

- This 5-clique is balanced.
- Partition: $X = \{x, y, z\}$ and $Y = \{p, q\}$. 
Proof Sketch for the Cartwright-Harary Theorem

Notes:

- Ignore the (trivial) case where all edge labels are ‘+’.
- So, assume that at least one edge has the label ‘-’.
- The proof actually constructs the **battling factions** partition.

Construction:

- Choose any node $a$ of $G$.
- Let the set $X$ consist of $a$ and all the nodes which are **friends** of $a$.
- Let $Y$ be the remaining set of nodes (i.e., the **enemies** of $a$).

An Illustration:

- Not all nodes/edges are shown.
Proof Sketch ... (continued)

**Part 1:** We must show that each internal edge in $X$ has the label ‘+’.

- Consider any two nodes $p$ and $q$ in $X$.
- If one of $p$ and $q$ is the node $a$, the conclusion follows since all nodes in $X$ are friends of $a$.
- So, assume that $p$ and $q$ are different from $a$.
- If $p$ and $q$ are enemies, we get the following **unbalanced** triangle in $G$:

```
X
+  +  +
p  a  q
```

- This contradicts the assumption that $G$ is balanced.
Part 2: We must show that each internal edge in $Y$ has the label ‘+’.

- Consider any two nodes $p$ and $q$ in $Y$.
- If $p$ and $q$ are enemies, we get the following unbalanced triangle in $G$:

\[
\begin{array}{c}
X \\
\hline
- \\
\hline
\hline
- \\
\hline
a \\
\hline
\end{array} 
\begin{array}{c}
Y \\
\hline
- \\
\hline
\hline
- \\
\hline
p \\
\hline
\hline
q \\
\hline
\end{array}
\]

- A contradiction.
Part 3: We must show that each external edge has the label ‘-’.

- Consider any two nodes $p \in X$ and $q \in Y$.

- If $p$ and $q$ are friends, we get the following unbalanced triangle in $G$:

```
  X
 / \
 a--p
 |   |
 |   +
 |   q
 |   +
 |---Y
```

- A contradiction. (This completes the proof.)

Notes:

- The Cartwright-Harary Theorem leads to an $O(n^2)$ algorithm for the problem. (See Handout 4.1.)

- The running time is linear in the size of the input.
An Application – International Relations


- There was a war between India and Pakistan.

- USA was trying to improve its relationship with China.

- The perception was that China and Pakistan were friends (since India was their common ‘enemy’).

- The structural balance theory suggests that USA should support Pakistan.
Some online networks allow people to express positive/negative sentiments.

**Examples:**

- Slashdot ([http://slashdot.org](http://slashdot.org)): Allows people to designate each other as ‘friend’ or ‘foe’.
- Epinions ([http://www.epinions.com](http://www.epinions.com)): A consumer review website where people could ‘trust’ or ‘distrust’ reviews. (These features were removed in 2014.)
Evolving models of signed graphs (e.g. [Antal et al. 2006] – Ref [20] in the text).

1. Start with a random labeling.
2. Look for an unbalanced triangle and flip one of the labels to make it balanced.
3. Repeat Step 2 until all triangles are balanced (or until the number of repetitions exceeds a set limit).

Capture situations where people update their likes/dislikes as they strive for structural balance.
Two forms of structural imbalance:

- (i) 
  \[ \begin{array}{ccc} 
  & A & \\
  B & - & C \\
  \end{array} \]

- (ii) 
  \[ \begin{array}{ccc} 
  & A & \\
  B & + & C \\
  \end{array} \]

Some sociologists (e.g. James Davis, University of Chicago) have argued that (ii) is a stronger form of imbalance than (i).

**Definition:** [Weaker form of Imbalance]
A clique with signed edges is weakly balanced if and only if there is no triangle with exactly two edges labeled ‘+’.

**Note:** One should expect a larger collection of possible structures that are weakly balanced.
A Weaker Form of Structural Balance (continued)

Example:

- This structure **allows** triangles with three edges labeled ‘-’.
- However, triangles with only one edge labeled ‘-’ are **not allowed**.

Characterization of Weakly Balanced Cliques:

**Theorem:** (Also due to Cartwright & Harary)

Let $G$ be a weakly balanced clique. Then the nodes of $G$ **can be partitioned into groups** such that for any pair of nodes $x$ and $y$

1. if $x$ and $y$ are in the same group, then $x$ and $y$ are friends and
2. if $x$ and $y$ are in different groups, then $x$ and $y$ are enemies.
Weak Balance for Cliques: Proof Idea

- Let $G$ be the weakly balanced clique.
- Choose any node $x$ of $G$ and construct set $V_1$ consisting of $x$ and all the friends of $x$.
- Let $V_2$ denote the set of remaining nodes.

- Each pair of nodes in $V_1$ must be friends. (Otherwise, will have a triangle in $V_1$ with exactly one edge labeled ‘-’, which is not weakly balanced.)
- Also, for any node $z \in V_1$ and any node $w \in V_2$, $z$ and $w$ are enemies.

- Think of $V_1$ as the first group.
- The complete graph on $V_2$ is also **weakly balanced**. So, one can continue the process with $V_2$, leading to several groups.
So far: Balance conditions for cliques.

Now: Strong structural balance for graphs which are not necessarily cliques.

There are two possible definitions.

Definition 1: Let $G$ a graph with each edge labeled ‘+’ or ‘-’. $G$ is balanced if signs can be assigned to the missing edges so that the resulting clique is (strongly) balanced.

Example:

- The graph on the right assigns the ‘+’ label to each missing edge.
- So, the graph on the left is balanced.
Definition 2: Let $G$ a graph with each edge labeled ‘+’ or ‘-’. $G$ is balanced if the nodes of $G$ can be partitioned into two sets $V_1$ and $V_2$ such that

1. Each edge inside $V_1$ or $V_2$ has the ‘+’ label and
2. each edge that joins a node in $V_1$ to a node in $V_2$ has the ‘-’ label.

Example:

Note: There need not be any internal edges.

Fact: Definitions 1 and 2 are equivalent; that is, a graph $G$ is balanced according to Definition 1 and if and only if it is balanced according to Definition 2.
Balance for General Graphs (continued)

Reason for the Equivalence of Definitions:

- If it is possible to assign labels to missing edges so that the graph becomes balanced (by Definition 1), then we can obtain a “battling factions” partition that satisfies Definition 2.

- If the graph satisfies Definition 2, then all internal edges can be labeled ‘+’ and all external edges can be labeled ‘-’ to satisfy Definition 1.

Note: Unfortunately, these definitions don’t directly lead to an efficient algorithm for checking the balance condition for general graphs.

Theorem: [Harary]

A signed graph is balanced if and only if it does not contain any cycle with an odd number of edges with label ‘-’.
**Example:** The following graph has a cycle with an **odd** number of edges labeled ‘-’.

- In any “battling factions” decomposition, nodes a and e must be on the same side.
- Likewise, nodes b and c must be on the same side, but **different** from the side that contains a and e.
- Now, we can’t add node d to either side.
- So, the above graph is **not** balanced.

**Note:** Harary’s theorem leads to an efficient algorithm for testing the strong balance condition for general graphs.

**Algorithm Description:** See Handout 4.2.
An Illustration for the Algorithm

Given signed graph $G$:

Graph $G^+$ after Step 2:

Note: $G$ does not contain any edge labeled ‘-’ joining two nodes in the same connected component.

Graph $H$ after Step 4:

- $H$ is not bipartite; it contains a cycle with 3 nodes.
- So, $G$ is not balanced.
Notion of Approximate Balance  (Brief Discussion)

- **So far:** “Perfect balance” (i.e., all triangles are balanced).

- Suppose we allow 0.1% of “unbalanced” triangles; that is, in the given signed clique $G$, 99.9% of the triangles are balanced. Then, the following result holds.

**Theorem:** Suppose $G$ is a signed clique such that 99.9% of the triangles in $G$ satisfy the strong balance condition. Then **at least one** of the following conditions hold:

- There is a subset $V'$ with at least 90% of the nodes of $G$ such that at least 90% of the edges inside $|V'|$ are labeled ‘+’.

- The nodes of $G$ can be partitioned into two sets $V_1$ and $V_2$ such that
  
  1. at least 90% of the internal edges are labeled ‘+’ and
  2. at least 90% of the external edges are labeled ‘-’.

**Note:** A proof of the above result is given in the text.