Finite Field Problems

1. Let \( f = x^2 + 1 \in \mathbb{Z}_{11}[x] \) and let \( \mathbf{F} = \mathbb{Z}_{11}[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements in \( \mathbf{F}^\times \)?
   b) What is the order of \( \alpha \) in \( \mathbf{F}^\times \)?
   c) What is the order of \( \alpha + 1 \) in \( \mathbf{F}^\times \)?
   d) What is the order of \( (\alpha + 1)^{392} \) in \( \mathbf{F}^\times \)?
   e) What is the order of \( \alpha + 2 \) in \( \mathbf{F}^\times \)?

2. Let \( f = x^2 + 1 \in \mathbb{Z}_{19}[x] \) and let \( \mathbf{F} = \mathbb{Z}_{19}[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What is the order of \( \alpha \) in \( \mathbf{F}^\times \)?
   b) What is the order of \( \alpha + 1 \) in \( \mathbf{F}^\times \)?

3. Let \( f = x^2 - x + 1 \in \mathbb{Z}_5[x] \) and let \( \mathbf{F} = \mathbb{Z}_5[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements in \( \mathbf{F}^\times \)?
   b) What is the order of \( \alpha \) in \( \mathbf{F}^\times \)?
   c) What is the order of \( \alpha + 1 \) in \( \mathbf{F}^\times \)?

4. Let \( f = x^2 - x - 1 \in \mathbb{Z}_7[x] \) and let \( \mathbf{F} = \mathbb{Z}_7[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements in \( \mathbf{F}^\times \)?
   b) What is the order of \( \alpha \) in \( \mathbf{F}^\times \)?
   c) What is the order of \( \alpha + 1 \) in \( \mathbf{F}^\times \)?

5. Let \( f = x^2 - 2 \in \mathbb{Z}_{13}[x] \) and let \( \mathbf{F} = \mathbb{Z}_{13}[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements in \( \mathbf{F}^\times \)?
   b) What is the order of \( \alpha \) in \( \mathbf{F}^\times \)?
   c) What is the order of \( \alpha + 1 \) in \( \mathbf{F}^\times \)?

6. Let \( f = x^4 - x - 1 \in \mathbb{Z}_3[x] \) and let \( \mathbf{F} = \mathbb{Z}_3[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements in \( \mathbf{F}^\times \)?
   b) What is the order of \( \alpha + 1 \) in \( \mathbf{F}^\times \)?
   c) What is the order of \( \alpha \) in \( \mathbf{F}^\times \)?