COST-MINIMIZING INPUT CHOICES

Mathematically, this is a constrained minimization problem. But before proceeding with a rigorous solution, it is useful to state the result to be derived with an intuitive argument. To minimize the cost of producing a given level of output, a firm should choose that point on the $q_0$ isoquant at which the rate of technical substitution of $l$ for $k$ is equal to the ratio $w/v$. It should equate the rate at which $k$ can be traded for $l$ in production to the rate at which they can be traded in the marketplace. Suppose that this were not true. In particular, suppose that the firm were producing output level $q_0$ using $k = 10, l = 10$, and assume that the RTS were 2 at this point. Assume also that $w = $1, $v = $1, and hence that $w/v = 1$ (which is unequal to 2). At this input combination, the cost of producing $q_0$ is $20. It is easy to show this is not the minimal input cost. For example, $q_0$ can also be produced using $k = 8$ and $l = 11$; we can give up two units of $k$ and keep output constant at $q_0$ by adding one unit of $l$. But at this input combination, the cost of producing $q_0$ is $19$, and hence the initial input combination was not optimal. A contradiction similar to this one can be demonstrated whenever the RTS and the ratio of the input costs differ.

Mathematical analysis

Mathematically, we seek to minimize total costs given $q = f(k, l) = q_0$. Setting up the Lagrangian

$$\mathcal{L} = wl + vk + \lambda[q_0 - f(k, l)],$$  \hspace{1cm} (10.3)

the first-order conditions for a constrained minimum are

$$\frac{\partial \mathcal{L}}{\partial l} = w - \lambda \frac{\partial f}{\partial l} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial k} = v - \lambda \frac{\partial f}{\partial k} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q_0 - f(k, l) = 0,$$  \hspace{1cm} (10.4)

or, dividing the first two equations,

$$\frac{w}{v} = \frac{\partial f/\partial l}{\partial f/\partial k} = \text{RTS (of } l \text{ for } k).$$  \hspace{1cm} (10.5)

This says that the cost-minimizing firm should equate the RTS for the two inputs to the ratio of their prices.

Further interpretations

These first-order conditions for minimal costs can be manipulated in several different ways to yield interesting results. For example, cross-multiplying Equation 10.5 gives

$$\frac{f_k}{v} = \frac{f_l}{w},$$  \hspace{1cm} (10.6)

That is, for costs to be minimized, the marginal productivity per dollar spent should be the same for all inputs. If increasing one input promised to increase output by a greater amount per dollar spent than did another input, costs would not be minimal—the firm should hire more of the input that promises a bigger "bang per buck" and less of the more costly (in terms of productivity) input. Any input that cannot meet the common benefit–cost ratio defined in Equation 10.6 should not be hired at all.
Equation 10.6 can, of course, also be derived from Equation 10.4, but it is more instructive to derive its inverse:

$$\frac{w}{f_l} = \frac{v}{f_k} = \lambda.$$ \hspace{1cm} (10.7)

This equation reports the extra cost of obtaining an extra unit of output by hiring either added labor or added capital input. Because of cost minimization, this marginal cost is the same no matter which input is hired. This common marginal cost is also measured by the Lagrange multiplier from the cost-minimization problem. As is the case for all constrained optimization problems, here the Lagrange multiplier shows how much in extra costs would be incurred by increasing the output constraint slightly. Because marginal cost plays an important role in a firm’s supply decisions, we will return to this feature of cost minimization frequently.

**Graphical analysis**

Cost minimization is shown graphically in Figure 10.1. Given the output isoquant $q_0$, we wish to find the least costly point on the isoquant. Lines showing equal cost are parallel straight lines with slopes $-w/v$. Three lines of equal total cost are shown in Figure 10.1; $C_1 < C_2 < C_3$. It is clear from the figure that the minimum total cost for producing $q_0$ is given by $C_1$, where the total cost curve is just tangent to the isoquant. The associated inputs are $l^*$ and $k^*$, where the superscripts emphasize that these input levels are a solution to a cost-minimization problem. This combination will be a true minimum if the isoquant is convex (if the RTS diminishes for decreases in $k/l$). The mathematical and graphic analyses arrive at the same conclusion, as follows.

A firm is assumed to choose $k$ and $l$ to minimize total costs. The condition for this minimization is that the rate at which $k$ and $l$ can be traded technically (while keeping $q = q_0$) should be equal to the rate at which these inputs can be traded in the market. In other words, the RTS (of $l$ for $k$) should be set equal to the price ratio $w/v$. This tangency is shown in the figure; costs are minimized at $C_1$ by choosing inputs $k^*$ and $l^*$. 

![Figure 10.1 Minimization of Costs](image-url)
Contingent demand for inputs

Figure 10.1 exhibits the formal similarity between the firm’s cost-minimization problem and the individual’s expenditure-minimization problem studied in Chapter 4 (see Figure 4.6). In both problems, the economic actor seeks to achieve his or her target (output or utility) at minimal cost. In Chapter 5 we showed how this process is used to construct a theory of compensated demand for a good. In the present case, cost minimization leads to a demand for capital and labor input that is contingent on the level of output being produced. Therefore, this is not the complete story of a firm’s demand for the inputs it uses because it does not address the issue of output choice. But studying the contingent demand for inputs provides an important building block for analyzing the firm’s overall demand for inputs, and we will take up this topic in more detail later in this chapter.

The firm’s expansion path

A firm can follow the cost-minimization process for each level of output: For each \( q \), it finds the input choice that minimizes the cost of producing it. If input costs (\( w \) and \( v \)) remain constant for all amounts the firm may demand, we can easily trace this locus of cost-minimizing choices. This procedure is shown in Figure 10.2. The curve \( 0E \) records the cost-minimizing tangencies for successively higher levels of output. For example, the minimum cost for producing output level \( q_1 \) is given by \( C_1 \), and inputs \( k_1 \) and \( l_1 \) are used. Other tangencies in the figure can be interpreted in a similar way. The locus of these points is the firm’s expansion path.
tangencies is called the firm’s expansion path because it records how input expands as output expands while holding the prices of the inputs constant.

As Figure 10.2 shows, the expansion path need not be a straight line. The use of some inputs may increase faster than others as output expands. Which inputs expand more rapidly will depend on the shape of the production isoquants. Because cost minimization requires that the $RTS$ always be set equal to the ratio $w/v$, and because the $w/v$ ratio is assumed to be constant, the shape of the expansion path will be determined by where a particular $RTS$ occurs on successively higher isoquants. If the production function exhibits constant returns to scale (or, more generally, if it is homothetic), then the expansion path will be a straight line because in that case the $RTS$ depends only on the ratio of $k$ to $l$. That ratio would be constant along such a linear expansion path.

It would seem reasonable to assume that the expansion path will be positively sloped; that is, successively higher output levels will require more of both inputs. This need not be the case, however, as Figure 10.3 illustrates. Increases of output beyond $q_2$ cause the quantity of labor used to decrease. In this range, labor would be said to be an inferior input. The occurrence of inferior inputs is then a theoretical possibility that may happen, even when isoquants have their usual convex shape.

Much theoretical discussion has centered on the analysis of factor inferiority. Whether inferiority is likely to occur in real-world production functions is a difficult empirical question to answer. It seems unlikely that such comprehensive magnitudes as “capital” and “labor” could be inferior, but a finer classification of inputs may bring inferiority to light. For example, the use of shovels may decrease as production of building foundations (and the use of backhoes) increases. In this book we shall not be particularly concerned with the analytical issues raised by this possibility, although complications raised by inferior inputs will be mentioned in a few places.

With this particular set of isoquants, labor is an inferior input because less $l$ is chosen as output expands beyond $q_2$.

![Figure 10.3: Input Inferiority](image)