CSI 503 – Data Structures and Algorithms – Fall 2009
Solutions to Homework II

Problem 1:

Given: \( f(n) = [0.2n^2] - 2[n^{3/2}] - 9 \) and \( g(n) = 9n^2 \).

Using the facts that \( \lfloor x \rfloor \geq x - 1 \) and \( \lceil x \rceil \leq x + 1 \) for any \( x \geq 0 \), we have

\[
\begin{align*}
f(n) &\geq (0.2n^2 - 1) - 2(n^{3/2} + 1) - 9 \\
&= 0.2n^2 - 2n^{3/2} - 12 \\
&= 0.1n^2 + (0.1n^2 - 2n^{3/2} - 12) \\
&\geq 0.1n^2 + (0.1n^2 - 14n^{3/2}) \quad (\text{since } n^{3/2} \geq 1 \text{ for all } n \geq 1)
\end{align*}
\]

The term \( (0.1n^2 - 14n^{3/2}) \) is non-negative when \( 0.1n^{1/2} \geq 14 \) or \( n \geq 19,600 \). Therefore,

\[
f(n) \geq 0.1n^2 \quad \text{for all } n \geq 19,600.
\]

Since \( g(n) = 9n^2 \), we have \( f(n) \geq (0.1/9)(9n^2) = (1/90)g(n) \) for all \( n \geq 19,600 \). Therefore, \( f(n) = \Omega(g(n)) \) with \( c = 1/90 \) and \( n_0 = 19,600 \).

Problem 2:

Part (a): Given that \( f(n) = (n^{1/2} + n^{1/3})/2 \) and \( g(n) = 8n^{1/3} + (\log_2 n)^2 \), we have

\[
\frac{g(n)}{f(n)} = \frac{2[8n^{1/3} + (\log_2 n)^2]}{n^{1/2} + n^{1/3}}
\]

\[
= \frac{16n^{1/3}}{n^{1/2} + n^{1/3}} + \frac{2(\log_2 n)^2}{n^{1/2} + n^{1/3}}
\]

\[
= \frac{16}{n^{1/6} + 1} + \frac{2(\log_2 n)^2}{n^{1/2} + n^{1/3}}
\]

Let \( T_1 \) and \( T_2 \) denote the two terms on the right side of the last expression above. We have

\[
\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} T_1 + \lim_{n \to \infty} T_2.
\]

Clearly, \( \lim_{n \to \infty} T_1 = 0 \) since the numerator of \( T_1 \) is a constant, while the denominator approaches \( \infty \) as \( n \to \infty \).

To find the limit of \( T_2 \), we note that \( \log_2 n = (\log_2 e) \ln n \). Therefore,

\[
T_2 = \frac{2(\log_2 e)^2 (\ln n)^2}{n^{1/2} + n^{1/3}}.
\]

Let \( p(n) = c(\ln n)^2 \) and \( q(n) = n^{1/2} + n^{1/3} \), where \( c = 2(\log_2 e)^2 \) is a constant. Since both \( p(n) \) and \( q(n) \) approach \( \infty \) as \( n \to \infty \), we need to use L’Hospital’s rule.
Part (b): We note that \( g(n) = 2c\ln n/n \) and \( q'(n) = n^{-1/2}/2 + n^{-2/3}/3 \). (Here, \( p'(n) \) and \( q'(n) \) are respectively the derivatives of \( p(n) \) and \( q(n) \).) Now, \( p'(n)/q'(n) = 2c\ln n/(n^{1/2}/2 + n^{1/3}/3) \).

When we take the limit of \( p'(n)/q'(n) \), once again, we have the \( \infty/\infty \) form. So, we need to use L'Hospital's rule again. The derivative of \( 2c\ln n \) is \( 2c/n \) and the derivative of \( n^{-1/2}/2 + n^{1/3}/3 \) is \( n^{-1/2}/4 + n^{-2/3}/9 \). The ratio of these derivatives is now \( 2c/(n^{1/2}/4 + n^{2/3}/9) \).

In the last expression, the numerator is a constant while the while the denominator approaches \( \infty \) as \( n \to \infty \). Therefore, \( \lim_{n \to \infty} g(n) = o(f(n)) \).

Part (b): We will use the fact that for any constant \( a \), the derivative of \( a^n \) with respect to \( n \) is \( a^n \ln a \). Since \( g(n) = n^2\ln n \) and \( f(n) = 8^n \), we have \( g(n)/f(n) = n^2/(8/7)^n \).

Here, \( \lim_{n \to \infty} g(n)/f(n) \) has the form \( \infty/\infty \). Therefore, we must use L'Hospital's rule. Note that \( g'(n) = 2n \) and \( f'(n) = (8/7)^n \ln a \).

Again, \( \lim_{n \to \infty} g'(n)/f'(n) \) has the form \( \infty/\infty \). Therefore, we must use L'Hospital's rule again. Note that \( g''(n) = 2 \) and \( f''(n) = (8/7)^n (\ln a)^2 \). (Here, \( g''(n) \) and \( f''(n) \) are respectively the second derivatives of \( g(n) \) and \( f(n) \).)

Now, \( \lim_{n \to \infty} g''(n)/f''(n) \) is zero since the numerator is a constant while the denominator approaches \( \infty \) as \( n \to \infty \).

Hence, we conclude that \( g(n) = o(f(n)) \).

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Problem 3:

Part (a): We are given that \( T(1) = 3 \) and \( T(n) = 3T(n-1) + 3 \) for all \( n \geq 2 \). Now,

\[
T(n) = 3T(n-1) + 3 \\
= 3[3T(n-2) + 3] + 3 \\
= 3^2T(n-2) + (3^2 + 3) \\
= 3^2[3T(n-3) + 3] + (3^2 + 3) \\
= 3^3T(n-3) + (3^3 + 3^2 + 3) \\
\vdots \\
= 3^{n-1}T(1) + \sum_{i=1}^{n-1} 3^i
\]

In the last step above, the first term \( 3^{n-1}T(1) = 3^n \) (since \( T(1) = 3 \)). The second term is a geometric series whose sum is \( 3(3^{n-1} - 1)/2 \). Therefore, \( T(n) = (3/2)3^n - 1 \) for all \( n \geq 1 \).

Part (b): We note that \( T(n) < (3/2)3^n \) for all \( n \geq 1 \). Therefore, \( T(n) = O(3^n) \), with \( c_2 = 3/2 \) and \( n_0 = 1 \).
Also, $T(n) = 3^n + (3^n/2 - 3/2)$ for all $n \geq 1$. Since $3^n \geq 3$ for all $n \geq 1$, we have $T(n) \geq 3^n$ for all $n \geq 1$. That is, $T(n) = \Omega(3^n)$, with $c_1 = 1$ and $n_0 = 1$.

In other words, $3^n \leq T(n) \leq (3/2)3^n$ for all $n \geq 1$; that is, $T(n) = \Theta(3^n)$.

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Problem 4:

**Data structure:** We use two integer arrays. One of these arrays, denoted by $U[1 .. B]$, has $B$ elements. The other array, denoted by $\text{KEYS}[1 .. n]$, has $n$ elements.

**Preprocessing step:** (The given keys are $k_1, k_2, \ldots, k_n$.)

for $j = 1$ to $n$ do

(a) $\text{KEYS}[j] = k_j$

(b) $U[k_j] = j$

endfor

The time preprocessing is $O(n)$ since the for loop iterates $n$ times and the time for each iteration is $O(1)$. (Array $U$ has $B$ elements. We modify the values of just $n$ of the elements.)

**Idea for implementing the Member function:** In the preprocessing step, we stored key $k_j$ in position $j$ of the $\text{KEYS}$ array and stored the index $j$ in $U[k_j]$. Now, to check whether a key $i$ is in the $\text{KEYS}$ array, we first check the value of $U[i]$. If the value of $U[i]$ is not in the range $1$ to $n$, obviously $i$ is not in the $\text{KEYS}$ array. Otherwise, the value $i$ is one of the given keys only when $\text{KEYS}[U[i]]$ contains the value $i$.

**Pseudocode for Member operation:**

\[
\text{MEMBER}(i) \quad /* \text{The value of } i \text{ is in the range } [1 .. B]. */
\]

(a) Let $t = U[i]$. 

(b) if $((1 \leq t \leq n) \text{ and } (\text{KEYS}[t] = i))$

then return “Yes”

else return “No”.

The running time for the MEMBER operation is $O(1)$ since each of the two steps uses $O(1)$ time.