Lemma 1: Every pair of tasks in the set $A$ returned by the algorithm is compatible.

Proof: To begin with, the algorithm adds $a_1$ to the set $A$. At that time, $A = \{a_1\}$. So, compatibility is trivial. Subsequently, every time the algorithm adds a new task $a_i$ to $A$ in Step 4, the starting time of $a_i$ is at least as large as the largest finish time among all the tasks that are currently in $A$. Thus, the new task $a_i$ is compatible with every task that is currently in $A$. Since this property holds every time a new task is added to $A$, at the end of the algorithm, every pair of tasks in $A$ is compatible. ■

Lemma 2 (Greedy Choice Property): Let $(a_1, a_2, \ldots, a_n)$ denote the task ordering after the sorting step. There is an optimal solution to the problem that includes task $a_1$.

Proof: Suppose $A_0 = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ is an optimal solution, also arranged according to the sorted order of finish times.

If $i_1 = 1$, then $A'$ already includes $a_1$ and so the lemma follows trivially. So, assume that $i_1 \neq 1$. Consider the set $A'' = (A' - \{a_{i_1}\}) \cup \{a_1\}$. Since the finish time of $a_1$ is no more than that of $a_{i_1}$, any task with which $a_{i_1}$ is compatible is also compatible with $a_1$. In other words, $A''$ also contains tasks that are pairwise compatible. Further, $|A''| = |A'|$. Therefore $A''$ is also an optimal solution to the problem. Since $A''$ contains $a_1$, the lemma is proven. ■

Lemma 3 (Optimal Substructure Property): If $A$ is an optimal solution to the problem containing $a_1$, then $A - \{a_1\}$ is an optimal solution to the problem consisting of the set $S_2$ of tasks where $S_2 = \{a_i : s_i \geq f_1\}$. (That is, $A - \{a_1\}$ is an optimal solution to the subproblem specified by $S_2$ containing all the tasks that are compatible with $a_1$.)

Proof: The proof is by contradiction. Suppose there is a solution $A'$ to the subproblem specified by $S_2$ such that $|A'| > |A - \{a_1\}|$. Since all the tasks in $S_2$ are compatible with $A_1$, the set $A'' = A' \cup \{a_1\}$ is a solution to the original problem containing $a_1$. However, $|A''|$ is larger than $|A|$, and this contradicts the assumption that $A$ is an optimal solution to the original problem. Thus, $A - \{a_1\}$ is an optimal solution to the problem specified by $S_2$. ■