Suggested Solutions to Problem Set 3

1. Consider a Cobb-Douglas economy in which TFP is constant and equal to 2. Each year, owners of capital receive 1/3 of what is produced while workers receive the remainder. The number of workers increases by 3 percent per year. Capital depreciates at a rate of 10 percent per year. The saving rate is 15 percent.

   a. Write down the equation for investment in per-worker variables. $k_{t+1} = 0.9 \cdot k_t + \frac{0.30k^{1/3}}{1.03}$

   b. Solve for the steady state level of per worker output, capital, saving, and consumption. What is the intuition behind the condition you used to solve for the steady state values?

      $\left(n + d\right)k = sAk^\alpha \Rightarrow 0.13k = 0.3k^{1/3} \Rightarrow k = 3.51$

      $y = Ak^\alpha \Rightarrow y = 2(3.51)^{1/3} = 3.04$

      $c = (1 - s)y \Rightarrow c = 0.85 \times 2.93 = 2.58$

      $sy = 0.15 \times 2.93 = 0.44$

      The intuition behind the steady state condition is that in the steady state, the depletion of capital per worker (population growth and depreciation) must be offset by the replacement of capital per worker (investment, or saving).

   c. Now you get to choose the saving rate. Compute the Golden Rule levels of per worker output, capital, consumption, and saving. What is the saving rate?

      $\alpha Ak^{\alpha - 1} = n + d \Rightarrow \frac{1}{3} 2k^{-2/3} = 0.13 \Rightarrow k = 11.61$

      $y = 2(11.61)^{1/3} = 4.52$

      $sy = (n + d)k \Rightarrow s 4.52 = 0.13 \times 11.61 \Rightarrow s = 0.33 = 1/3$

      $c = (1 - s)y \Rightarrow c = 0.66 \times 4.52 = 3.04$

      $sy = 0.34 \times 4.52 = 1.51$

2. Consider this statement: “Devoting a larger share of national output to investment would help to restore rapid productivity growth and rising living standards.” Under what conditions is the statement accurate?

   - Increased saving will only boost productivity growth when the economy is away from the steady state. In the steady state, per-worker variables increase at the growth rate of worker effectiveness. This growth rate does not depend on the rate of saving.

   - The part of the statement about living standards depends on whether the economy is relative to the golden rule level of capital. If the capital stock is higher than the golden rule, it is possible to increase consumption per worker in the steady state by decreasing the saving rate. If the capital stock is lower than the golden rule level, then devoting more resources to investment (increasing saving) can indeed raise living standards (consumption) in the steady state.

3. Suppose that the economy’s production function is given by $Y_t = K_t^{1/2} \left(E_t N_t \right)^{1/2}$ and that the saving rate is equal to 16 percent and that the rate of depreciation is equal to 10 percent. Further, suppose
that the number of workers grows at 2 percent per year and that effectiveness per worker grows at 4 percent per year.
a. Find the steady-state values of
   • The capital stock per effective worker.
     \[(n + d + g) = s\tilde{k}^{-a-i} \Rightarrow (0.02 + 0.10 + 0.04) = 0.16\tilde{k}^{-1/2}\]
     \[\tilde{k} = 1,\]
   • Output per effective worker.
     \[\tilde{y} = \tilde{k}^{1/2} = 1\]
   • The growth rate of output per effective worker. The growth rate is 0 in the steady state.
   • The growth rate of output per worker. Output per worker grows at the growth rate of effectiveness per worker = 0.04.
   • The growth rate of output. Output grows at the growth rate of effectiveness plus the growth rate of workers = 0.04 + 0.02 = 0.06.

b. Suppose that the growth rate of effectiveness per worker doubles to 8 percent per year. Recompute the answers to (a). Explain.
   • The capital stock per effective worker.
     \[(n + d + g) = s\tilde{k}^{-a-i} \Rightarrow (0.02 + 0.10 + 0.08) = 0.16\tilde{k}^{-1/2}\]
     \[\tilde{k} = 16/25,\]
   • Output per effective worker.
     \[\tilde{y} = \tilde{k}^{1/2} = 4/5,\]
   • The growth rate of output per effective worker. The growth rate is 0 in the steady state.
   • The growth rate of output per worker. Output per worker grows at the growth rate of effectiveness per worker = 0.08.
   • The growth rate of output. Output grows at the growth rate of effectiveness plus the growth rate of workers = 0.08 + 0.02 = 0.10.
   The point here is that capital per effective worker and output per effective worker are lower with a higher growth rate of effectiveness, but the steady state per-worker variables (e.g., output, capital, and consumption) all grow at a faster rate.
   Graphically, the increase in the growth rate of effectiveness (g) pivots the replacement investment line upwards reducing capital per effective worker.

c. Now suppose that the growth rate of effectiveness per worker is still 4 percent per year, but the number of workers now grows at 3 percent per year. Recompute the answers to (a). Show the effect on capital per effective worker graphically. Compare output per effective worker and growth rates of output per worker in (a) and (c).
   • The capital stock per effective worker.
     \[(n + d + g) = s\tilde{k}^{-a-i} \Rightarrow (0.03 + 0.10 + 0.04) = 0.16\tilde{k}^{-1/2}\]
     \[\tilde{k} = (16/17)^2,\]
   • Output per effective worker.
     \[\tilde{y} = \tilde{k}^{1/2} = 16/17,\]
   • The growth rate of output per effective worker. The growth rate is 0 in the steady state.
• The growth rate of output per worker. Output per worker grows at the growth rate of effectiveness per worker = 0.04.

• The growth rate of output. Output grows at the growth rate of effectiveness plus the growth rate of workers = 0.04 + 0.03 = 0.07.

In part (a), output per effective worker is larger and growth rates are the same. The increase in population growth pivots the replacement investment curve upwards reducing equilibrium capital per effective worker.

4. Consider the following production function that does not exhibit a diminishing marginal product of capital: $y_t = Ak_t$.

   a. In the standard Solow model, the per-worker level of capital converges toward a steady state. Show that this does not occur in this model.

      In the AK model, the growth rate of capital is constant and equal to $\frac{k_{t+1}}{k_t} - 1 = g_k = sA - d$. A constant growth rate implies that the economy will never reach a steady state.

   b. Show that a higher saving rate leads to a permanently higher growth rate. Explain.

      See the equation above. As the saving rate rises, so too does the growth rate. In the absence of diminish returns to capital, a higher saving rate can sustain a permanently higher growth rate.

   c. Why does this conclusion differ from that in the Solow model?

      In the Solow model, a steady state is reached because of the diminishing marginal product of capital. In the AK model, the MPK is constant. Per worker variables will grow forever as long as the MPK is constant.

   d. Do you think the production function is reasonable? What theories would lead to this type of production function?

      We have to think about situations in which the MPK might reasonably be constant. Possibilities include the Y=AK model in which the acquisition of new capital has externalities which raise the level of technology so much that MPK does not fall. Perhaps R&D and IT behaves like this, but it is doubtful that MPK is constant in standard mature industries.

   e. Define absolute and conditional convergence. In the standard Solow model, what must be true for absolute convergence to hold? Conditional convergence?

      Absolute convergence: each economy ends up with the same steady state. Conditional convergence: each similar economy ends up with the same steady state – countries converge to their own steady state.

   f. What does the AK model imply about absolute and conditional convergence?
Growth rates will diverge as long as saving rates differ. Neither absolute nor conditional convergence need occur.

5. Abel & Bernanke Ch. 6, **Numerical Problem #7** (p. 243)

First, derive saving per worker as

\[ sy = y - c - g = [1 - .5(1 - t) - t] 8k^5 = .5(1 - t)8k^5 = 4 (1 - t)k^5 \]

(a) When \( t = 0 \), \( sy = 4 (1 - 0)k^5 = 4k^5 \) = national saving per worker

Investment per worker = \( (n + d)k = .1k \)

In steady state, \( sy = (n + d)k \), so 4\( k^5 \) = .1\( k \), or 40\( k^5 \) = \( k^2 \), so \( k = 1600 \). Since \( k = 1600 \), \( y = 8 \times 1600^5 = 320 \), \( c = .5(1 - 0)320 = 160 \), and \( (n + d)k = .1 \times 1600 = 160 \) = investment per worker.

(b) When \( t = 0.5 \), \( sy = 4 (1 - 0.5)k^5 = 2k^5 \) = national saving per worker

Investment per worker = \( (n + d)k = .1k \)

In steady state, \( sy = (n + d)k \), so 2\( k^5 \) = .1\( k \), or 20\( k^5 \) = \( k^2 \), so \( k = 400 \). Since \( k = 400 \), \( y = 8 \times 400^5 = 160 \), \( c = .5(1 - 0.5)160 = 40 \), and \( (n + d)k = .1 \times 400 = 40 \) = investment per worker, \( g = ty = .5 \times 160 = 80 \).

6. Abel & Bernanke Ch. 6, **Analytical Problem #3** (p. 243)

a. With a balanced budget \( T / N = g \). National saving is \( S = s(Y - T) = sN[(Y / N) - (T / N)] = sN(y - g) \).

Setting saving equal to investment gives

\[ S = I, \]

\[ sN(y - g) = (n + d)K, \]

\[ s(y - g) = (n + d)k, \]

\[ s[f(k) - g] = (n + d)k. \]

This equilibrium point \( k^* \) is shown in the figure below.

b. If the government permanently increases purchases per worker, the \( s[f(k) - g] \) curve shifts down from \( s[f(k) - g^1] \) to \( s[f(k) - g^2] \) in the figure below. In steady-state equilibrium, the capital-labor ratio is lower. Output per worker, capital per worker, and consumption per worker are lower in the steady state. The optimal level of government purchases is not zero—it depends on the benefits of the government purchases as well as on the costs of these purchases.