Stocks and Bonds
1. Present Value

2. Bonds

3. Stocks
1 Present Value = today’s value of income at a future date

- Income at one future date
  - value today of $X$ dollars in one year
    \[ V_t = \frac{X_{t+1}}{(1 + i_t)} \]
    where $i_t$ is the nominal interest rate on assets held during period $t$
  - value today of $X$ dollars in two years
    \[ V_t = \frac{X_{t+2}}{(1 + i_t)(1 + i_{t+1}^e)} \]
    where $i_{t+1}^e$ is the interest rate expected to prevail in one year
- Stream of income in the future, $z_t$ today, $z_{t+1}$ in one year, $z_{t+2}$ in two years and $z_{t+3}$ in three years

- known stream

$$V_t = z_t + \frac{z_{t+1}}{1 + i_t} + \frac{z_{t+2}}{(1 + i_t)(1 + i_{t+1}^e)} + \frac{z_{t+3}}{(1 + i_t)(1 + i_{t+1}^e)(1 + i_{t+2}^e)}$$

- unknown stream - replace known values of the future income with expectations

$$V_t = z_t + \frac{z_{t+1}^e}{1 + i_t} + \frac{z_{t+2}^e}{(1 + i_t)(1 + i_{t+1}^e)} + \frac{z_{t+3}^e}{(1 + i_t)(1 + i_{t+1}^e)(1 + i_{t+2}^e)}$$
– stream of income with interest rates constant

\[ V_t = z_t + \frac{z_{t+1}^c}{1 + i_t} + \frac{z_{t+2}^c}{(1 + i_t)^2} + \frac{z_{t+3}^c}{(1 + i_t)^3} \]

– constant interest rates and constant payments of \( iz \) (interest multiplied by principle) forever beginning one year from now

\[ V_t = iz \left( \frac{1}{1 + i} + \left( \frac{1}{1 + i} \right)^2 + \left( \frac{1}{1 + i} \right)^3 + \ldots \right) = z \]

note that the value of paying interest on debt forever (present-value sum) and the value of paying off the debt today (\( z \)) are equal!
- Real stream of income - use real interest to take present-value

  - Divide both sides of PV equation by current price to express real present value

  \[
  \frac{V_t}{P_t} = \frac{z_t}{P_t} + \frac{z_{t+1}^e}{1 + i_t P_t} \frac{1}{P_t} + \frac{z_{t+2}^e}{(1 + i_t)(1 + i_{t+1}^e)} \frac{1}{P_t} \\
  + \frac{z_{t+3}^e}{(1 + i_t)(1 + i_{t+1}^e)(1 + i_{t+2}^e)} \frac{1}{P_t}
  \]

  - Note that

  \[
  (1 + i_t) \frac{P_t}{P_{t+1}^e} = 1 + r
  \]
Using this, real present value is stated as:

\[
\frac{V_t}{P_t} = \frac{z_t}{P_t} + \frac{z_{t+1}}{1 + r_t P_{t+1}} + \frac{z_{t+2}^e}{(1 + r_t)(1 + r_{t+1}) P_{t+2}}
\]

\[
+ \frac{z_{t+3}^e}{(1 + r_t)(1 + r_{t+1})(1 + r_{t+2}) P_{t+3}} \frac{1}{P_{t+3}}
\]

- Definitions
  
  - discount rate = \( i \)
  
  - discount factor = \( \frac{1}{1+i} \)
2 Bonds

2.1 Definitions

- Maturity - the length of time the bond promises payments to its holder

- Coupon bonds - bonds that promise multiple payments before maturity and one payment at maturity

- Discount bonds - bonds that promise a single payment at maturity

- Face value - the payment at maturity
2.2 Price of a discount bond

- Assume the bond will pay $50 at the end of two years and that the interest rate is 4% and is expected to rise to 4.5% after one year. The price of the bond is the present value of that final payment.

\[
P_{2t} = \frac{$50}{(1 + i_t) (1 + i_{t+1}^e)} = \frac{$50}{(1.04)(1.045)} = $46.01
\]

- Arbitrage - Will the investor prefer the two year bond above or two one year bonds?

  - For every dollar you put in a two year bond, get \( \frac{1}{P_{2t}} \) two year bonds. At the end of the first year, now you have \( \frac{1}{P_{2t}} \) one year bonds valued at \( P_{1t+1}^e \) for a total value of \( \frac{P_{1t+1}^e}{P_{2t}} \).
– For every dollar you put in a one year bond, get $1 + i_t$.

– Which do you prefer?

\[ \frac{P_{1t+1}^e}{P_{2t}} = 1 + i_t \]

or equivalently

\[ P_{2t} = \frac{P_{1t+1}^e}{1 + i_t} \]

– The price of a two year bond must be the present-value of the expected price of the one-year bond. Note this ignores risk. Which option has risk in the first period?
Yield to maturity on an n-year bond or equivalently the n-year interest rate is defined as that constant annual interest rate that makes the bond price today equal to the present value of future payments on the bond.

– What is the yield to maturity on the two year bond above?

\[
$46.01 = \frac{$50}{(1 + i_{2t})^2}
\]

solving for \(i_{2t}\) yields 4.25%.

– Note that

\[
\frac{$50}{(1 + i_t)\left(1 + i_{t+1}^e\right)} = \frac{$50}{(1 + i_{2t})^2}
\]

Rearranging yields

\[
1 + 2i_{2t} + i_{2t}^2 = 1 + i_t + i_{t+1}^e + i_t i_{t+1}^e
\]
Long-term rates are the average of expected future short-term rates (ignoring risk)

- Why are long-term interest rates currently so low?

Risk - Since the two-year bond has a risky value in period 1 (why?) the alternatives of a two-year bond and two one year bond are not equally attractive if you don’t like risk. Therefore, the two-year bond alternative must pay a slightly higher return, given by the risk premium \((rp)\).

\[
\frac{P_{1t+1}^e}{P_{2t}} = 1 + i_t + (rp)_t
\]
performing the same steps as before, we find that with risk:

\[ i_{2t} \approx \frac{1}{2} \left( i_t + i_{t+1}^e + rp_t \right) \]

- With risk, long-term rates are a little above the average of expected future short-term rates.

- Yield curve = interest rates on n-period bonds as n (maturity) increases. Due to risk, the yield curve generally slopes upward, implying that longer term bonds have higher interest rates. Under what circumstances might we have an inverted (slopes downward) yield curve?
3 Stock Prices

- The value of a stock (and hence its price) is determined by the present value of its dividends. The ex-dividend price (after the current dividend has been paid) is therefore:

\[
Q_t = \frac{D_{t+1}^e}{1 + i_t + rp} + \frac{D_{t+2}^e}{(1 + i_t + rp)(1 + i_{t+1}^e + rp)}
+ \frac{D_{t+3}^e}{(1 + i_t + rp)(1 + i_{t+1}^e + rp)(1 + i_{t+2}^e + rp)} + \ldots
\]

- This expression is based on the fundamental value of the stock (the determinates of its true value).
• Dividends tend to move with profits, implying that higher expected future profits raise a stock’s price.

• Higher interest rates reduce the present-value of the stream of dividends, thereby reducing the stock’s price.

• An alternative expression:

• Note that the price expected to prevail in one period is given by:

$$Q_{t+1}^e = \frac{D_{t+2}^e}{(1 + i_{t+1}^e + rp)} + \frac{D_{t+3}^e}{(1 + i_{t+1}^e + rp)(1 + i_{t+2}^e + rp)} + \ldots$$
Substituting this above yields:

\[ Q_t = \frac{D_{t+1}^e}{1 + i_t + rp} + \frac{Q_{t+1}^e}{(1 + i_t + rp)} \]

- Rational speculative bubbles - occur when asset prices are driven from fundamental values by expectations of every-rising asset prices
4 Summary

- The price of an asset is the present-value of the assets future payments

- Payment of interest on a bond forever equals the value of the bond

- Long-term interest rates are approximately equal to the average of expected future short-term rates plus a risk premium

- The yield curve typically slopes upward due to the risk premium on long-term assets
• The price of a stock is the expected present value of future dividends
  – Higher profits raise a stock’s price
  – Higher interest rates reduce a stock’s price