SYNTAX

We describe the syntax of a family of languages $L$, the first order ones. All the languages of this type share a common logical vocabulary, comprising:

1) Logical Operators
   a) connectives: $\neg$, $\land$, $\lor$ and $\rightarrow$
   b) quantifiers: $\forall$ and $\exists$

2) Identity
   a two-place relation symbol $\approx$

3) Auxiliary Symbols
   a) an infinite list of individual variables
   b) three punctuation symbols: (, ) and ,

In addition, each member of the family may contain an extralogical vocabulary comprising symbols of one or more of the following types:

1) A list of individual constants

2) For each $n$, a list of $n$-place function symbols

3) For each $n$, a list of $n$-place predicate symbols

The difference between logical and non-logical vocabulary items is reflected in the semantics of languages of type $L$. The definition of what it means for an $L$-sentence to be true involves the meanings of the logical vocabulary, so that these meanings remain fixed when we consider different possible interpretations of a sentence. Our choice of logical vocabulary is motivated by intuitions about what ought to count as logical, but they are not sufficiently precise to determine uncontroversially how the distinction should be drawn. The decision to study first order languages commits us to one view of the domain of logic; others are possible.

We represent the extralogical vocabulary $E$ of a language as an ordered triple

$\langle (P_n)_{n<\omega}, (f^n_m)_{n<\omega}, (c^n_m)_{n<\omega} \rangle$

where $(P_n)_{n<\omega}$ is a list of predicate symbols, $(f^n_m)_{n<\omega}$ is a list of function symbols and $(c^n_m)_{n<\omega}$ is a list of individual constants. Let $E$ be some fixed extralogical vocabulary, then $L_E$, the first order language based upon $E$, is given by the following rules of syntax:

**Definition**

The (individual) **terms** of $L_E$ are defined by induction as follows:

*Base clause*: each individual variable and each individual constant of $E$ is a term of $L_E$.

*Inductive clause*: if $f$ is an $n$-place function symbol of $E$ and $t_1,\ldots, t_n$ are terms of $L_E$, then $f(t_1,\ldots, t_n)$ is a term of $L_E$.

*Exclusionary clause*: The only terms of $L_E$ are those given by the preceding clauses.
Definition

The (well-formed) formulas of \( L_E \) are defined by induction as follows:

Base clauses: 1) If \( t_1, \ldots, t_n \) are terms of \( L_E \) and \( P \) is an \( n \)-place \( P \) predicate symbol of \( V \), then \( P(t_1, \ldots, t_n) \) is a formula of \( L_E \)

2) If \( t \) and \( u \) are terms of \( L_E \), then \( t \neq u \) is a formula of \( L_E \)

Inductive clauses:

1) If \( \phi \) is a formula of \( L_E \), then so is \( \neg \phi \)
2) If \( \phi \) and \( \psi \) are formulas of \( L_E \), then so is \( (\phi \land \psi) \)
3) If \( \phi \) and \( \psi \) are formulas of \( L_E \), then so is \( (\phi \lor \psi) \)
4) If \( \phi \) and \( \psi \) are formulas of \( L_E \), then so is \( (\phi \rightarrow \psi) \)
5) If \( x \) is a variable and \( \phi \) is a formula of \( L_E \), then so is \( \forall x \phi \)
6) If \( x \) is a variable and \( \phi \) is a formula of \( L_E \), then so is \( \exists x \phi \)

Exclusionary clause: The only formulas of \( L_E \) are those given by the preceding clauses.

Some Syntactic Lemmas

1) (Unique Readability for Terms) For every term \( t \) of \( L_E \), exactly one of the following holds:
   i) \( t \) is a variable
   ii) \( t \) is a constant
   iii) \( t \) is of the form \( f(t_1, \ldots, t_n) \) for exactly one function symbol \( f \) and terms \( t_1, \ldots, t_n \).

Proof
   By induction on terms, using the fact that no initial segment of a term can be a term.

2) (Unique Readability for Formulas) For every formula \( \phi \) of \( L_E \), exactly one of the following holds:
   i) \( \phi \) is of the form \( P(t_1, \ldots, t_n) \), for unique \( P, t_1, \ldots, t_n \)
   ii) \( \phi \) is of the form \( t \neq u \), for unique \( t, u \)
   iii) \( \phi \) is of the form \( \neg \theta \), for unique \( \theta \)
   iv) \( \phi \) is of the form \( (\theta \land \psi) \), for unique \( \theta, \psi \)
   v) \( \phi \) is of the form \( (\theta \lor \psi) \), for unique \( \theta, \psi \)
   vi) \( \phi \) is of the form \( (\theta \rightarrow \psi) \), for unique \( \theta, \psi \)
   vii) \( \phi \) is of the form \( \forall x \theta \), for unique \( \theta \)
   viii) \( \phi \) is of the form \( \exists x \theta \), for unique \( \theta \)

In cases i) and iii), \( \phi \) is an atomic formula of \( L_E \).
In case ii), the main logical operator of \( \phi \) is negation and its immediate subformula is \( \theta \).
In case iv), the main logical operator of \( \phi \) is conjunction and its immediate subformulas are \( \theta \) and \( \psi \).
In case v), the main logical operator of \( \phi \) is disjunction and its immediate subformulas are \( \theta \) and \( \psi \).
In case vi), the main logical operator of \( \phi \) is implication and its immediate subformulas are \( \theta \) and \( \psi \).
In case vii), the main logical operator of \( \phi \) is the universal quantifier and its immediate subformula is \( \theta \).
In case viii), the main logical operator of \( \phi \) is the universal quantifier and its immediate subformula is \( \theta \).
Definition
For each formula $\phi$ of $L_E$, we define $S_\phi$, the set of subformulas of $\phi$, by induction as follows:

*Base clause:* If $\phi$ is of the form $P(t_1,\ldots,t_n)$ or $t = u$, $S_\phi = \{ \phi \}$

*Inductive clauses:*
- If $\phi$ is of the form $\neg \theta$, $S_\phi = S_{\theta} \cup \{ \phi \}$
- If $\phi$ is of the form $(\theta \land \psi)$, $S_\phi = S_{\theta} \cup S_{\psi} \cup \{ \phi \}$
- If $\phi$ is of the form $(\theta \lor \psi)$, $S_\phi = S_{\theta} \cup S_{\psi} \cup \{ \phi \}$
- If $\phi$ is of the form $(\theta \rightarrow \psi)$, $S_\phi = S_{\theta} \cup S_{\psi} \cup \{ \phi \}$
- If $\phi$ is of the form $\forall x \theta$, $S_\phi = S_{\theta} \cup \{ \phi \}$
- If $\phi$ is of the form $\exists x \theta$, $S_\phi = S_{\theta} \cup \{ \phi \}$

For any term $t$ of $L_E$, let $t_V$ be the set of variables occurring in $t$.

Definition
For each formula $\phi$ of $L_E$, we define $Fv_\phi$, the set of free variables of $\phi$, by induction as follows:

*Base clauses:*
- If $\phi$ is of the form $P(t_1,\ldots,t_n)$, $Fv_\phi = t_{1V} \cup \ldots \cup t_{nV}$
- If $\phi$ is of the form $t = u$, $Fv_\phi = t_V \cup u_V$

*Inductive clauses:*
- If $\phi$ is of the form $\neg \theta$, $Fv_\phi = Fv_{\theta}$
- If $\phi$ is of the form $(\theta \land \psi)$, $Fv_\phi = Fv_{\theta} \cup Fv_{\psi}$
- If $\phi$ is of the form $(\theta \lor \psi)$, $Fv_\phi = Fv_{\theta} \cup Fv_{\psi}$
- If $\phi$ is of the form $(\theta \rightarrow \psi)$, $Fv_\phi = Fv_{\theta} \cup Fv_{\psi}$
- If $\phi$ is of the form $\forall x \theta$, $Fv_\phi = Fv_{\theta} - \{ x \}$
- If $\phi$ is of the form $\exists x \theta$, $Fv_\phi = Fv_{\theta} - \{ x \}$

Definition
A sentence of $L_E$ is a formula $\phi$ such that $Fv_\phi = \emptyset$.

**SEMANTICS**

As in propositional logic, an interpretation of a language is supposed to specify whether its formulas are true or false and this is accomplished by specifying how truth values are to be assigned to atomic formulas together with a set of clauses which explain how the truth value of a compound is determined by the truth values of its components. However, in the case of propositional language the simplest formulas are atoms which are simply assigned truth values, in the case of first-order logic, the simplest formulas are themselves compounds—albeit of sub-sentential units—predicates and terms—and their interpretation, quite reasonably, is determined by the interpretation of these components. Furthermore, terms themselves may be compounds, whose interpretation will be determined by the interpretations of their components. It makes no sense to assign a truth value to a term or a predicate: we have to specify what individual the term denotes and to which individuals the predicate applies, and then use this information to determine the truth value of an atomic formula.
So, the interpretation of a first order language involves the following steps:

1) Specify which individuals the simplest terms denote.
2) Specify how to determine which individuals the compound terms denote in terms of the denotations of their components.
3) Specify to which individuals the predicates apply.
4) Determine the truth values of the atomic formulas in terms of the interpretations of their components.
5) Specify how to determine the truth values of compound formulas in terms of the truth values of their components.

So, instead of simply specifying a truth assignment, we must begin with something which tells us which individuals the terms denote and to which individuals the predicates apply. That thing is a **structure**.

**Definition**

A **structure** for \( L_E \) (or, an \( L_E \)-structure) is an ordered pair \( A = \langle A, v_A \rangle \) where:

1) \( A \) is a non-empty set, called to **domain** or **universe** of \( A \) (whose members are called the **individuals** of \( A \)).
2) \( v_A \) is a function with domain \( E \) such that:
   i) If \( c \) is a constant, then \( v_A(c) \in A \)
   ii) If \( f \) is an \( n \)-place function symbol, then \( v_A(f) \) is a function from \( A^n \) to \( A \) (i.e. a single-valued \( n+1 \)-place relation on \( A \))
   iii) If \( P \) is an \( n \)-place relation symbol, then \( v_A(P) \subseteq A^n \)

Note that we allow the function \( v_A \) to be empty.

**Examples:**

1) Let \( A = \langle \mathbb{N}, \{ <0, 0>, <s, s>, <+>, <\times> \rangle \rangle \), the structure of the natural numbers.
2) Let \( A = \langle U, \{ \in \} \rangle \), the structure whose universe is the power set of \( \mathbb{N} \) and whose only relation is set membership.

**Definition**

If \( A = \langle A, v_A \rangle \) is an \( L_E \)-structure, then a **variable interpretation** (or **assignment**) in \( A \) is a function from the set of individual variables to \( A \).

We now proceed to interpret the expressions of \( L_E \) in the structure \( A \) relative to a variable interpretation in \( A \).

For the terms \( t \) of \( L_E \), this amounts to specifying its denotation, i.e. which member of \( A \) it picks out or refers to.

**Definition**

Let \( A = \langle A, v_A \rangle \) be an \( L_E \)-structure, \( v \) be a variable interpretation in \( A \), and \( t \) be a term of \( L_E \), we define the denotation of \( t \) in \( A \) **relative to** the variable interpretation \( v \), \( \text{den}_A(t) [v] \) by induction on terms as follows:

**Base clauses:** If \( t \) is a constant, then \( \text{den}_A(t) [v] = v_A(t) \)
If \( t \) is a variable, then \( \text{den}_A(t) [v] = v(t) \)

**Inductive clause:** If \( t \) is of the form \( f(t_1, ..., t_n) \), then
\[
\text{den}_A(t) [v] = v_A(f)(\text{den}_A(t_1) [v], ..., \text{den}_A(t_n) [v])
\]
For the formulas $\phi$ of $L_n$, it amounts to assigning a truth value to $\phi$ in $A$.

**Definition**

Let $A = \langle A, v_A \rangle$ be an $L_E$-structure, $v$ be a variable interpretation in $A$, and $\phi$ be a formula of $L_E$. We define the truth value of $\phi$ in $A$ relative to the variable interpretation $v$, $v_A(\phi)[v]$, by induction on formulas as follows:

**Base clauses:**

- If $\phi$ is of the form $P(t_1, \ldots, t_n)$, $v_A(\phi)[v] = T$ iff $<\text{den}_A(t_1), \ldots, \text{den}_A(t_n)> [v] \in v_A(P)$
- If $\phi$ is of the form $t = u$, $v_A(\phi)[v] = T$ iff $\text{den}_A(t)[v] = \text{den}_A(u)[v]

**Inductive clauses:**

- If $\phi$ is of the form $\neg \theta$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v] = F$
- If $\phi$ is of the form $(\theta \land \psi)$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v] = T$ and $v_A(\psi)[v] = T$
- If $\phi$ is of the form $(\theta \lor \psi)$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v] = T$ or $v_A(\psi)[v] = T$
- If $\phi$ is of the form $(\theta \to \psi)$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v] = F$ or $v_A(\psi)[v] = T$
- If $\phi$ is of the form $\forall x \theta$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v'] = T$ for all variable interpretations $v'$ which differ from $v$ at most in assigning a different individual from $A$ to the variable $x$.
- If $\phi$ is of the form $\exists x \theta$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v'] = T$ for some variable interpretation $v'$ which differs from $v$ at most in assigning a different individual from $A$ to the variable $x$.

(In particular, we do not exclude the possibility that $v' = v$.)

**Notation**

If $v$ is a variable interpretation in $A$ and $a \in A$, we write $v_{(x=a)}$ for the variable interpretation which is just like $v$ except that $v_{(x=a)}(x) = a$. (Notice that, if $v(x) = a$, then $v_{(x=a)}$ is just $v$.)

Then, we can rewrite the last two clauses of this definition as follows:

- If $\phi$ is of the form $\forall x \theta$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v_{(x=a)}] = T$, for every $a \in A$.
- If $\phi$ is of the form $\exists x \theta$, $v_A(\phi)[v] = T$ iff $v_A(\theta)[v_{(x=a)}] = T$, for some $a \in A$.

**Example:** Let $L_E$ be the language whose only non-logical vocabulary consists of the constant $0$, the one-place function symbol $s$, and the two-place function symbols $+$ and $\times$. Let $A = \langle N, \langle 0, \bullet, < s, >, < +, >, < \times, > \rangle >$ be the structure of the natural numbers described above, and let $v(x_i) = i$, for all $i \in N$. Then we can use the preceding definition to interpret various formulas of this language:

1. $s(0) + s(0) = ss(0)$
2. $\forall x_1 \forall x_3 (x_1 \times (x_3 + s(0))) = ((x_1 \times x_3) + x_1)$
3. $\forall x_1 \exists x_2 x_2 = s(x_1)$